AN OPERADIC MODEL FOR A MAPPING SPACE AND ITS ASSOCIATED SPECTRAL SEQUENCE

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Abstract. Let X and Y be simplicial sets and $\mathbb{K}$ a field. In [13], Fresse has constructed an algebra model over an $E_\infty\mathbb{K}$-operad $\mathcal{E}$ for the mapping space $\mathcal{F}(X,Y)$, whose source X is finite, provided the homotopy groups of the target Y are finite. In this paper, we show that if the underlying field $\mathbb{K}$ is the closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ and the given mapping space is connected, then the finiteness assumption of the homotopy group of Y can be dropped in constructing the $\mathcal{E}$-algebra model. Moreover, we give a spectral sequence converging to the cohomology of $\mathcal{F}(X,Y)$ with coefficients in $\overline{\mathbb{F}}_p$, whose E_2-term is expressed via Lannes' division functor in the category of unstable $\overline{\mathbb{F}}_p$-algebra over the Steenrod algebra.

1. introduction

Let X and Y be spaces (or simplicial sets) and $\mathcal{F}(X,Y)$ denote the mapping space. In [17], Haefliger has given a rational model for a mapping space $\mathcal{F}(X,Y)$ for which Y is a nilpotent space. Subsequently, Bousfield, Peterson and Smith [9] have constructed another rational model for a mapping space with a functorial way, more precisely, their model is expressed via a division functor in the category of commutative differential $\mathbb{Z}$ graded algebras over the rational field. In the same paper, we are also aware of an interesting spectral sequence (henceforth BPS spectral sequence) converging to $H^*(\mathcal{F}(X,Y);\mathbb{Q})$, which is constructed with the algebraic model. Brown and Szczarba [10] have derived an accessible rational model for $\mathcal{F}(X,Y)$ by computing the division functor explicitly. The construction renders the model more computable.

As for a p-adic model for a space, Mandell [22] has proved that the homotopy category of nilpotent, p-complete spaces of finite p-type is equivalent to a full subcategory of the homotopy category of algebras over an $\overline{\mathbb{F}}_p$-operad $\mathcal{E}$. Here $\overline{\mathbb{F}}_p$ denotes the closure of the finite field $\mathbb{F}_p$. This motivates us to construct an $\mathcal{E}$-algebra model for a mapping space $\mathcal{F}(X,Y)$. Recently, Fresse [13] has given such a model by means of a division functor in the category of algebras over an $\mathbb{F}_p$-operad under some finiteness condition on the homotopy group of X. One of the purposes of this article is to improve Fresse's model for a mapping space. Another one is to

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construct a spectral sequence converging to $H^*(\mathcal{F}(X,Y);\overline{\mathbb{F}}_p)$, which is regarded as a p-adic version of the BPS spectral sequence.

It is worth to mention that Chataur and Thomas [12] have constructed a $\mathcal{C}_\infty$-algebra model for a mapping space which is the normalization of a simplicial $\mathcal{C}_\infty$-algebra, where $\mathcal{C}_\infty$ is a cofibrant replacement of the commutative operad. When the source space is the circle, the model for the mapping space, namely for the loop space, is related with the Hochschild chain complex. We also mention that our spectral sequence is a generalization of that due to Bökstedt and Ottosen [6], which converges to the cohomology of a free loop space.

We recall briefly the algebraic model for a mapping space over an operad due to Fresse. Let $\mathcal{E}$ denote the Barratt-Eccles operad over a field $\mathbb{K}$, which is an E_∞-operad. Then we can regard the normalized cochain functor $C^*(-;\mathbb{K})$ as a functor from the category of simplicial sets to $\mathcal{E}$-Alg the category of $\mathcal{E}$-algebras ([4, 1.5],[26]). Let A be an $\mathcal{E}$-algebra and K an $\mathcal{E}$-coalgebra. The diagonal map on $\mathcal{E}$ makes the dg-module $\mathrm{Hom}_{\mathbb{K}}(K,-)$ of homogeneous morphisms into an $\mathcal{E}$-algebra (see [13, 1.5] for details). We denote by $\mathcal{E}(\ ,\ )$ the hom set in $\mathcal{E}$-Alg.

Proposition 1.1. [13, 1.6.Proposition] *Let K be an $\mathcal{E}$-coalgebra. Then the functor $\mathrm{Hom}_{\mathbb{K}}(K,-)$ has a left adjoint. More explicitly, for A an $\mathcal{E}$-algebra, there is an $\mathcal{E}$-algebra $A \oslash K$ such that $\mathcal{E}(A\oslash K,-)\cong\mathcal{E}(A,\mathrm{Hom}_{\mathbb{K}}(K,-))$.*

Let K^* be an $\mathcal{E}$-algebra of finite type and K_* the $\mathcal{E}$-coalgebra which is the dual to K^*. Then, by definition, $A\oslash K_*$ is regarded as Lannes' functor $(A:K^*)_{\mathcal{E}\text{-Alg}}$ in the category of $\mathcal{E}$-algebras (see [30, 3.2 and 3.8] for the existence of the division functor, such as Lannes' T-functor). Moreover, if A is an almost free algebra $\mathcal{E}(V)$, then $A\oslash K$ is also an almost free algebra of the form $\mathcal{E}(V\otimes K)$. Since $\mathrm{Hom}_{\mathbb{K}}(K,-)$ preserves fibrations and acyclic fibrations, the total left derived functor $-\oslash^L K$ of $-\oslash K$ can be defined; that is, we have a natural bijection $\bar{h}\mathcal{E}(A\oslash^L K,-)\cong \bar{h}\mathcal{E}(A,\mathrm{Hom}_{\mathbb{K}}(K,-))$ for any $\mathcal{E}$-algebra A. Here $\bar{h}\mathcal{E}(\ ,\ )$ denotes the hom set in the homotopy category of $\mathcal{E}$-algebras. The functor $-\oslash^L K$ provides an $\mathcal{E}$-algebra model for a mapping space.

Theorem 1.2. [13, 1.10.Theorem] *Let X and Y be simplicial sets. We assume that X is finite and that $\pi_n(Y)$ is a finite p-group for $n\geq 0$. We have a quasi-isomorphism between $C^*(\mathcal{F}(X,Y);\mathbb{K})$ and $C^*(Y;\mathbb{K})\oslash^L C_*(X;\mathbb{K})$, which is functorial with respect to X and Y.*

Henceforth, we work in the category of algebras over the Barratt-Eccles operad $\mathcal{E}$ defined in the field $\overline{\mathbb{F}}_p$, unless otherwise specify mentioned. The chain and cochain complexes $C_*(X;\overline{\mathbb{F}}_p)$ and $C^*(X;\overline{\mathbb{F}}_p)$ are written as $C_*(X)$ and $C^*(X)$, respectively. In this paper, we first show that $C^*(\mathcal{F}(X,Y))$ can be connected with $C^*(Y)\oslash^L C_*(X)$ by quasi-isomorphisms without assuming that $\pi_n(Y)$ is a finite p-group, subject to the connectedness of the mapping space $\mathcal{F}(X,Y)$. More precisely, we establish the following theorem.

Theorem 1.3. *Let X be a finite simplicial set and Y a connected nilpotent simplicial set of finite type. Assume that the connectivity of Y is greater than or equal to the dimension of X. Then there exists an isomorphism between $C^*(\mathcal{F}(X,Y))$ and $C^*(Y)\oslash^L C_*(X)$, which is functorial with respect to X and Y, in the homotopy category of $\mathcal{E}$-algebras.*

As mentioned above, the functor $-\oslash K_*$ is regarded as Lannes' division functor $(-:K^*)_{\mathcal{E}\text{-Alg}}$. This fact enables us to construct a spectral sequence converging to the cohomology $H^*(\mathcal{F}(X,Y))$. In order to describe the spectral sequence, we recall that the generalized Steenrod algebra $\mathcal{B}$ is the free associative $\mathbb{F}_p$-algebra generated by the P^s and (if $p>2$) the βB^s for $s\in\mathbb{Z}$ over the two sided ideal generated by the Adem relations (see [22, Section 11]). The result [22, Theorem 1.4] states that the quotient algebra $\mathcal{B}/(Id-P^0)$ is the usual Steenrod algebra $\mathcal{A}$. Let $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$ be the category of unstable $\overline{\mathbb{F}}_p$-algebras over the generalized Steenrod algebra $\mathcal{B}$. We have a spectral sequence.

Theorem 1.4. (Compare with [9, Corollary 3.5]) *Let X be a finite simplicial set and Y a connected nilpotent simplicial set of finite type. Assume that the connectivity of Y is greater than or equal to the dimension of X. Then there exists a left-half plane spectral sequence $\{E_r,d_r\}$ with*

$$E_2^{s,*}\cong L_s(H^*(Y):H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$$

converging strongly to $H^*(\mathcal{F}(X,Y))$. Here $L_s(-:H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ denotes the s^{th} left derived functor of the division functor $(-:H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ in the category $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$. Moreover the spectral sequence is natural with respect to X and Y.

In what follows, we shall refer to the spectral sequence in Theorem 1.4 as the mod p BPS spectral sequence. For a $\mathcal{B}$-algebra B and a $\mathcal{B}$-algebra A of finite type, one can define the derived functor $L_s(B:A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ using a simplicial resolution of B in the category $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$. Since the resolution is a complex in the category of unstable $\mathcal{B}$-modules, the functor $L_s(B:A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ for any s inherits the $\mathcal{B}$-module structure from that of the complex. The same derived functor can be defined in the category $\mathcal{S}\text{-}\overline{\mathbb{F}}_p$ of unstable $\overline{\mathbb{F}}_p$-algebras over the usual Steenrod algebra $\mathcal{A}$. Observe that an object in $\mathcal{S}\text{-}\overline{\mathbb{F}}_p$ is regarded as one in $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$ with the natural projection $\mathcal{B}\to\mathcal{B}/(Id-P^0)=\mathcal{A}$. The following theorem allows us to work in the more familiar category $\mathcal{S}\text{-}\overline{\mathbb{F}}_p$ than $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$ when computing the mod p BPS spectral sequence.

Theorem 1.5. *Let A and B be $\mathcal{A}$-algebras of finite type. Then $L_s(B:A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is isomorphic to $L_s(B:A)_{\mathcal{S}\text{-}\overline{\mathbb{F}}_p}$ as a $\mathcal{B}$-module for any s.*

This theorem implies that the mod p BPS spectral sequence is reducible in the second quadrant. Moreover we have

Assertion 1.6. *The mod p BPS spectral sequence possesses an unstable module structure on $\mathcal{B}$ and hence on $\mathcal{A}$.*

For the more precise statement concerning the Steenrod operations on the spectral sequence, see Theorem 7.7.

The rest of the paper is organized as follows. We recall Mandell's work for p-adic homotopy theory in Section 2 since our proof of Theorem 1.3 relies on the work. In section 3, we prove Theorem 1.3 and give a result (Theorem 3.6) concerning the homotopy type of mapping space. Section 4 is devoted to proving Theorem 1.4. The edge homomorphism of the mod p BPS spectral sequence in Theorem 1.4 is also considered. In Section 5, we prove that, for any unstable $\mathcal{A}$-algebra A of finite type, $L_s(H^*(K(\mathbb{Z}/p,n):A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}=0$ when $s<0$. This fact is a key to proving Theorem 1.5. In Section 6, we first clarify the algebra structure of the division functor $(B:A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ over $\mathcal{B}$ for given unstable $\mathcal{B}$-algebras B and A. Moreover, we give a few examples in which the division functor $(H^*(Y) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ for some spaces X and Y is compared with the cohomology of the function space $\mathcal{F}(X,Y)$ with coefficients in $\overline{\mathbb{F}}_p$ via the edge homomorphism. In particular, an interesting calculation of the Steenrod operation in $H^*(\mathcal{F}(\Sigma_g, BSU(2)); \overline{\mathbb{F}}_3)$ is made in terms of the Lannes' division functor, where Σ_g is a Riemann surface of genus g and $BSU(2)$ denotes the classifying space of the Lie group $SU(2)$. We mention that the calculation is based on a more general result (Theorem 6.6). Section 7, Appendix, is devoted to defining well-behaved Steenrod operations in the spectral sequence which is constructed out of a simplicial $\mathcal{E}$-algebra. As a consequence, we will see that the mod p BPS spectral sequence possesses an unstable module structure on $\mathcal{A}$.

It is convenient to fix terminology for this article. An operad $\mathcal{E} = \{\mathcal{E}(r)\}_{r\geq 0}$ over a field $\mathbb{K}$ is called an E_∞-operad if each complex $\mathcal{E}(r)$ is acyclic and consists of free modules over the group ring $\mathbb{K}[\Sigma_r]$, where Σ_r is the symmetric group of order r. We refer to an E_∞-operad as an $E_\infty\mathbb{K}$-operad when emphasizing the underlying field $\mathbb{K}$. An almost free algebra is an $\mathcal{E}$-algebra of the form $(\mathcal{E}(V), d_0 + d_1)$, for which $(\mathcal{E}(V), d_0)$ is the free $\mathcal{E}$-algebra generated by a differential graded module V and d_1 is a derivation associated with a morphism $h : V \to \mathcal{E}(V)$ of degree $+1$. More precisely, we can write $d_1(\rho\otimes v_1 \otimes \cdots \otimes v_r) = \sum_{i=1}^{r} \pm\rho(v_1, ..., h(v_i), ..., v_r)$ for any $\rho \otimes v_1 \otimes \cdots \otimes v_r \in \mathcal{E}(V) = \oplus_{r\geq 0}\mathcal{E}(r) \otimes_{\mathbb{K}[\Sigma_r]} V^{\otimes r}$ (see [13, Section 2.3]).

2. Overview of Mandell's work for p-adic homotopy theory

As mentioned in the introduction, our proof of Theorem 1.3 relies on Mandell's work for p-adic homotopy theory [22]. We recall it briefly in this short section.

Let $C_*(X)$ denote the normalized chain complex of a simplicial set X with coefficients in the field $\overline{\mathbb{F}}_p$ and let $C^*(X)$ be the dual to $C_*(X)$, namely the normalized cochain complex.

Let $\mathcal{E}$ be an $E_\infty\overline{\mathbb{F}}_p$-operad and $\mathcal{E}(\ ,\ )$ the hom set in the category of $\mathcal{E}$-algebras $\mathcal{E}$-Alg. The hom set of the category of the simplicial sets $\Delta^{op}Set$ is denoted by Simpl(,). Let Δ be a category consisting of ordered sets $[n] = \{0, 1, ..., n\}$ and non-decreasing maps $[n] \to [m]$. Observe that the standard simplicial set $\Delta[n] = \hom_\Delta(\ , [n])$ defines a cosimplicial simplicial set $\Delta[\bullet]$ and gives the simplicial $\mathcal{E}$-algebra $C^*(\Delta[\bullet])$.

In [22], Mandell has defined the contravariant functor U from the category of $\mathcal{E}$-algebras $\mathcal{E}$-Alg to the category $\Delta^{op}Set$ by $UA = \mathcal{E}(A, C^*(\Delta[\bullet]))$ for any $\mathcal{E}$-algebra A. An important property of the functor U is stated as follows.

Proposition 2.1. [22, Proposition 4.2] *The functors U and C^* are contravariant right adjoints between the category of simplicial sets and the category of $\mathcal{E}$-algebras:*

$$\mathrm{Simpl}(X, UA) \cong \mathcal{E}(A, C^*(X)).$$

Let $\mathcal{H}_0$ be the homotopy category obtained from the category of simplicial sets by formally inverting the weak equivalences and $\bar{h}\mathcal{E}$ denote the homotopy category of $\mathcal{E}$-algebras.

Proposition 2.2. [22, Proposition 4.3] *The derived functor $\mathbf{U}$ of U exists and gives an adjunction $\mathcal{H}_0(X, \mathbf{U}A) \cong \bar{h}\mathcal{E}(A, C^*(X))$.*

3. Proof of Theorem 1.3

Before proving Theorem 1.3, we first recall a result due to Fresse, which is a key to constructing the $\mathcal{E}$-algebra model $C^*(Y) \oslash^L C_*(X)$ for the mapping space $\mathcal{F}(X,Y)$. Observe that, by definition, $\mathcal{F}(X,Y)_q = \mathrm{Simpl}(X \times \Delta[q], Y)$.

For any simplicial set K, let $F_{C^*(K)} \to C^*(K)$ denote the universal cofibrant resolution of the $\mathcal{E}$-algebra $C^*(K)$. Observe that the resolution is constructed with the bar and cobar construction [14, Theorem 2.19].

Theorem 3.1. [13, 5.1.Theorem] *Let X and Y be a simplicial sets. We have a morphism of $\mathcal{E}$-algebras $q : F_{C^*(X\times Y;\mathbb{K})} \to C^*(X;\mathbb{K})\widehat{\otimes}C^*(Y;\mathbb{K})$, functorial in X and Y, which makes the diagram*

$$\begin{array}{ccc} F_{C^*(X\times Y;\mathbb{K})} & & \\ {\scriptstyle p}\downarrow\simeq & \overset{q}{\searrow} & \\ C^*(X\times Y;\mathbb{K}) & \xrightarrow[EZ]{\simeq} & C^*(X;\mathbb{K})\widehat{\otimes}C^*(Y;\mathbb{K}) \end{array}$$

commutative in cohomology, where EZ is the classical shuffle morphism.

We can assume that Y is a connected nilpotent p-complete simplicial set. In fact it is immediate that a p-completion $\phi : Y \longrightarrow Y_p^\wedge$ induces a quasi-isomorphism $C^*(\mathcal{F}(*, Y_p^\wedge)) \longrightarrow C^*(\mathcal{F}(*, Y))$. Inductive application of the Eilenberg-Moore spectral sequence mentioned in [21, Theorem 2.1] enables us to conclude that the p-completion ϕ gives rise to a quasi-isomorphism $C^*(\mathcal{F}(X, Y_p^\wedge)) \longrightarrow C^*(\mathcal{F}(X,Y))$. Observe that X is a finite simplicial set.

Let $\iota : A \xrightarrow{\simeq} C^*(Y)$ be a cofibrant resolution. Since Y is resolvable, the unit $Y \longrightarrow UA$, which is the adjoint to ι, is a weak equivalence. In order to prove Theorem 1.3, we first consider the following sequence (3.1) of morphisms of simplicial sets:

$$\begin{array}{ccc} \mathcal{F}(X,Y) & \xrightarrow[\simeq]{} & \mathcal{F}(X,UA) = \mathrm{Simpl}(X\times\Delta[\bullet], UA) \\ & & \uparrow\cong \\ & & \mathcal{E}(A, C^*(X\times\Delta[\bullet])) \\ & & \uparrow p_* \\ & & \mathcal{E}(A, F_{C^*(X\times\Delta[\bullet])}) \\ & & \downarrow q_* \\ & & \mathcal{E}(A, C^*(X)\otimes C^*(\Delta[\bullet])) \\ & & \downarrow\cong \\ & & U(A\oslash C_*(X)) = \mathcal{E}(A\oslash C_*(X), C^*(\Delta[\bullet])). \end{array}$$

We define functors C^*_X, F_X and T^*_X from the category $\Delta^{op}Set$ to $\mathcal{E}$-Alg by $C^*_X(-) = C^*(X\times -)$, $F_X(-) = F_{C^*(X\times -)}$ and $T^*_X(-) = C^*(X)\otimes C^*(-)$, respectively. Define an $\mathcal{E}$-algebra map $\psi_1 : C^*(X\times K) \to \mathrm{Simpl}(K, C^*_X(\Delta[\bullet]))$ by $\psi_1(\alpha)(\sigma) = (1_X\times\sigma)^*(\alpha)$, where $1_X\times\sigma : X\times\Delta[n] \to X\times K$ for $\sigma \in K_n \cong \mathrm{Simpl}(\Delta[n], K)$ (see [22, Section 4]). In similar fashion, $\mathcal{E}$-algebra maps $\psi_2 : F_{C^*(X\times K)} \to \mathrm{Simpl}(K, F_X(\Delta[\bullet]))$ and $\psi_3 : C^*(X)\otimes C^*(K) \to \mathrm{Simpl}(K, T^*_X(\Delta[\bullet]))$ are defined.

Lemma 3.2. *The maps ψ_1, ψ_2 and ψ_3 are isomorphisms which make the following diagram commutative:*

$$\begin{array}{ccccc} C^*(X\times K) & \xleftarrow[\simeq]{p} & F_{C^*(X\times K)} & \xrightarrow[\simeq]{q} & C^*(X)\otimes C^*(K) \\ \psi_1 \downarrow \cong & & \cong \downarrow \psi_2 & & \cong \downarrow \psi_3 \\ \mathrm{Simpl}(K, C_X^*(\Delta[\bullet])) & \xleftarrow[p_*]{} & \mathrm{Simpl}(K, F_X(\Delta[\bullet])) & \xrightarrow[q_*]{} & \mathrm{Simpl}(K, T_X^*(\Delta[\bullet])). \end{array}$$

Proof. The inverse $\eta_1 : \mathrm{Simpl}(K, C_X^*(\Delta[\bullet])) \to C^*(X \times K)$ of ψ_1 is given by $\eta(\gamma)(\sigma) = \gamma(\sigma^2)(\sigma^1 \times c_n)$ for $\gamma \in K$, where $\sigma = \sigma^1 \times \sigma^2 \in X_n \times K_n$ and $c_n = id_{[n]} \in \Delta[n]_n$. Similarly, we can define the inverses of ψ_2 and ψ_3. The commutativity follows from the functorality of p and of q. □

Theorem 3.3. *In the sequence (3.1) of morphisms of simplicial sets, the maps $p_* : \mathcal{E}(A, F_{C^*(X\times\Delta[\bullet])}) \to \mathcal{E}(A, C^*(X \times \Delta[\bullet]))$ and $q_* : \mathcal{E}(A, F_{C^*(X\times\Delta[\bullet])}) \to \mathcal{E}(A, C^*(X)\otimes C^*(\Delta[\bullet]))$ are weak equivalences for any cofibrant object A.*

Thus the sequence (3.1) allows us to obtain an operadic model for $\mathcal{F}(X, Y)$.

Corollary 3.4. *In the homotopy category $\mathcal{H}_0$, the simplicial set $U(A \oslash C_*(X))$ is isomorphic to $\mathcal{F}(X, Y)$.*

Proof of Theorem 3.3. We write $U_{C_X^*}A = \mathcal{E}(A, C^*(X\times\Delta[\bullet]))$, $U_{F_X}A = \mathcal{E}(A, F_{C^*(X\times\Delta[\bullet])})$ and $U_{T_X^*}A = \mathcal{E}(A, C^*(X)\otimes C^*(\Delta[\bullet]))$. By using Lemma 3.2 and the same argument as in the proof of Proposition 2.1 due to Mandell, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Simpl}(K, U_{C_X^*}A) & \xrightarrow{\cong} & \mathcal{E}(A, C^*(X\times K)) \\ (p_*)_* \uparrow & & \uparrow \mathcal{E}(A,p) \\ \mathrm{Simpl}(K, U_{F_X}A) & \xrightarrow{\cong} & \mathcal{E}(A, F_{C^*(X\times K)}) \\ (q_*)_* \downarrow & & \downarrow \mathcal{E}(A,q) \\ \mathrm{Simpl}(K, U_{T_X^*}A) & \xrightarrow[\cong]{} & \mathcal{E}(A, C^*(X)\otimes C^*(K)) \end{array}$$

in which horizontal arrows are bijective. Here $\mathcal{E}(A, p)$ and $\mathcal{E}(A, q)$ are maps induced by p and q, respectively. If A is a cofibrant object, the diagram obtained by replacing $\mathcal{E}$ and Simpl with $\bar{h}\mathcal{E}$ and $\mathcal{H}_0$ is also commutative. In the diagram, $\mathcal{E}(A, p)$ and $\mathcal{E}(A, q)$ are bijective because p and q are quasi-isomorphisms. Thus we see that p_* and q_* are weak equivalences. □

We here recall the definition of the B_*-complex introduced by Mandell. Let $B_{m,n}$ ($1 \le m < \infty$, $n > 1$) be an $\mathcal{E}$-algebra model for $K(\mathbb{Z}/p^m, n)$, that is, there exists a quasi-isomorphism $B_{m,n} \to C^*(K(\mathbb{Z}/p^m, n))$. Let $B_{\infty,n}$ be an $\mathcal{E}$-algebra model for $K(\mathbb{Z}_p^\wedge, n)$. A B-cell $(CB_{m,n}, B_{m,n})$ is an augmented $\mathcal{E}$-algebra $CB_{m,n}$ together with $B_{m,n} \to CB_{m,n}$ a cofibration of augmented $\mathcal{E}$-algebras such that the augmentation $CB_{m,n} \to \overline{\mathbb{F}}_p$ is a quasi-isomorphism. We can assume $B_{m,n}$ and $CB_{m,n}$ are almost free algebras. A B_*-complex is an $\mathcal{E}$-algebra $A = \mathrm{Colim}\ A_j$ such that $A_0 = \overline{\mathbb{F}}_p$ and for each $j > 0$, A_j fits in a push out diagram

$$\begin{array}{ccc} B_{m_j,n_j+1} & \rightarrowtail & CB_{m_j,n_j+1} \\ f \downarrow & & \downarrow \\ A_j & \longrightarrow & A_{j+1}, \end{array}$$

where $\{n_j\}$ is a non-decreasing sequence in which positive numbers repeat at most finitely many times.

By assumption, the simplicial set Y has a Postnikov tower whose fibres are all $K(\mathbb{Z}/p^m, n)$'s and $K(\mathbb{Z}_p^\wedge, n)$'s with only finitely many for each n. From the tower, we have a B_*-complex $A = \mathrm{Colim}\ A_j \overset{\simeq}{\to} C^*(Y)$. Applying the functor $- \oslash C_*(X)$ to each stage of the B_*-complex, we get a push out diagram (4.2):

$$\begin{array}{ccc} B \oslash C_*(X) & \rightarrowtail & CB \oslash C_*(X) \\ {\scriptstyle f\oslash 1}\downarrow & & \downarrow \\ A_j \oslash C_*(X) & \longrightarrow & A_{j+1} \oslash C_*(X), \end{array}$$

and a pull back diagram (4.3):

$$\begin{array}{ccc} U(B \oslash C_*(X)) & \longleftarrow & U(CB \oslash C_*(X)) \\ {\scriptstyle U(f\oslash 1)}\uparrow & & \uparrow \\ U(A_j \oslash C_*(X)) & \longleftarrow & U(A_{j+1} \oslash C_*(X)). \end{array}$$

Here (CB, B) is a B-cell with $B = B_{m_j, n_j+1}$ $(1 \le m_j \le \infty)$. From Corollary 3.4 and the assumption on the dimension of X and the connectivity of Y, it follows that the simplicial set $U(B_{m_j,n_j+1} \oslash C_*(X))$ is simply connected. Thus the diagram (4.3) enables us to obtain the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ converging strongly to $H(C^*(U(A_{j+1} \oslash C_*(X))))$ with

$$E_2 \cong \mathrm{Tor}_{H(C^*(U(B\oslash C_*(X))))}(H(C^*(U(A_i \oslash C_*(X)))), H(C^*(U(CB \oslash C_*(X))))).$$

From [22, Corollary 3.6] and the diagram (4.2), we have a left half-plane cohomological spectral sequence $\{\hat{E}_r, \hat{d}_r\}$ with

$$\hat{E}_2 \cong \mathrm{Tor}_{H^*(B\oslash C_*(X))}(H^*(A_i \oslash C_*(X)), H^*(CB \oslash C_*(X)))$$

converging strongly to $H^*(A_{j+1} \oslash C_*(X))$. Comparing $\{\hat{E}_r, \hat{d}_r\}$ and $\{E_r, d_r\}$ with the morphism of spectral sequences induced by the adjoint maps $A_i \oslash C_*(X) \to C^*(U(A_i \oslash C_*(X)))$, $B \oslash C_*(X) \to C^*(U(B \oslash C_*(X)))$ and $CB \oslash C_*(X) \to C^*(U(CB \oslash C_*(X)))$, we have the following theorem.

Theorem 3.5. *Let A be the B_*-complex mentioned above. Then the adjoint $A \oslash C_*(X) \to C^*U(A \oslash C_*(X))$ is a quasi-isomorphism.*

Proof. In order to obtain the result, it suffices to prove that the adjoint map $ad : B_{m_j,n_j+1} \oslash C_*(X) \to C^*(U(B_{m_j,n_j+1} \oslash C_*(X)))$ is a quasi-isomorphism for any $1 \le m_j \le \infty$ and n_j.

Since $\mathcal{H}_0(X, UCB) \cong \bar{h}\mathcal{E}(CB, C^*(X)) = *$ for any simplicial set X, it follows that UCB is contractible and hence so is $U(CB \oslash C_*(X)) \simeq \mathcal{F}(X, UCB)$. We see that $\bar{h}\mathcal{E}(CB\oslash C_*(X), -) \cong \bar{h}\mathcal{E}(CB, C^*(X)\otimes -) \cong \bar{h}\mathcal{E}(\overline{\mathbb{F}}_p, C^*(X)\otimes -) = *$. Thus the $\mathcal{E}$-algebra $CB \oslash C_*(X)$ is acyclic. Hence $ad : CB \oslash C_*(X) \to C^*(U(CB \oslash C_*(X)))$ is a quasi-isomorphism.

The result [13, 4.1.Lemma] asserts that the $\mathcal{E}$-algebra $B_{1,n} \oslash C_*(X)$ is quasi-isomorphic to $C^*(\mathcal{F}(X, K(\mathbb{Z}/p, n)))$. Therefore, it follows from [22, Theorem 7.3] that $B_{1,n} \oslash C_*(X)$ is quasi-isomorphic to a B_*-complex. The argument of the proof of the implication (ii) $\Rightarrow$ (i) in [22, Theorem 7.3 (ii)] allows us to deduce that the adjoint $ad : B_{1,n} \oslash C_*(X) \to C^*(U(B_{1,n} \oslash C_*(X)))$ is a quasi-isomorphism.

Consider the push out diagram

$$\begin{array}{ccc} B_{1,n_j} & \rightarrowtail & CB_{1,n_j+1} \\ \downarrow & & \downarrow \\ B_{m-1,n_j+1} & \longrightarrow & B_{m,n_j+1}, \end{array}$$

associated with the fibre square

$$\begin{array}{ccc} K(\mathbb{Z}/p, n_j) & \longleftarrow & * \\ \uparrow & & \uparrow \\ K(\mathbb{Z}/p^{m-1}, n_j+1) & \longleftarrow & K(\mathbb{Z}/p^m, n_j+1). \end{array}$$

By using inductively the comparison of the spectral sequences mentioned above with the adjoint maps, we see that for $2 \le m < \infty$, $ad : B_{m,n_j+1} \oslash C_*(X) \to C^*(U(B_{m,n_j+1} \oslash C_*(X)))$ is also a quasi-isomorphism. We can regard B_{∞,n_j+1} as the colimit of the cofibre sequence $B_{1,n_j+1} \to B_{2,n_j+1} \to \cdots \to B_{m,n_j+1} \to \cdots$. This implies that $ad : B_{\infty,n_j+1} \oslash C_*(X) \to C^*(U(B_{\infty,n_j+1} \oslash C_*(X)))$ is a quasi-isomorphism. □

Proof of Theorem 1.3. Combining Corollary 3.4 with Theorem 3.5, we have the result. □

Thanks to Corollary 3.4, the following significant result on the homotopy type of mapping spaces is also deduced.

Theorem 3.6. *Let X and X' be connected finite simplicial sets and Y a connected p-complete nilpotent simplicial set of finite type. Suppose that $\dim X$ and $\dim X'$ are less than or equal to the connectivity of Y and $C^*(X)$ is quasi-isomorphic to $C^*(X')$ as an $\mathcal{E}$-algebra. Then $\mathcal{F}(X, Y)$ and $\mathcal{F}(X', Y)$ are weak equivalent.*

Proof. Let $A \to C^*(Y)$ be a cofibrant resolution. It follows that $\bar{h}\mathcal{E}(A \oslash C_*(X), -) \cong \bar{h}\mathcal{E}(A, C^*(X) \otimes -) \cong \bar{h}\mathcal{E}(A, C^*(X') \otimes -) \cong \bar{h}\mathcal{E}(A \oslash C_*(X'), -)$. Thus we can conclude that $A \oslash C_*(X) \cong A \oslash C_*(X')$ as an $\mathcal{E}$-algebra and hence $U(A \oslash C_*(X)) \simeq U(A \oslash C_*(X'))$. The result follows from Corollary 3.4. □

4. Proof of Theorem 1.4

Before proving Theorem 1.4, we here consider the homology of a free $\mathcal{E}$-algebra $\mathcal{E}(V) = \oplus_{r\ge 0}\mathcal{E}(r) \otimes_{\overline{\mathbb{F}}_p[\Sigma_r]} V^{\otimes r}$. The vector space V is decomposed as $V = H(V) \oplus d_V S \oplus S$. Then it follows from [27, Lemma 1.1 (iii)] that the inclusion $i : H(V) \to V$ induces a quasi-isomorphism $\varphi = \oplus_r(1 \otimes i^{\otimes r}) : \mathcal{E}(H(V)) \to \mathcal{E}(V)$ in the category of differential graded modules. The $\mathcal{E}$-algebra structure of a free $\mathcal{E}$-algebra comes from the operad structure of $\mathcal{E}$. This implies that φ is a morphism of $\mathcal{E}$-algebras. Let M be an unstable $\mathcal{B}$-$\overline{\mathbb{F}}_p$-module and $U_{en}M$ denote the enveloping algebra, which is an unstable $\mathcal{B}$-$\overline{\mathbb{F}}_p$-algebra. From [22, Proposition 12.4], we see that the canonical map $H(V) \to H(\mathcal{E}(H(V)))$ is extendable to an isomorphism on $\mathcal{K}H(V) := U_{en}\mathcal{B}H(V)$, where $\mathcal{B}H(V)$ denotes the free $\mathcal{B}$-$\overline{\mathbb{F}}_p$-module generated by $H(V)$. We regard $\mathcal{K}$ as a functor from the category of $\overline{\mathbb{F}}_p$-vector spaces to the category $\mathcal{K}$-$\overline{\mathbb{F}}_p$. Consequently we have the following lemma.

Lemma 4.1. *The composition map*

$$u_{\mathcal{E}(V)} : \mathcal{K}H(V) = U_{en}\mathcal{B}H(V) \to H(\mathcal{E}(H(V))) \stackrel{H(\varphi)}{\to} H(\mathcal{E}(V))$$

is an isomorphism of unstable $\mathcal{B}$-$\overline{\mathbb{F}}_p$-algebras.

In the remainder of this section, we construct the spectral sequence in Theorem 1.4.

Let $\mathcal{E}(W) \to C^*(Y)$ be an almost free resolution in which $\mathcal{E}(W)$ is a B_*-complex. From Theorem 1.3, it follows that $\mathcal{E}(W) \oslash C_*(X) \cong C^*(\mathcal{F}(X,Y))$ in the homotopy category $\bar{h}\mathcal{E}$. Consider a simplicial $\mathcal{E}$-resolution $\mathcal{R}$ of $\mathcal{E}(W)$:

$$\cdots \to \mathcal{E}(V_{-s}) \stackrel{\partial_{-s}}{\to} \cdots \to \mathcal{E}(V_{-1}) \stackrel{\partial_{-1}}{\to} \mathcal{E}(V_0) \stackrel{\varepsilon}{\to} \mathcal{E}(W)$$

equipped with contractions $h : \mathcal{E}(V_{-s+1}) \to \mathcal{E}(V_{-s})$ in the category of differential graded modules over $\overline{\mathbb{F}}_p$. (As an example of such a resolution, we can give the standard simplicial resolution described in [3] and [30, 3.8].)

Lemma 4.2. *The map*

$$\varepsilon \oslash 1 : \mathrm{Total}(\mathcal{E}(V_\bullet) \oslash C_*(X)) \to \mathcal{E}(W) \oslash C_*(X)$$

is a quasi-isomorphism.

Proof. Put $A = \mathrm{Total}(\mathcal{E}(V_\bullet) \oslash C_*(X))$. We define a decreasing filtration $\widetilde{F} = \{\widetilde{F}^i A\}$ of A with internal degrees: $\widetilde{F}^i A^n = \oplus_{-s+k=n, k\geq i}(\mathcal{E}(V_{-s}) \oslash C^*(X))^k$. It is easy to check that the filtration $\widetilde{F}$ is exhaustive and weakly convergent; that is, $A = \cup_{s\geq 0} F^{-s}A$ and $F^pA \cap \mathrm{Ker}\ d = \cap_r(F^pA \cap d^{-1}(F^{p+r}A))$. Let us consider the spectral sequence $\{\widetilde{E}_r, \widetilde{d}_r\}$ associated with the filtration $\widetilde{F}$. Then the E_1-term has the form

$$\mathcal{C} : \qquad \cdots \to \mathcal{O}(\mathcal{E}(V_{-s}) \oslash C_*(X)) \stackrel{\mathcal{O}(\partial_{-s}\oslash 1)}{\to} \cdots \to \mathcal{O}(\mathcal{E}(V_0) \oslash C_*(X)) \to 0.$$

Here $\partial_\bullet \oslash 1 = \sum(-1)^i d_i \oslash 1$ and $\mathcal{O}$ denotes the forgetful functor from the category of differential graded modules to the category of graded vector spaces. We view $\mathcal{O}(\mathcal{E}(V_{-s}))$ as an $\mathcal{O}\mathcal{E}$-algebra. Observe that $(\mathcal{O}(\mathcal{E}(V_{-s}) \oslash C_*(X)), \mathcal{O}(\partial_{-s} \oslash 1)) = (\mathcal{O}(\mathcal{E}(V_{-s}) \oslash \mathcal{O}C_*(X), \mathcal{O}(\partial_{-s}) \oslash 1)$. We see that the free resolution $\mathcal{R}$ of $\mathcal{E}(W)$ gives rise to a free resolution $\mathcal{L} : \mathcal{O}(\mathcal{E}(V_\bullet)) \to \mathcal{O}(\mathcal{E}(W)) \to 0$ of $\mathcal{O}(\mathcal{E}(W))$. The complex $\mathcal{L} \oslash 1 : \mathcal{O}(\mathcal{E}(V_\bullet)) \oslash \mathcal{O}C_*(X)$ is nothing but the complex $\mathcal{C}$ mentioned above. Since $\mathcal{O}(\mathcal{E}(W))$ is a free $\mathcal{O}\mathcal{E}$-algebra, we can take a constant simplicial resolution $\mathcal{V} : \cdots \to \mathcal{O}(\mathcal{E}(W)) \stackrel{=}{\to} \mathcal{O}(\mathcal{E}(W)) \stackrel{=}{\to} \mathcal{O}(\mathcal{E}(W)) \to 0$ as a free simplicial resolution of $\mathcal{O}(\mathcal{E}(W))$. Since $\widetilde{E}_2^{-i,*} \cong H^{-i}(\mathcal{C}) = H^{-i}(\mathcal{L} \oslash 1) \cong H^{-i}(\mathcal{V} \oslash 1)$, it follows that $\widetilde{E}_2^{-i,*} = 0$ for $i > 0$ and that $\widetilde{E}_1^{0,*} = \mathcal{O}(\mathcal{E}(W)) \oslash \mathcal{O}C_*(X)$. Moreover we see that $\widetilde{E}_2^{0,*} \cong \widetilde{E}_\infty^{0,*} \cong H(\mathrm{Total}(\mathcal{E}(V_\bullet) \oslash C_*(X)))$.

We next define the filtration ‘F of the $\mathcal{E}$-algebra $\mathcal{E}(W) \oslash C_*(X)$ with internal degrees. This filtration gives a spectral sequence $\{‘E_r, ‘d_r\}$ converging to $H(\mathcal{E}(W) \oslash C_*(X))$ with

$$‘E_1^{i,*} = \begin{cases} \mathcal{O}(\mathcal{E}(W)) \oslash \mathcal{O}C_*(X) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

It is immediate that $‘E_2^{0,*} \cong ‘E_\infty^{0,*} \cong H(\mathcal{E}(W) \oslash C_*(X))$. The map $\varepsilon \oslash 1$, which preserves the filtrations, induces $\{f_r\} : \{\widetilde{E}_r, \widetilde{d}_r\} \to \{‘E_r, ‘d_r\}$ a morphism of spectral sequences. This allows us to obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{E}_2^{0,*} & \xrightarrow{f_2} & {}^{`}E_2^{0,*} \\ \cong\downarrow & & \downarrow\cong \\ H(\mathrm{Total}(\mathcal{E}(V_\bullet)\oslash C_*(X))) & \xrightarrow[H(\varepsilon\oslash 1)]{} & H(\mathcal{E}(W)\oslash C_*(X)). \end{array}$$

Since f_1 is an isomorphism, we have the result. □

We define a decreasing filtration $F^\bullet = \{F^{-s}A\}$ of the total complex A of the double complex $\mathcal{E}(V_\bullet)\oslash C_*(X) = (\mathcal{E}(V_\bullet) : C^*(X))_{\mathcal{E}\text{-Alg}}$ by $F^{-s}A = \oplus_{-s\leq i}\mathcal{E}(V_i)\oslash C^*(X)$. Let $\{E_r, d_r\}$ be the spectral sequence associated with the filtration $F^\bullet$.

Theorem 4.3. *The spectral sequence $\{E_r, d_r\}$ converges strongly to $H^*(\mathcal{F}(X,Y))$.*

Proof. It is readily seen that the filtration $F^\bullet$ is exhaustive and weakly convergent. Since the filtration is bounded below, it follows that the natural map $u : HA \to \lim_{\leftarrow} HA/F^pHA$ is an isomorphism; that is, the filtration is strongly convergent. The complex $\mathcal{E}(W)\oslash C_*(X)$ is quasi-isomorphic to $C^*(\mathcal{F}(X,Y))$. Therefore the result follows from Lemma 4.2. □

We will describe the E_2-term of the mod p BPS spectral sequence in terms of the derived functor of the division functor in the category $\mathcal{K}$-$\overline{\mathbb{F}}_p$.

Since the E_1-term of the spectral sequence is induced by the internal differential, namely the differentials of the complexes $\mathcal{E}(V_{-s+1})$, it follows that the E_1-term has the form

$$\cdots \to H(\mathcal{E}(V_{-s})\oslash C_*(X)) \overset{H(\partial\oslash 1)}{\to} \cdots \to H(\mathcal{E}(V_{-1})\oslash C_*(X)) \to H(\mathcal{E}(V_0)\oslash C_*(X)) \to 0.$$

Since the free $\mathcal{E}$-resolution $\mathcal{R}$ of $\mathcal{E}(W)$ has contractions, by taking the internal differential, we have a simplicial resolution

$$\begin{array}{ccccccc} \cdots \longrightarrow & H(\mathcal{E}(V_{-1})) & \xrightarrow{H(\partial_{-1})} & H(\mathcal{E}(V_0)) & \xrightarrow{H(\varepsilon)} & H^*(Y) & \longrightarrow 0 \\ & \uparrow\cong & & \uparrow\cong & & & \\ & \mathcal{K}H(V_{-1}) & & \mathcal{K}H(V_0) & & & \end{array}$$

in the category $\mathcal{K}$-$\overline{\mathbb{F}}_p$. The following proposition completes the proof of the Theorem 1.4. Observe that the naturality of the spectral sequence also follows from the same proposition.

Proposition 4.4. *Let $\mathcal{E}(V)$ and $\mathcal{E}(V')$ be free $\mathcal{E}$-algebras and let $\partial : \mathcal{E}(V) \to \mathcal{E}(V')$ be a morphism of $\mathcal{E}$-algebras.*
(i) *Then the diagram*

$$\begin{array}{ccc} (H(\mathcal{E}(V)) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & \xrightarrow{(H(\partial):1)} & (H(\mathcal{E}(V')) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \\ {\scriptstyle (u_{\mathcal{E}(V)}:1)}\uparrow\cong & & \cong\uparrow{\scriptstyle (u_{\mathcal{E}(V')}:1)} \\ (\mathcal{K}H(V) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & & (\mathcal{K}H(V') : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \\ \| & & \| \\ \mathcal{K}(H(V)\otimes H_*(X)) & & \mathcal{K}(H(V')\otimes H_*(X)) \\ {\scriptstyle u_{\mathcal{E}(V\otimes C_*(X))}}\downarrow\cong & & \cong\downarrow{\scriptstyle u_{\mathcal{E}(V'\otimes C_*(X))}} \\ H\mathcal{E}(V\otimes C_*(X)) & \xrightarrow[H(\partial\oslash 1)]{} & H\mathcal{E}(V'\otimes C_*(X)) \end{array}$$

is commutative in $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$.
(ii) *Let* $f : X \to X'$ *be a map. Then the following diagram is commutative:*

$$\begin{array}{ccc}
(H(\mathcal{E}(V)) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & \xrightarrow{(1:H^*(f))} & (H(\mathcal{E}(V)) : H^*(X'))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \\
{\scriptstyle (u_{\mathcal{E}(V)}:1)}\uparrow\cong & & \cong\uparrow{\scriptstyle (u_{\mathcal{E}(V')}:1)} \\
(\mathcal{K}H(V) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & & (\mathcal{K}H(V) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \\
\| & & \| \\
\mathcal{K}(H(V) \otimes H_*(X)) & & \mathcal{K}(H(V) \otimes H_*(X')) \\
{\scriptstyle u_{\mathcal{E}(V\otimes C_*(X))}}\downarrow\cong & & \cong\downarrow{\scriptstyle u_{\mathcal{E}(V'\otimes C_*(X))}} \\
H\mathcal{E}(V \otimes C_*(X)) & \xrightarrow[H(1\oslash C_*(f))]{} & H\mathcal{E}(V \otimes C_*(X')).
\end{array}$$

In order to prove Proposition 4.4, we prepare a lemma. Let $\mathcal{K}(\quad,\quad)$ denote the hom set in the category $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$.

Lemma 4.5. *For any* $\mathcal{E}$*-algebra* N*, the diagram*

$$\begin{array}{ccc}
\mathcal{E}(\mathcal{E}(V) \oslash C_*(X), N) & \xrightarrow[\cong]{\mathrm{ad}_{\mathcal{E}(V)}} & \mathcal{E}(\mathcal{E}(V), N \otimes C^*(X)) \\
{\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\
\mathcal{K}(H(\mathcal{E}(V) \oslash C_*(X)), HN) & & \mathcal{K}(H(\mathcal{E}(V), HN \otimes C^*(X))) \\
{\scriptstyle u^*_{\mathcal{E}(V\otimes C_*(X))}}\downarrow\cong & & \cong\downarrow{\scriptstyle u^*_{\mathcal{E}(V)}} \\
\mathcal{K}((\mathcal{K}H(V) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}, HN) & \xrightarrow[\mathrm{ad}_{\mathcal{K}H(V)}]{\cong} & \mathcal{K}((\mathcal{K}H(V)), HN \otimes H^*(X)))
\end{array}$$

is commutative. Here $\mathrm{ad}_{\mathcal{E}(V)}$ *and* $\mathrm{ad}_{H\mathcal{E}(V)}$ *are adjoint isomorphisms and* H *is the map which sends a morphism* f *of chain complexes to the induced homomorphism* $H(f)$ *on homology.*

Observe that, by definition, $\mathcal{E}(V) \oslash C_*(X) = \mathcal{E}(V \otimes C_*(X))$ and $(\mathcal{K}H(V) : H^*(X)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} = \mathcal{K}(H(V) \otimes H_*(X))$.

Proof of Lemma 4.5. The given diagram is decomposed as follows:

$$\begin{array}{ccccc}
\mathcal{E}(\mathcal{E}(V \otimes C_*(X)), N) & \xrightarrow[\cong]{} & \mathrm{Hom}_{dgm}(V \otimes C_*(X), N) & \xrightarrow[\cong]{} & \\
{\scriptstyle u^*_{\mathcal{E}(V\otimes C_*(X))}\circ H}\downarrow & & {\scriptstyle H}\downarrow & & \\
\mathcal{K}(\mathcal{K}(H(V) \otimes H_*(X)), HN) & \xrightarrow[\cong]{} & \mathrm{Hom}_{dgm}(H(V) \otimes H_*(X), HN) & \xrightarrow[\cong]{} &
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{dgm}(V, N \otimes C^*(X)) & \xrightarrow[\cong]{} & \mathcal{E}(\mathcal{E}(V), N \otimes C^*(X)) \\
{\scriptstyle H}\downarrow & & \downarrow{\scriptstyle u^*_{\mathcal{E}(V)}\circ H} \\
\mathrm{Hom}_{dgm}(H(V), HN \otimes H^*(X)) & \xrightarrow[\cong]{} & \mathcal{K}(\mathcal{K}(H(V)), HN \otimes H^*(X)),
\end{array}$$

where $\mathrm{Hom}_{dgm}(\quad,\quad)$ denotes the hom set of the category of differential graded modules and the compositions of horizontal isomorphisms are the adjoints. By a fairly straightforward manner, we can check the commutativity of each square. The details are left to the reader. □

Proof of Proposition 4.4. (i) Put $a_U = (u^*_{\mathcal{E}(V)})^{-1} \circ \mathrm{ad}_{\mathcal{K}H(V)} \circ u^*_{\mathcal{E}(V \otimes c_*(X))}$ for $U = V$ and V'. From Lemma 4.5, we have a diagram (4.1):

$$
\begin{array}{ccccccc}
\mathcal{E}(\mathcal{E}(V\otimes C_*),N) & & \xrightarrow{ad_{\mathcal{E}(V)}} & & \mathcal{E}(\mathcal{E}(V),N\otimes C^*) & & \\
& \searrow^{H} & & & \downarrow{\scriptstyle (\partial)^*} & \searrow^{H} & \\
\downarrow{\scriptstyle (\partial\oslash 1)^*} & & \mathcal{K}(H\mathcal{E}(V\otimes C_*),HN) & \xrightarrow{a_V} & & & \mathcal{K}(H\mathcal{E}(V),HN\otimes H^*(X)) \\
& & \downarrow{\scriptstyle H(\partial\oslash 1)^*} & & & & \downarrow{\scriptstyle H(\partial)^*} \\
\mathcal{E}(\mathcal{E}(V'\otimes C_*),N) & & \xrightarrow{ad_{\mathcal{E}(V')}} & & \mathcal{E}(\mathcal{E}(V'),N\otimes C^*) & & \\
& \searrow^{H} & & & & \searrow^{H} & \\
& & \mathcal{K}(H\mathcal{E}(V'\otimes C_*),HN) & \xrightarrow{a_{V'}} & & & \mathcal{K}(H\mathcal{E}(V'),HN\otimes H^*(X))
\end{array}
$$

whose squares are commutative except for the front one, where $C^* = C^*(X)$ and $C_* = C_*(X)$. We also have a diagram

$$
\begin{array}{ccccc}
\mathcal{K}(H\mathcal{E}(V\otimes C_*),HN) & \xrightarrow{a_V} & \mathcal{K}(H\mathcal{E}(V),HN\otimes H^*(X)) & \xleftarrow[\cong]{\mathrm{ad}_{H\mathcal{E}(V)}} & \mathcal{K}((H\mathcal{E}(V):H^*(X),HN) \\
\downarrow{\scriptstyle H(\partial\oslash 1)^*} & & \downarrow{\scriptstyle H(\partial)^*} & & \downarrow{\scriptstyle (H(\partial):1)^*} \\
\mathcal{K}(H\mathcal{E}(V'\otimes C_*),HN) & \xrightarrow[a_{V'}]{} & \mathcal{K}(H\mathcal{E}(V'),HN\otimes H^*(X)) & \xleftarrow[\cong]{\mathrm{ad}_{H\mathcal{E}(V')}} & \mathcal{K}((H\mathcal{E}(V):H^*(X),HN)
\end{array}
$$

in which the right square is commutative. The commutativity of diagram (4.1) implies that $H(\partial)^* a_V(id_{H\mathcal{E}(V\otimes C_*)}) = a_{V'} H(\partial \oslash 1)^*(id_{H\mathcal{E}(V\otimes C_*)})$ and hence

$$(H(\partial):1)^*(ad_{H\mathcal{E}(V)})^{-1} a_V(id_{H\mathcal{E}(V\otimes C_*)}) = (ad_{H\mathcal{E}(V')})^{-1} a_{V'} H(\partial \oslash 1)(id_{H\mathcal{E}(V\otimes C_*)}).$$

Put $\Psi_U = u_{\mathcal{E}(U\otimes C_*(X))} \circ (u_{\mathcal{E}(U)} : 1)^{*-1}$ for $U = V$ and V'. Then the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{K}(H\mathcal{E}(U\otimes C_*),HN) & \xrightarrow{u^*_{\mathcal{E}(U\otimes C_*(X))}} & \mathcal{K}((\mathcal{K}H(U):H^*(X)),HN) & \xrightarrow[\cong]{\mathrm{ad}_{\mathcal{K}H(U)}} & \mathcal{K}(\mathcal{K}H(U),HN\otimes H^*(X)) \\
& \searrow^{\Psi_U^*} & \uparrow{\scriptstyle (u_{\mathcal{E}(U)}:1)^*}\,\cong & & \uparrow{\scriptstyle u^*_{\mathcal{E}(U)}}\,\cong \\
& & \mathcal{K}((H\mathcal{E}(U):H^*(X)),HN) & \xrightarrow[\cong]{\mathrm{ad}_{H\mathcal{E}(U)}} & \mathcal{K}((H\mathcal{E}(U),HN\otimes H^*(X))
\end{array}
$$

enables us to deduce that $(\mathrm{ad}_{H\mathcal{E}(U)})^{-1} \circ a_U = \Psi^*_U$. Thus by choosing $\mathcal{E}(V \otimes C_*)$ as N, we can conclude that $(H(\partial) : 1)^* \Psi^*_V(id_{H(\mathcal{E}(V\otimes C_*))}) = \Psi^*_{V'} H(\partial \oslash 1)^*(id_{H(\mathcal{E}(V\otimes C_*))})$. This completes the proof.
(ii) The map $1 \oslash C_*(f) : \mathcal{E}(V \otimes C_*(X)) \to \mathcal{E}(V \otimes C_*(X'))$ is defined by the extension of the linear map $1 \otimes C_*(f) : V \otimes C_*(X) \to V \otimes C_*(X')$. Therefore it is easy to check the commutativity. □

We end this section with consideration on the edge homomorphism of the spectral sequence $\{E_r, d_r\}$.

Let $\iota : * \to X$ be the inclusion map. Observe that the map ι induces the evaluation map $ev_0 = \mathcal{F}(\iota, id) : \mathcal{F}(X, Y) \to \mathcal{F}(*, Y) = Y$. Define a map

$$\theta : H^*(Y) \to (H^*(Y) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} = L_0(H^*(Y) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$$

by composing the natural isomorphism $H^*(Y) \xrightarrow{\cong} (H^*(Y) : \overline{\mathbb{F}}_p)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ with the induced map $(1 : \iota^*) : (H^*(Y) : \overline{\mathbb{F}}_p)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \to (H^*(Y) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$.

Proposition 4.6. *The induced map $ev_0^* : H^*(Y) \to H^*(\mathcal{F}(X, Y)))$ is decomposed via the edge homomorphism of the spectral sequence $\{E_r, d_r\}$, that is, the following* *diagram is commutative:*

$$\begin{array}{ccc} H^*(Y) & \xrightarrow{\quad ev_0^* \quad} & H^*(\mathcal{F}(X,Y)) \\ \theta\downarrow & & \uparrow \\ E_2^{0,*} = L_0(H^*(Y):H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \twoheadrightarrow E_3^{0,*} & \twoheadrightarrow \cdots \longrightarrow & E_\infty^{0,*}. \end{array}$$

Proof. Consider the spectral sequence $\{\widehat{E}_r, \widehat{d}_r\}$ with $\widehat{E}_2^{s,*} \cong L_s(H^*(Y):H^*(*))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ converging to the cohomology $H^*(\mathcal{F}(*,Y)) = H^*(Y)$. We regard that this spectral sequence is constructed from the complex $\mathcal{R} \oslash \overline{\mathbb{F}}_p$, where $\mathcal{R}$ denotes a simplicial resolution of a cofibrant model for Y. Since $\mathcal{R} \oslash \overline{\mathbb{F}}_p$ is isomorphic to $\mathcal{R}$ as a complex, it follows that $\widehat{E}_2^{s,*} = 0$ if $s \neq 0$ and that $\widehat{E}_2^{0,*} = L_0(H^*(Y):\overline{\mathbb{F}}_p)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} = H^*(Y)$. This implies that the spectral sequence $\{\widehat{E}_r, \widehat{d}_r\}$ collapses at the E_2-term. Moreover the edge homomorphism $\widehat{E}_2^{0,*} \to \cdots \to \widehat{E}_\infty^{0,*}$ is viewed as the identity map. The inclusion map $\iota : * \to X$ induces a morphism $\{f_r\}$ of spectral sequences from $\{\widehat{E}_r, \widehat{d}_r\}$ to $\{E_r, d_r\}$ on account of the naturality of the spectral sequence. We then see that the morphism f_2 is nothing but the map θ. This completes the proof. □

Remark 4.7. From the construction of the spectral sequence, we see that the edge homomorphism coincides with the composition

$$\begin{array}{ccc} (H^*(Y):H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & & \\ \| & & \\ \mathrm{Coker}\{H(\partial \oslash 1) : H(\mathcal{E}(V_{-1} \otimes C_*(X))) \to H(\mathcal{E}(V_0 \otimes C_*(X)))\} & & \\ \downarrow H(\varepsilon \oslash 1) & & \\ H(\mathcal{E}(W) \oslash C_*(X)) & \xrightarrow{\quad \cong \quad} & H^*(\mathcal{F}(X,Y)). \end{array}$$

This implies that the edge homomorphism is a morphism of unstable $\mathcal{B}$-algebras.

Remark 4.8. We here give a model for the based mapping space $\mathcal{F}_*(X,Y)$.

Let $\iota : * \to X$ be the inclusion and $\mathcal{E}(W) \xrightarrow{\simeq} C^*(Y)$ a cofibrant model. Recall that the maps in the sequence (3.1) and the adjoint in Theorem 3.5 are functorial with respect to X. Therefore we have a commutative diagram

$$\begin{array}{ccc} C^*(Y) = C^*(\mathcal{F}(*,Y)) & \xrightarrow{\mathcal{F}(\iota,Y)} & C^*(\mathcal{F}(X,Y)) \\ \simeq\uparrow & & \uparrow\simeq \\ \mathcal{E}(W) = \mathcal{E}(W) \oslash \overline{\mathbb{F}}_p & \xrightarrow[id \oslash in]{} & \mathcal{E}(W) \oslash C_*(X) = \mathcal{E}(W \otimes C_*(X)) \end{array}$$

for which $in : \overline{\mathbb{F}}_p \to C_*(X)$ is the inclusion and the vertical arrows are quasi-isomorphisms. The map $id \oslash in$ is regarded as the inclusion $\mathcal{E}(W) \to \mathcal{E}(W \otimes C_*(X))$ and hence as a cofibation.

Consider the evaluation fibration

$$\mathcal{F}_*(X,Y) \to \mathcal{F}(X,Y) \xrightarrow{ev_0} Y.$$

The map ev_0 is nothing but the induced map $\mathcal{F}(\iota, Y)$ so that the map $id \oslash in$ is a model for the evaluation map ev_0. By virtue of [11, Théorème 4.2], we have a model $\mathcal{E}(W \otimes C^*(X)^+) \xrightarrow{\simeq} C^*(\mathcal{F}_*(X,Y))$, where $C^*(X)^+ = C^*(X)/\overline{\mathbb{F}}_p$.

So far we adhere to working with cochain complexes in order to get topological results. However, one can reconsider the construction of the mod p BPS spectral sequence by replacing the cochain complexes $C^*(Y)$ and $C^*(X)$ with $\mathcal{E}$-algebras B and A, respectively. Then a more algebraic spectral sequence appears. More precisely, we can establish the following theorem.

Theorem 4.9. *Let A and B be $\mathcal{E}$-algebras over a field $\mathbb{F}$. Suppose that A is of finite type. Then there exists a left-half plane spectral sequence $\{E_r, d_r\}$ converging strongly to $H(B \oslash^L A_*)$ with $E_2^{s,*} \cong L_s(H(B) : H(A))_{\mathcal{K}\text{-}\mathbb{F}}$, the left derived functors of the division functor $(\ : H(A))_{\mathcal{K}-\mathbb{F}}$ in the category $\mathcal{K}$-$\mathbb{F}$ of unstable $\mathcal{B}$-algebras over $\mathbb{F}$.*

Remark 4.10. Let $\mathbb{F}_p^{\bullet}Y$ be the cosimplicial resolution of Y. It seems that the Bousfield spectral sequence [7] for the cosimplicial space $\mathcal{F}(X, \mathbb{F}_p^{\bullet}Y)$ gives rise to the spectral sequence converging to $H(\mathcal{F}(X,Y))$ with the same E_2-term as that of our spectral sequence (see [5, Sections 3 and 7] for details of the case where $X = S^1$).

5. Derived functors of division functors

Let A be an unstable $\mathcal{A}$-algebra of finite type over $\overline{\mathbb{F}}_p$. To simplify, the derived functor $L_s(\ : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ may be denoted by $L_s(\ : A)$. Our goal in this short section is to prove Theorem 1.5. We will first prove the following theorem.

Theorem 5.1. *For any $s < 0$, $L_s(H^*(K(\mathbb{Z}/p, n)) : A) = 0$ and $L_0(H^*(K(\mathbb{Z}/p, n)) : A)$ is isomorphic to $U_{en}\mathcal{A}(e_n \otimes A_*)$ as a graded vector space.*

Before proving Theorem 5.1, we prepare a proposition and a spectral sequence.

Proposition 5.2. *$L_0(\overline{\mathbb{F}}_p : A) = \overline{\mathbb{F}}_p$ and $L_s(\overline{\mathbb{F}}_p : A) = 0$ for $s < 0$.*

Proof. Let $\cdots \longrightarrow \mathcal{K}(V_{-1}) \longrightarrow \mathcal{K}(\overline{\mathbb{F}}_p) \xrightarrow{\varepsilon} \overline{\mathbb{F}}_p \longrightarrow 0$ be the standard simplicial resolution of $\overline{\mathbb{F}}_p$. Here $\mathcal{K}$ denotes the functor $U_{en}\mathcal{B}$ from the category of graded $\overline{\mathbb{F}}_p$-vector spaces to $\mathcal{K}$-$\overline{\mathbb{F}}_p$ (see Section 4). We then define a morphism $r : \overline{\mathbb{F}}_p \to \mathcal{K}(\overline{\mathbb{F}}_p)$ in $\mathcal{K}$-$\overline{\mathbb{F}}_p$ by $r(1) = 1_{\mathcal{K}}$. It is immediate that the composite $\varepsilon \circ r$ is the identity. From [30, Lemma 7.1.3], we see that the complex $\cdots \longrightarrow (\mathcal{K}(V_{-1}) : A) \xrightarrow{d_0 - d_1} (\mathcal{K}(\overline{\mathbb{F}}_p) : A) \longrightarrow 0$, which computes the derived functor $L_s(\overline{\mathbb{F}}_p : A)$, is acyclic. The result for $s = 0$ follows from Theorem 6.1 in the next section. □

It is known that the cohomology $H^*(K(\mathbb{Z}/p, n))$ is isomorphic to $U_{en}\mathcal{A}(e_n)$ as an unstable $\mathcal{A}$-algebra (see also [30, Proposition 1.6.2], [8]). Moreover, by virtue of [22, Proposition 12.5], we obtain a pushout diagram

$$\begin{array}{ccc} U_{en}\mathcal{B}e_n & \xrightarrow{\alpha} & U_{en}\mathcal{B}e_n \ =: \alpha_* U_{en}\mathcal{B}e_n \\ \downarrow & & \downarrow \\ \overline{\mathbb{F}}_p & \longrightarrow & U_{en}\mathcal{A}e_n \cong H^*(K(\mathbb{Z}/p, n)) \end{array}$$

in the category $\mathcal{K}$-$\overline{\mathbb{F}}_p$, where $\alpha = U_{en}(1 - P^0)$.

By using a spectral sequence [28, II §6 Theorem 6 (b)] due to Quillen, Bökstedt and Ottosen have construct a spectral sequence converging to a derived functor applied to a pushout in an appropriate category (see [6, Proposition 6.3]). The way of their construction of the spectral sequence does work well in the category $\mathcal{K}$-$\overline{\mathbb{F}}_p$. The division functor $(- : A)$ preserves colimits, and hence pushouts, in $\mathcal{K}$-$\overline{\mathbb{F}}_p$

since the functor is the left adjoint of the tensor product functor $A \otimes -$. Thus [6, Proposition 6.3] enables one to deduce the following proposition.

Proposition 5.3. *Let* $B' \leftarrow B \rightarrow B''$ *be a diagram in the category* $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$ *such that* $\mathrm{Tor}_i^B(B', B'') = 0$ *for any* $i > 0$ *and let* $B' \otimes_B B''$ *denote the pushout. Then there exists a third quadrant spectral sequence* $\{E_r, d_r\}$ *converging to* $L_*(B' \otimes_B B'' : A)$ *with* $E_2^{s,t} \cong \mathrm{Tor}_s^{L_*(B:A)}(L_*(B' : A), L_*(B'' : A))_t$.

Proof of Theorem 5.1. We observe that $L_s(\mathcal{K}(e_n) : A) = 0$ if $s < 0$ and that $L_0(\mathcal{K}(e_n) : A) = \mathcal{K}(e_n \otimes A_*)$. Moreover we see that $\alpha_*\mathcal{K}(e_n)$ is a free $\mathcal{K}(e_n)$-module. This follows from [22, Proposition 12.5]. Proposition 5.3 enables us to obtain a spectral sequence $\{E_r, d_r\}$ such that

$$E_2^{s,t} \cong \mathrm{Tor}_s^{L_*(\mathcal{K}(e_n):A)}(L_*(\alpha_*\mathcal{K}(e_n) : A), L_*(\overline{\mathbb{F}}_p : A))_t, \quad E_r^{s,t} \Rightarrow L_{s+t}(U_{en}\mathcal{A}(e_n) : A).$$

By virtue of [13, 4.2.7 Lemma], we see that $L_0(\alpha_*\mathcal{K}(e_n) : A) = (\alpha_*\mathcal{K}(e_n) : A) \cong \mathcal{K}(e_n \otimes A_*) \otimes U_{en}\mathcal{A}(e_n \otimes A_*)$ as a $\mathcal{K}(e_n \otimes A_*)$-module. Therefore it follows from Proposition 5.2 that

$$\begin{aligned} & \mathrm{Tor}_*^{L_*(\mathcal{K}(e_n):A)}(L_*(\alpha_*\mathcal{K}(e_n) : A), L_*(\overline{\mathbb{F}}_p : A))_* \\ \cong\ & \mathrm{Tor}_0^{L_*(\mathcal{K}(e_n):A)}(L_*(\alpha_*\mathcal{K}(e_n) : A), \overline{\mathbb{F}}_p)_0 \\ \cong\ & U_{en}\mathcal{A}(e_n \otimes A_*). \end{aligned}$$

This completes the proof. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We take the standard simplicial resolution $U_\bullet \xrightarrow{\varepsilon} B \to 0$ of B in the category $\mathcal{S}\text{-}\overline{\mathbb{F}}_p$ (see [3] and [30, 3.8]). Let $B_\bullet \xrightarrow{\eta} B \to 0$ and $V_{i\bullet} \xrightarrow{\eta} U_i \to 0$ be the standard simplicial resolutions of B and U_i, respectively, in $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$. Observe that $U_i = (U_{en}\mathcal{A})^{i+1}(B)$, $B_k = \mathcal{K}^{k+1}(B)$ and $V_{ik} = \mathcal{K}^{k+1}(U_i)$. Since the standard simplicial resolution is functorial, we have a double complex :

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & U_i & \xrightarrow{\sum(-1)^i d_i} & U_{i+1} & \longrightarrow & \cdots & \longrightarrow & U_0 & \xrightarrow{\varepsilon} & B & \longrightarrow & 0 \\
 & & \eta\uparrow & & \eta\uparrow & & & & \eta\uparrow & & \eta\uparrow & & \\
\cdots & \longrightarrow & V_{i0} & \xrightarrow{\sum(-1)^i\mathcal{K}(d_i)} & V_{(i+1)0} & \longrightarrow & \cdots & \longrightarrow & V_{00} & \xrightarrow{\mathcal{K}(\varepsilon)} & B_0 & \longrightarrow & 0 \\
 & & d_0-d_1\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & V_{i1} & \xrightarrow{\sum(-1)^i\mathcal{K}^2(d_i)} & V_{(i+1)1} & \longrightarrow & \cdots & \longrightarrow & V_{01} & \xrightarrow{\mathcal{K}^2(\varepsilon)} & B_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \cdots & & \vdots & & \vdots & &
\end{array}$$

The resolution $U_\bullet \xrightarrow{\varepsilon} B \to 0$ has a contraction $h_i : U_i \to U_{i-1}$ for any i in the category of graded $\overline{\mathbb{F}}_p$-vector spaces. Thus, for any k, the horizontal sequence $V_{\bullet k} \xrightarrow{\mathcal{K}^{k+1}(\varepsilon)} B_k \to 0$ is interpreted as a simplicial resolution of $B_k = \mathcal{K}^{k+1}(B)$. Therefore the spectral sequence arising from the horizontal filtration of the double complex $D_{\bullet\bullet} := \mathrm{Total}((V_{\bullet\bullet} : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p})$ gives an isomorphism of $\mathcal{B}$-modules from $H_*(D_{\bullet\bullet})$ to $L_*(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$. The vertical filtration of the double complex $D_{\bullet\bullet}$ defines another spectral sequence converging to $H_*(D_{\bullet\bullet})$ with $E_1^{s,t} \cong L_t(U_s : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$. By virtue of Theorem 5.1, we see that $E_1^{s,0} \cong L_0(U_s : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ and $E_1^{s,t} = 0$ for $t < 0$. Moreover we have an epimorphism

$$(\eta : 1) : L_0(U_s : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} = (V_{s0} : A)/\mathrm{Im}\{(\partial_0 : 1) - (\partial_1 : 1)\} \longrightarrow (U_s : A)_{\mathcal{S}\text{-}\overline{\mathbb{F}}_p},$$

which is induced by η, in the category of $\mathcal{B}$-modules. Theorem 5.1 implies that $L_0(U_s : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} = (U_s : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is isomorphic to $(U_s : A)_{\mathcal{S}\text{-}\overline{\mathbb{F}}_p}$ as a vector space and hence $(\eta : 1)$ is an isomorphism. Thus it follows that $E_2^{s,0} \cong L_s(B : A)_{\mathcal{S}\text{-}\overline{\mathbb{F}}_p}$ and $E_2^{s,t} = 0$ for $t < 0$. We have an isomorphism of $\mathcal{B}$-modules from $H_*(D_{\bullet\bullet})$ to $L_s(B : A)_{\mathcal{S}\text{-}\overline{\mathbb{F}}_p}$. This completes the proof. □

Remark 5.4. Theorem 5.1 is refined. In fact, Theorem 1.5 yields that the division functor $(H^*(K(\mathbb{Z}/p, n)) : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is isomorphic to $U_{en}\mathcal{A}(e_n \otimes A_*)$ as a $\mathcal{B}$-module.

6. Computation of the division functor $(H^*(Y) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$

Let A and B be unstable $\mathcal{B}$-algebras. The fact described in Remark 4.7 motivates us to compute the division functor $(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ as an unstable $\mathcal{B}$-algebra.

In order to describe the structure of the functor explicitly, we first define a bracket $\langle\ ,\ \rangle : \{\mathcal{K}(B \otimes A_*) \otimes A\}^{\otimes r} \otimes A_*^{\otimes r} \to \mathcal{K}(B \otimes A_*)$ by

$$\langle \alpha_1 \otimes x_1 \otimes \cdots \otimes \alpha_r \otimes x_r, a_{1*} \otimes \cdots \otimes a_{r*}\rangle = \pm\alpha_1\langle x_1, a_{1*}\rangle \cdots \alpha_r\langle x_r, a_{r*}\rangle.$$

Here A_* is the dual space to A and $\langle\ ,\ \rangle : A \otimes A_* \to \overline{\mathbb{F}}_p$ denotes the usual pairing.

Let $1_{\mathcal{K}}$ and 1_B are units of $\mathcal{K}(B)$ and B respectively. The counit of A_* is denoted by 1_{A_*}. We use the usual notation P^I for $\beta^{\varepsilon_1}P^{s_1}\cdots\beta^{\varepsilon_k}P^{s_k}$, where $I = (\varepsilon_1, s_1, ..., \varepsilon_k, s_k)$.

Theorem 6.1. *Suppose that A is of finite type. Let $\mathfrak{A}$ is the ideal of the unstable $\mathcal{B}$-algebra $\mathcal{K}(B \otimes A_*)$ generated by the elements $1_{\mathcal{K}} - 1_B \otimes 1_{A_*}$ and*

$$\langle \delta(P^{I_1})(y_1 \otimes x_{i_1}) \otimes x^{i_1} \otimes \cdots \otimes \delta(P^{I_r})(y_r \otimes x_{i_r}) \otimes x^{i_r}, \Delta^{(r-1)}x\rangle - (P^{I_1}y_1 \cdots P^{I_r}y_r) \otimes x,$$

where $\Delta^{(r-1)}$ is the r-fold iterated coproduct on the dual algebra A_*, $\{x^i\}$ and $\{x_i\}$ denote a basis of A and its dual basis, respectively. Then Lannes' division functor $(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is isomorphic to $\mathcal{K}(B \otimes A_*)/\mathfrak{A}$ as an unstable $\mathcal{B}$-algebra.

Proof. Let us consider the standard simplicial resolution of B:

$$\cdots \longrightarrow \mathcal{K}(\mathcal{K}(B)) \xrightarrow{d_0 - d_1} \mathcal{K}(B) \xrightarrow{\varepsilon} B \longrightarrow 0.$$

By a straightforward calculation, we can get explicit forms of $(d_0 : 1)$ and $(d_1 : 1)$ as follows:

$$(d_0 : 1)(1_{\mathcal{K}} \otimes 1) = 1_{\mathcal{K}}, \quad (d_1 : 1)(1_{\mathcal{K}} \otimes 1_{A_*}) = 1_B \otimes 1_{A_*}$$
$$(d_0 : 1)(1_{\mathcal{K}} \otimes x_*) = 0, \quad (d_1 : 1)(1_{\mathcal{K}} \otimes x_*) = 1_B \otimes x_* \quad \text{for} \quad x_* \neq 1$$
$$(d_0 : 1)((P^{I_1}y_1 \cdot_{\mathcal{K}} \cdots_{\mathcal{K}} P^{I_r}y_r) \otimes x)$$
$$= \langle \delta(P^{I_1})(y_1 \otimes x_{i_1}) \otimes x^{i_1} \otimes \cdots \otimes \delta(P^{I_r})(y_r \otimes x_{i_r}) \otimes x^{i_r}, \Delta^{(r-1)}x\rangle,$$
$$(d_1 : 1)((P^{I_1}y_1 \cdot_{\mathcal{K}} \cdots_{\mathcal{K}} P^{I_r}y_r) \otimes x) = (P^{I_1}y_1 \cdots P^{I_r}y_r) \otimes x,$$

where $\cdot_{\mathcal{K}}$ denotes the product on $\mathcal{K}(B)$. The division functor $(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is realized as the quotient algebra $\mathcal{K}(B \otimes A_*)/\mathrm{Im}\{(d_0 : 1) - (d_1 : 1)\}$. In fact, the existence of degeneracy operator $s_0 : \mathcal{K}(B) \to \mathcal{K}\mathcal{K}(B)$ tells us that $\mathfrak{A}' = \mathrm{Im}\{(d_0 : 1) - (d_1 : 1)\}$ is an ideal. We see that the algebra $\mathcal{K}((\mathcal{K}(B) \otimes A_*)$ is generated by elements $q_1 \cdots q_r \otimes x_*$ for $q_i \in \mathcal{K}(B)$ and $1_{\mathcal{K}} \otimes x_*$. Moreover it is readily seen that the elements $1_B \otimes x_*$ for $x_* \neq 1$ belong to $\mathfrak{A}$. Thus we have $\mathfrak{A}' = \mathfrak{A}$. □

The following proposition states one of important properties of the division functor $(\ \ :\ \ )_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$.

Proposition 6.2. *Let A and B be unstable algebras over the usual Steenrod algebra $\mathcal{A}$. Then the division functor $(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is generated, as an algebra, by the elements of the form $y \otimes x_*$ such that y is indecomposable and $x_* \in A_*$. Moreover, the division functor is an unstable $\mathcal{A}$-algebra.*

Proof. We use the same notation as in Theorem 6.1. The second assertion follows from Theorem 1.5.

Let $\{x^{i(N)}\}$ and $\{x_{i(N)}\}$ be a basis of A^N and its dual basis of A_{*N}, where $x_{1(0)}$ denotes the unit 1_A of A. We write $\{x^i\} = \cup_N \{x^{i(N)}\}$ and $\{x_i\} = \cup_N \{x_{i(N)}\}$. Let QB be the vector space of indecomposable elements and $S(QB \otimes A_*)$ denote the symmetric algebra generated by the vector space $QB \otimes A_*$. Consider the natural map $\eta : S(QB \otimes A_*) \to (B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$. In order to complete the proof, it suffices to show that the map η is surjective. Recall from Theorem 6.1 the formula (6.1):

$$\langle \delta(P^{I_1})(y_1 \otimes x_{i_1}) \otimes x^{i_1} \otimes \cdots \otimes \delta(P^{I_r})(y_r \otimes x_{i_r}) \otimes x^{i_r}, \Delta^{(r-1)} x\rangle - (P^{I_1} y_1 \cdots P^{I_r} y_r) \otimes x.$$

This enables us to conclude that an element $y \otimes x_i$ for $y \in B$ is in the image of η. We have to prove that $\beta^\varepsilon P^s(y \otimes x_j)$ with $\varepsilon + s > 0$ is in Im η. Assume that $\beta^\varepsilon P^t(y \otimes x_j) \in$ Im η for t and ε such that $\varepsilon + t < M$. Consider the element $\beta^\varepsilon P^s(y \otimes x_j)$ with $\varepsilon + s = M$. Then by applying (6.1) again, we see that

$$\begin{aligned} & (\beta^\varepsilon P^s y) \otimes x_j \\ = \ & \langle \delta(\beta^\varepsilon P^s)(y \otimes x_i) \otimes x^i, x_j \rangle \\ = \ & \langle \sum_{\substack{\varepsilon_0 + \varepsilon_1 = \varepsilon, s_0 + s_1 = s \\ \varepsilon_1 + s_1 > 0}} \pm \beta^{\varepsilon_0} P^{s_0}(y \otimes x_i) \otimes \beta^{\varepsilon_1} P^{s_1} x^i, x_j \rangle + \langle \beta^\varepsilon P^s(y \otimes x_i) \otimes x^i, x_j \rangle. \end{aligned}$$

It turns out that the element $\langle \beta^\varepsilon P^s(y \otimes x_i) \otimes x^i, x_j \rangle = \beta^\varepsilon P^s(y \otimes x_j)$ is in the image of η. We have the result. □

Theorem 6.3. *Let X be an $n-1$-dimensional finite simplicial set. Then the edge homomorphism*

$$(H^*(K(\mathbb{Z}/p, n)) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \to H^*(\mathcal{F}(X, K(\mathbb{Z}/p, n)))$$

is an isomorphism.

Proof. Consider the spectral sequence $\{E_r, d_r\}$ in Theorem 1.4 converging to the cohomology $H^*(\mathcal{F}(X, K(\mathbb{Z}/p, n)))$. Theorem 5.1 yields that $E_r^{s,*} = 0$ for $s < 0$. It is readily seen that $E_\infty^{0,*} \cong E_2^{0,*} \cong (H^*(K(\mathbb{Z}/p, n)) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ and $E_\infty^{s,*} = 0$ if $s < 0$. We have the result. □

Assume that $X = S^1$ and $H^*(Y;\mathbb{F}_p)$ is a polynomial algebra. Let $\mathcal{F}$ be the the homotopy fibre square

$$\begin{array}{ccc} \mathcal{F}(S^1, Y) & \longrightarrow & Y \\ {\scriptstyle ev_*}\downarrow & & \downarrow{\scriptstyle \Delta} \\ Y & \xrightarrow[\Delta]{} & Y \times Y, \end{array}$$

where Δ is the diagonal map. Then the mod p cohomology algebra of the free loop space $LY = \mathcal{F}(S^1, Y)$ can be determined explicitly by using the Eilenberg-Moore spectral sequence (EMSS) $\{E_r(Y), d_r(Y)\}$ associated with $\mathcal{F}$ (see [18, Remarks 3.4, 3.5], [20, Theorem 1.6]). To be exact, if $H^*(Y;\mathbb{F}_p) = \mathbb{F}_p[y_1, .., y_l]$, as an $H^*(Y;\mathbb{F}_p)$-algebra, then

$$H^*(LY;\mathbb{F}_p) \cong \mathbb{F}_p[y_1, .., y_l] \otimes \Lambda(\bar{y}_1, \bar{y}_2, .., \bar{y}_l) \quad \text{if} \quad p \neq 2,$$

where $\deg \bar{y}_i = \deg y_i - 1$. In the case $p = 2$, we see that

$$H^*(LY;\mathbb{F}_2) \cong \mathbb{F}_2[y_1, .., y_l] \otimes \mathbb{F}_2[\bar{y}_1, \bar{y}_2, .., \bar{y}_l] \big/ (\bar{y}_i^2 + \mathfrak{D}Sq^{\deg y_i - 1} y_i; i = 1, 2, .., l)$$

as an $H^*(Y;\mathbb{F}_2)$-algebra, for which $\mathfrak{D}$ is the derivation defined by $\mathfrak{D}(y_i) = \bar{y}_i$ (see also [5] for the algebra structure of $H^*(LY;\mathbb{F}_2)$).

Since the derivation $\mathfrak{D}$ is compatible with the Steenrod operations ([18, Remark 3.5]), we can determine explicitly the $\mathcal{A}$-algebra structure of $H^*(LY;\mathbb{F}_p)$ from that of the polynomial algebra $H^*(Y;\mathbb{F}_p)$ (see, for example, [18, Example 3.6]). The following theorem asserts that the $\mathcal{A}$-algebra structure of $H^*(LY;\overline{\mathbb{F}}_p)$ is also expressed via the Lannes' division functor.

Theorem 6.4. *Let Y be a simply-connected space whose mod p cohomology is a polynomial algebra. Then the edge homomorphism*

$$\mathrm{edge}_{(Y:S^1)} : (H^*(Y) : H^*(S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \to H^*(LY)$$

is an isomorphism.

Before proving Theorem 6.4, we consider some important relation which appears in the division functor $(H^*(Y) : H^*(S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$.

Lemma 6.5. *Let t_* be the base of $H_*(S^1)$ which is defined from a base of $H_*(S^1;\mathbb{F}_p)$ via the inclusion $H_*(S^1;\mathbb{F}_p) \to H_*(S^1)$.*
(i) *The algebra $(H^*(Y) : H^*(S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is generated by the element $y_i \otimes t_*$ and $y_i \otimes 1$.*
(ii) *For the case $p = 2$, we have $(y \otimes t_*)^2 = \widetilde{\mathfrak{D}}Sq^{\deg y - 1}(y \otimes 1)$, where $\widetilde{\mathfrak{D}}$ is the derivation defined by $\widetilde{\mathfrak{D}}(y \otimes 1) = y \otimes t_*$ for any y.*

Proof. Part (i) follows from Proposition 6.2.

In the case $p = 2$, it follows from the formula (6.1) that $(y \otimes t_*)^2 = Sq^{\deg y - 1}(y \otimes t_*) = (Sq^{\deg y - 1} y) \otimes t_*$. Since t_* is primitive, we see that, for any elements $y_1, .., y_r$ in $H^*(Y)$,

$$(y_1 \cdots y_r) \otimes t_* = (y_1 \otimes t_*)(y_2 \otimes 1) \cdots (y_r \otimes 1) + \cdots + (y_1 \otimes 1)(y_2 \otimes 1) \cdots (y_{r-1} \otimes 1)(y_r \otimes t_*).$$

in $(H^*(Y) : H^*(S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$. This completes the proof of (ii). □

Proof of Theorem 6.4. We fix an generator y_i of $H^*(Y : \mathbb{F}_p)$. Put $n_i = \deg y_i$. Let $f : Y \to K_{n_i} = K(\mathbb{Z}/p, n_i)$ be a representative of the element y_i. Then we see that there exists an element $s^{-1}\iota_{n_i}$ such that $\mathcal{F}(1, f)^*(s^{-1}\iota_{n_i}) = \bar{y}_i$. This follows from the naturality of the EMSS $\{E_r(Y), d_r(Y)\}$. Let t_* be the same base of $H_*(S^1)$ as in Lemma 6.5. Theorem 6.3 allows us to deduce that $\mathrm{edge}_{(K_{n_i}:S^1)}(e_{n_i} \otimes t_*) = \alpha_i s^{-1}\iota_{n_i}$ for some non-zero element $\alpha_i \in \overline{\mathbb{F}}_p$. By naturality of the edge homomorphism, we have a commutative diagram

$$\begin{array}{ccc} (H^*(K_{n_i}) : H^*(S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & \xrightarrow{\mathrm{edge}_{(K_{n_i}:S^1)}} & H^*(LK_{n_i}) \\ {\scriptstyle (f^*:1)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(1,f)^*} \\ (H^*(Y) : H^*(S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & \xrightarrow[\mathrm{edge}_{(Y:S^1)}]{} & H^*(LY). \end{array}$$

It is readily seen that $\mathrm{edge}_{(Y:S^1)}(y_i \otimes t_*) = \alpha_i \bar{y}_i$. Proposition 4.6 enables us to deduce that $\mathrm{edge}_{(Y:S^1)}(y_i \otimes 1) = y_i$. We define an algebra map φ from $H^*(LY)$ to $(H^*(Y) : H^*(S^1))$ by $\varphi(y_i) = y_i \otimes 1$ and $\varphi(\bar{y}_i) = (\alpha_i)^{-1} y_i \otimes t_*$. The well-definedness for $p = 2$ follows from that of the map $\mathrm{edge}_{(Y:S^1)}$ and Lemma 6.5(ii). By virtue of Lemma 6.5(i), we can conclude that φ is the inverse of the edge homomorphism $\mathrm{edge}_{(Y:S^1)}$. □

Next we look at the edge homomorphism in the case where X is a Riemann surface Σ_g of genus g and the target Y is the classifying space BG of a simply-connected Lie group G. Using generators $\alpha_1, \beta_1, ..., \alpha_g, \beta_g$ of $\pi_1(\Sigma_g)$, we can write $\Sigma_g = \vee_{l=1}^{2g} S^1 \cup_\alpha D^2$, where $\alpha = [\alpha_1, \beta_1] \cdots [\alpha_1, \beta_g]$. Let $\{E_r, d_r\}$ be the EMSS obtained from the cofibre square

$$\begin{array}{ccc} \Sigma_g & \xleftarrow{j} & D^2 \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \iota} \\ \vee_{l=1}^{2g} S^1 & \xleftarrow[\alpha]{} & S^1 \end{array}$$

converging to $H^*(\mathcal{F}(\Sigma_g, BG))$ (see [21, Theorem 2.1]). Assume that the cohomology $H^*(BG; \mathbb{F}_p)$ is isomorphic to a polynomial algebra $\mathbb{F}_p[c_i]$ generated by the elements with even degree, equivalently the integral cohomology of G is p-torsion free. Then the calculation of the integral cohomology of $\mathcal{F}(\Sigma_g, BG)$ due to Atiyah and Bott in [2, Proposition 2.10] allows us to deduce that the EMSS $\{E_r, d_r\}$ collapses at the E_2-term. Hence we have

$$E_\infty \cong \overline{\mathbb{F}}_p[c_i] \otimes \bigotimes_{l=1}^{2g} \Lambda(x_{il}) \otimes \Gamma[s^{-1}x_i]$$

as a bigraded algebra, where bideg $c_i = (0, \deg c_i)$, bideg $x_{il} = (0, \deg c_i - 1)$ and bideg $\gamma_t(s^{-1}x_i) = (-t, t(\deg c_i - 1))$ (for more details, see [21, Remark 3.6]). Moreover it follows that, as an $H^*(\mathcal{F}(\vee^{2g} S^1, BG))$-module,

$$H^*(\mathcal{F}(\Sigma_g, BG)) \cong H^*(\mathcal{F}(\vee^{2g} S^1, BG)) \otimes \Gamma[s^{-1}x_i].$$

Here the $H^*(\mathcal{F}(\vee^{2g} S^1, BG); \mathbb{F}_p)$-module structure is defined using the map $\mathcal{F}(i, 1)^*$. We observe that $H^*(\mathcal{F}(\vee^{2g} S^1, BG)) \cong \overline{\mathbb{F}}_p[c_i] \otimes \bigotimes_{l=1}^{2g} \Lambda(x_{il})$ (see [21, Proposition 3.1]).

Let $F^*H^* = \{F^iH^*\}_{i\le 0}$ be the filtration of $H^*(\mathcal{F}(\Sigma_g, BG))$ which is brought from the EMSS $\{E_r, d_r\}$. Let SA be the subalgebra generated by $H^*(\mathcal{F}(\vee^{2g}S^1, BG))$ and elements $s^{-1}x_i$. It follows from the bigraded algebra structure of E_∞ that SA is a submodule of F^pH^*. Moreover the $\mathcal{A}$-algebra structure of the EMSS tells us that SA is a sub $\mathcal{A}$-algebra of $H^*(\mathcal{F}(\Sigma_g, BG))$.

As is presumed from the integral cohomology calculation of $\mathcal{F}(\Sigma_g, BSU(2))$ due to Masbaum [24], it seems very difficult to determine the whole algebra structure of $H^*(\mathcal{F}(\Sigma_g, BG))$. Fortunately, we can determine the $\mathcal{A}$-algebra structure of SA explicitly using the edge homomorphism $\mathrm{edge}_{(BG:\Sigma_g)}$ subject to exact knowledge of $\mathcal{A}$-action on $H^*(BG;\mathbb{F}_p)$.

Theorem 6.6. *Assume that the integral cohomology of G is p-torsion free. Then the edge homomorphism*

$$\mathrm{edge}_{(BG:\Sigma_g)} : (H^*(BG) : H^*(\Sigma_g))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \to H^*(\mathcal{F}(\Sigma_g, BG))$$

factors through the sub $\mathcal{A}$-algebra SA of $H^*(\mathcal{F}(\Sigma_g, BG))$. Moreover the map into SA is an isomorphism.

Proof. Let $f_j : BG \to K_j = K(\mathbb{Z}/p, \deg c_j)$ be a representative of the element c_j. We consider the following commutative diagram

$$\begin{array}{ccc}
(H^*(K_j) : H^*(\Sigma_g))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & \xrightarrow[\cong]{\mathrm{edge}_{(K_j:\Sigma_g)}} & H^*(\mathcal{F}(\Sigma_g, K_j)) \\
{\scriptstyle (f_j^*:1)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(1,f_j)^*} \\
(H^*(BG) : H^*(\Sigma_g))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & \xrightarrow[\mathrm{edge}_{(BG:\Sigma_g)}]{} & H^*(\mathcal{F}(\Sigma_g, BG)) \\
{\scriptstyle (1:i^*)}\uparrow & & \uparrow{\scriptstyle \mathcal{F}(i,1)^*} \\
(H^*(BG) : H^*(\vee^{2g}S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} & \xrightarrow[\mathrm{edge}_{(BG:\vee^{2g}S^1)}]{\cong} & H^*(\mathcal{F}(\vee^{2g}S^1, BG)).
\end{array}$$

Let $\{t_k\}_{1\le j\le 2g}$ and $\{s_2\}$ be bases of $H_1(\Sigma_g)$ and $H_2(\Sigma_g)$, respectively. The proof of Theorem 6.4 does work well to verify that the edge homomorphism $\mathrm{edge}_{(BG:\vee^{2g}S^1)}$ is an isomorphism such that $\mathrm{edge}_{(BG:\vee^{2g}S^1)}(c_i \otimes t_k) = \alpha_{ik}x_{ik}$ for some non-zero element $\alpha_{ik} \in \overline{\mathbb{F}}_p$. Hence we have $\mathrm{edge}_{(BG:\Sigma_g)}(c_i \otimes t_k) = \alpha_{ik}x_{ik}$. Theorem 6.3 implies that the map $\mathrm{edge}_{(BG:\Sigma_g)}$ sends $c_i \otimes s_2$ to $\alpha_i s^{-1}x_i$ with a non-zero element $\alpha_i \in \overline{\mathbb{F}}_p$. By virtue of Proposition 6.2, we see that the algebra $(H^*(Y) : H^*(\Sigma_g))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is generated by the elements $c_i \otimes t_k$, $c_i \otimes s_2$ and $c_i \otimes 1$. It follows from the bigraded algebra structure of the E_∞-term that, as an $H^*(\mathcal{F}(\vee^{2g}S^1, BG))$-module,

$$SA \cong H^*(\mathcal{F}(\vee^{2g}S^1, BG)) \otimes \overline{\mathbb{F}}_p[s^{-1}x_i]/\left((s^{-1}x_i)^p\right).$$

We define an $H^*(\mathcal{F}(\vee^{2g}S^1, BG))$-module map $q : SA \to (H^*(Y) : H^*(\Sigma_g))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ by $q(x_{ik}) = \alpha_{ik}^{-1}c_i \otimes t_k$ and $q((s^{-1}x_j)^n) = (\alpha_j^{-1}c_j \otimes s_2)^n$ for $n < p$. It is immediate that q is the inverse of the edge homomorphism as an $H^*(\mathcal{F}(\vee^{2g}S^1, BG))$-module map. In fact, the edge homomorphism is a morphism of algebras over $\mathcal{A}$. Thus we have the result. □

We conclude the paper with an example concerning calculations of the Steenrod operations on the cohomology $H^*(\mathcal{F}(\Sigma_g, BG);\overline{\mathbb{F}}_p)$.

Example 6.7. We choose a base $\{1, t_i^{(l)*}, t_i^{(m)*}, s_2^*\}_{1\le i\le g}$ for the integral cohomology $H^*(\Sigma_g;\mathbb{Z})$ so that $t_i^{(l)*}t_j^{(m)*} = \delta_{ij}s_2^*$ and ${t_i^{(l)*}}^2 = {t_i^{(m)*}}^2 = 0$. Then, in the cohomology $H^*(\mathcal{F}(\Sigma_g, BSU(2)); \overline{\mathbb{F}}_3)$,

$$(c_2\otimes s_2)^3 = P^1(c_2\otimes s_2) = \sum_i (c_2\otimes t_i^{(l)})\cdot(c_2\otimes t_i^{(m)}) + 2(c_2\otimes s_2)(c_2\otimes 1).$$

Here $\{1, t_i^{(l)}, t_i^{(m)}, s_2\}$ denotes the dual basis for $H_*(\Sigma_g)$ and $c_2\in H^4(BSU(2))$ is the mod 3 reduction of the 2nd Chern class. In fact, by using (6.1), we see that

$$0 = \langle \delta(P^1)(c_2\otimes x_i)\otimes x^i, s_2\rangle - (P^1c_2)\otimes s_2 = P^1(c_2\otimes s_2) - c_2^2\otimes s_2$$

in $(H^*(BSU(2)) : H^*(\Sigma_g))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_3}$, where $\{x_i\} = \{1, t_i^{(l)}, t_i^{(m)}, s_2\}$. Moreover it follows that

$$\begin{aligned} c_2^2\otimes s_2 &= \langle c_2\otimes x^{i_1}\otimes x_{i_1}\otimes c_2\otimes x^{i_2}\otimes x_{i_2}, \Delta s_2\rangle \\ &= \langle c_2\otimes x^{i_1}\otimes x_{i_1}\otimes c_2\otimes x^{i_2}\otimes x_{i_2}, \\ &\qquad s_2\otimes 1 + 1\otimes s_2 - \sum_i t_i^{(l)}\otimes t_i^{(m)} + \sum_i t_i^{(m)}\otimes t_i^{(l)}\rangle \\ &= (c_2\otimes s_2)\cdot(c_2\otimes 1) + (c_2\otimes 1)\cdot(c_2\otimes s_2) \\ &\qquad -\sum_i (c_2\otimes t_i^{(l)})\cdot(c_2\otimes t_i^{(m)}) + \sum_i (c_2\otimes t_i^{(m)})\cdot(c_2\otimes t_i^{(l)}). \end{aligned}$$

Thus we have the above formula.

Remark 6.8. We refer the reader to [1] for remarkable properties which are reliable in explicitly computing Lannes' T-functor.

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7. Appendix

As mentioned in Introduction, this section is devoted to considering the Steenrod operations in the spectral sequence arising from a simplicial $\mathcal{E}$-algebra. We assume that the underlying field is $\mathbb{F}_p$ or $\overline{\mathbb{F}}_p$ throughout this section.

Let $A_\bullet = \{(A_i^*, \partial_A)\}$ be a simplicial $\mathcal{E}$-algebra. We put $A^{-i} = A_i$ and define the total complex $\mathrm{Total}(A^\bullet)$ by $\mathrm{Total}(A^\bullet)^n = \oplus_{s+t=n}A^{s,t}$. Let $\{E_r, d_r\}$ be the left-half plane spectral sequence associated with the filtration F defined by $F^{i,n} = \oplus_{s+t=n, s\ge i}A^{s,t}$. Put $F^sH = \mathrm{Im}\{H(\iota) : H(F^s\mathrm{Total}(A^\bullet)) \to H(\mathrm{Total}(A^\bullet))\}$, where $\iota : F^s\mathrm{Total}(A^\bullet) \to \mathrm{Total}(A^\bullet)$ is the natural inclusion. Then $\{F^*H\}$ is a strongly convergent decreasing filtration of $H(\mathrm{Total}(A^\bullet))$ (see [25, Chapter 3]). Observe that $E_\infty^{s,t} \cong F^sH^{s+t}/F^{s+1}H^{s+t}$ as a module. Thus, by definition, the spectral sequence converges strongly to $H^*(\mathrm{Total}(A^\bullet))$. Recall that

$$E_r^{s,t} = Z_r^{s,t}/(Z_{r-1}^{s+1,t-1} + B_{r-1}^{s,t}),$$

where $Z_r^{s,t} = F^{s,s+t}\cap d^{-1}(F^{s+r,s+t+1})$ and $B_{r-1}^{s,t} = F^{s,s+t}\cap dF^{s-r+1,s+t-1}$. As is known, there exists an isomorphism $\rho : E_2^{s,t} \cong H^s\big(H^t(A,\partial), \sum(-1)^iH(d_i)\big) =: H_{II}H_I(A)$ which sends the element $[z_s\oplus z_{s+1}\oplus\cdots\oplus z_0]$ of E_2 to $[[z_s]_\partial]$, where $[z_s]_\partial$ denotes the class in $H^t(A_s,\partial)$ represented by z_s. Since $H^*(A,\partial)$ is an unstable $\mathcal{B}$-algebra and $\sum(-1)^iH(d_i)$ is a $\mathcal{B}$-*module* map, we can define the Steenrod operation $\beta^\varepsilon P^i_{E_2}$ on $H_{II}H_I(A)$ by $\beta^\varepsilon P^i_{E_2}[[z_s]_\partial] = (-1)^{\varepsilon s}[\beta^\varepsilon P^i[z_s]_\partial]$.

Our main theorem in this section is stated as follows.

Theorem 7.1. *In the spectral sequence $\{E_r, d_r\}$ constructed above, each stage (E_r, d_r) possesses a differential bigraded algebra structure and an unstable module structure over the big Steenrod algebra $\mathcal{B}$ for each r such that:*
(1) $\beta^\varepsilon P^i d_r = (-1)^\varepsilon d_r \beta^\varepsilon P^i$,
(2) *for any operator* $\beta^\varepsilon P^i \in \mathcal{B}$, $\beta^\varepsilon P^i : E_r^{s,t} \to E_r^{s,t+2i(p-1)+\varepsilon}$,
(3) *the Cartan formula holds and*
(4) *for* $r = 2$, *the operation* $\beta^\varepsilon P^i : E_2^{s,t} \to E_2^{s,t+2i(p-1)+\varepsilon}$ *coincides with* $\beta^\varepsilon P^i_{E_2}$ *up to the isomorphism* $\rho : E_2^{*,*} \cong H_{II}H_I(A)$.

Suppose further that there exists an $\mathcal{E}$-algebra map η from A_0^* to an $\mathcal{E}$-algebra (B, ∂) such that the composition $\mathrm{Total}(A_\bullet) \xrightarrow{\pi} A_0 \xrightarrow{\eta} B^*$ *is a morphism of differential graded module and a quasi-isomorphism, where π denotes the natural projection. Then $F^sH(B,\partial) := H(\eta\circ\pi)\big(F^sH\big)$ is an unstable $\mathcal{B}$-submodule of $H(B,\partial)$ and, as a bigraded algebra and an unstable $\mathcal{B}$-module, $E_\infty^{*,*} \cong F^*H(B,\partial)/F^{*+1}H(B,\partial)$. In consequence, the spectral sequence $\{E_r, d_r\}$ converges strongly to $H(B,\partial)$ as an algebra and an unstable $\mathcal{B}$-module.*

Suppose that a given spectral sequence $\{E_r, d_r\}$ admits the differential bigraded algebra structure and an unstable module structure over the big Steenrod algebra $\mathcal{B}$ (resp. the usual Steenrod algebra $\mathcal{A}$) for which the conditions (1), (2), (3) and (4) in Theorem 7.1 are satisfied. Then we call such the structure on $\{E_r, d_r\}$ a well compatible DGA-$\mathcal{B}$-module (resp. DGA-$\mathcal{A}$-module) structure.

The rest of this section is devoted to proving Theorem 7.1.

Let $NA_\bullet$ be the normalized complex of $A_\bullet$; that is, $NA_\bullet = \cap_{i\neq 0}\mathrm{Ker}\ d_i$ with the differential d_0. Since d_i is a morphism of $\mathcal{E}$-algebras for any i, it follows that NA_n is a sub $\mathcal{E}$-algebra of A_n for any n. We define the spectral sequence $\{\widetilde{E}_r, \widetilde{d}_r\}$ converging to $H^*(\mathrm{Total}(NA))$ by the same way as in the construction of the spectral sequence $\{E_r, d_r\}$. It is immediate that the inclusion $i : NA \to A$ gives rise to the morphism $\{i_r\} : \{\widetilde{E}_r, \widetilde{d}_r\} \to \{E_r, d_r\}$ of the spectral sequences.

Lemma 7.2. *The morphism $i_r : \widetilde{E}_r \to E_r$ is an isomorphism of differential graded modules for any $r \geq 2$.*

Proof. We first observe that the degeneracy and face operators on $A^\bullet$ are compatible with the differential ∂_A. The inclusion $i : NA \to A$ defines the morphism $i_1 : \widetilde{E}_1 = H(NA,\partial) \to H(A,\partial) = \widetilde{E}_1$. Since the inclusion $i : NA \to A$ is a chain homotopy equivalence for which the chain homotopy is constructed from degeneracy and face maps, it follows that i_1 induces an isomorphism

$$i_2 = H(i_1) : \widetilde{E}_2 = H\big(H(NA,\partial), H(d_0)\big) \xrightarrow{\cong} H\big(H(A,\partial), \sum(-1)^i H(d_i)\big) = E_2$$

and hence i_r is an isomorphism for any $r \geq 2$. □

Remark 7.3. By construction, the homomorphisms i_1 and i_2 are a $\mathcal{B}$-algebra map and a $\mathcal{B}$-module map, respectively.

In order to prove Theorem 7.1, we define a $\mathcal{B}$-module structure on the spectral sequence $\{\widetilde{E}_r, \widetilde{d}_r\}$. Moreover we give an algebra structure on $\{E_r, d_r\}$ and will prove that the Cartan formula holds with respect to the Steenrod operations on $\{E_r, d_r\}$ which comes from a $\mathcal{B}$-module structure on $\{\widetilde{E}_r, \widetilde{d}_r\}$ via the isomorphism $\{i_r\}$ of the spectral sequences.

Let $\beta^\varepsilon P^i$ be the chain level operation on $NA_\bullet$ (see [27, Section 3]). We here define operation $\beta^\varepsilon P^i_{SS} : \mathrm{Total}(NA)^n \to \mathrm{Total}(NA)^{n+2i(p-1)+\varepsilon}$ for any i and $\varepsilon = 0, 1$ by $\beta^\varepsilon P^i_{SS} = \oplus_s (-1)^{\varepsilon s}\beta^\varepsilon P^i$.

Proposition 7.4. *The operation $\beta^\varepsilon P^i_{SS}$ gives a well-defined operation $\beta^\varepsilon P^i$: $\widetilde{E}^{s,t}_r \to \widetilde{E}^{s,t+2i(p-1)+\varepsilon}_r$ such that $\beta^\varepsilon P^i \widetilde{d}_r = (-1)^\varepsilon \widetilde{d}_r \beta^\varepsilon P^i$.*

The chain level operation is not homomorphism so that we have to verify the well-definedness of the operators very carefully.

We prepare a lemma to prove Proposition 7.4. Let ∂_1 and ∂_2 be the vertical differential and the horizontal differential d_0 on $\mathrm{Total}(NA)$, respectively.

Lemma 7.5. *For any element $u_{i-1} \in NA^{(i-1),n-(i-1)}$ and $u_i \in NA^{i,n-i}$, if $\partial_2 u_{i-1} + \partial_1 u_i = 0$, then $\partial_2 \beta^\varepsilon P^i u_{i-1} + (-1)^\varepsilon \partial_1 \beta^\varepsilon P^i u_i = 0$.*

Proof. We observe that ∂_1 is the differential $(-1)^i \partial_{NA_i}$ of NA_i and ∂_2 is the morphism of $\mathcal{E}$-algebras. Since $\partial_1 u_i$ is a ∂_1-cycle, it follows from the formula (10) in the proof of [27, Theorem 3.1] that $\beta^\varepsilon P^i(-\partial_1 u_i) = -\beta^\varepsilon P^i \partial_1 u_i$. This enables us to deduce that $\beta^\varepsilon P^i \partial_2 u_{i+1} = -\beta^\varepsilon P^i \partial_1 u_i$ from $\partial_2 u_{i+1} + \partial_1 u_i = 0$. By virtue of [27, Theorem 3.1 (i)], we have $\partial_2 \beta^\varepsilon P^i u_{i+1} = -(-1)^\varepsilon \partial_1 \beta^\varepsilon P^i u_i$. □

Proof of Proposition 7.4. Lemma 7.5 implies that $\beta^\varepsilon P^i_{SS}(Z^{s,t}_r)$ is contained in $Z^{s,2i(p-1)+\varepsilon}_r$ for any r. Therefore, in order to verify the well-definedness for the operator on $E^{*,*}_r$, it suffices to show that $\beta^\varepsilon P^i_{SS}(B^{s,*}_{r-1}) \subset Z^{s+1,*}_{r-1} + B^{s,*}_{r-1}$.

Put $P^I = \beta^\varepsilon P^i$. Let $\widetilde{u}$ be the element $(\partial_1 + \partial_2)(u_{s-r+1} \oplus \cdots \oplus u_{s-1} \oplus u_s \oplus \cdots u_0)$ of $B^{s,t}_{r-1}$. By definition, we see that $\partial_1 u_{s-r+1} = 0$ and $\partial_2 u_{i-1} + \partial_1 u_i = 0$ for $s - r + 2 \le i \le s - 1$. Thus it follows that

$$\widetilde{u} = (\partial_2 u_{s-1} + \partial_1 u_s) \oplus (\partial_2 u_s + \partial_1 u_{s+1}) \oplus \cdots \oplus (\partial_2 u_{-1} + \partial_1 u_0)$$

and hence

$$P^I \widetilde{u} = (-1)^{\varepsilon s} P^I(\partial_2 u_{s-1} + \partial_1 u_s) \oplus (-1)^{\varepsilon(s+1)} P^I(\partial_2 u_s + \partial_1 u_{s+1}) \oplus \cdots \oplus P^I(\partial_2 u_{-1} + \partial_1 u_0).$$

We write $u = u_{s-r+1} \oplus \cdots \oplus u_{s-1} \oplus u_s \oplus \cdots \oplus u_0$. Lemma 7.5 yields that

$$\begin{aligned}(\partial_1 + \partial_2)P^I u = 0 \oplus \cdots 0 \oplus ((-1)^{\varepsilon(s-1)} \partial_2 P^I u_{s-1} + (-1)^{\varepsilon s} \partial_1 P^I u_s) \\ \oplus((-1)^{\varepsilon s} \partial_2 P^I u_s + (-1)^{\varepsilon(s+1)} \partial_1 P^I u_{s+1}) \oplus \cdots .\end{aligned}$$

Then we have

$$\begin{aligned}&P^I \widetilde{u} - (-1)^\varepsilon (\partial_1 + \partial_2)(P^I u) \\ &\qquad = (-1)^{\varepsilon s}\{P^I(\partial_2 u_{s-1} + \partial_1 u_s) - (P^I \partial_2 u_{s-1} + P^I \partial_1 u_s)\} \oplus \\ &\qquad\quad (-1)^{\varepsilon(s-1)}\{P^I(\partial_2 u_s + \partial_1 u_{s+1}) - (P^I \partial_2 u_s + P^I \partial_1 u_{s+1})\} \oplus \cdots .\end{aligned}$$

By direct computation, it is readily seen that that, for $j \ge s$,

$$\begin{aligned}&(\partial_1 + \partial_2)\{P^I(\partial_2 u_j + \partial_1 u_{j+1}) - (P^I \partial_2 u_j + P^I \partial_1 u_{j+1})\} \\ &\qquad = \partial_1 P^I \partial_2 u_j + P^I \partial_2 \partial_1 u_{j+1} - \partial_1 P^I \partial_2 u_j - P^I \partial_2 \partial_1 u_{j+1} = 0.\end{aligned}$$

Thus it follows that $(-1)^{\varepsilon(s-1)}\{P^I(\partial_2 u_s + \partial_1 u_{s+1}) - (P^I \partial_2 u_s + P^I \partial_1 u_{s+1})\} \oplus \cdots$ is in $Z^{s+1,*}_{r-1}$. The elements $\partial_2 u_{s-1} + \partial_1 u_s$ and $\partial_2 u_{s-1}$ are ∂_1-cycles. In fact $\partial_1 \partial_2 u_{s-1} = -\partial_2 \partial_1 u_{s-1} = \partial_2 \partial_2 u_{s-2} = 0$. Since the operator P^I is a homomorphism on $H^{-s}(NA, \partial_1)$, we see that $[P^I(\partial_2 u_{s-1} + \partial_1 u_s)] = P^I[\partial_2 u_{s-1} + \partial_1 u_s] =$

$P^I[\partial_2 u_{s-1}]+P^I[\partial_1 u_s]=[P^I(\partial_2 u_{s-1})+P^I(\partial_2 u_s)]$ in $H^{-s}(NA,\partial_1)$ and hence there exists an element $b\in NA^{-s}$ such that

$$(\partial_1+\partial_2)(b)-\partial_2 b=P^I(\partial_2 u_{s-1}+\partial_1 u_s)-(P^I(\partial_2 u_{s-1})+P^I(\partial_2 u_s)).$$

Moreover we have

$$\begin{aligned} -(\partial_1+\partial_2)(\partial_2 b) &= (\partial_1+\partial_2)(\partial_1 b) \\ &= (\partial_1+\partial_2)(P^I(\partial_2 u_{s-1}+\partial_1 u_s)-P^I(\partial_2 u_{s-1})-P^I(\partial_2 u_s)) \\ &= \partial_1 P^I\partial_2 u_{s-1}+\partial_2 P^I\partial_1 u_s-\partial_1 P^I\partial_2 u_{s-1}-\partial_2 P^I\partial_1 u_s=0. \end{aligned}$$

This enables us to conclude that $\partial_2 b$ is in $Z^{s+1,*}_{r-1}$. Therefore it follows that the element $P^I\widetilde{u}-(-1)^\varepsilon(\partial_1+\partial_2)(P^I u)$ belongs to $B^{s,*}_{r-1}+Z^{s+1,*}_{r-1}$ and hence $P^I\widetilde{u}\in B^{s,*}_{r-1}+Z^{s+1,*}_{r-1}$.

The same computation as above is applicable to verify the formula $\beta^\varepsilon P^i\widetilde{d}_r=(-1)^\varepsilon\widetilde{d}_r\beta^\varepsilon P^i$. □

Proof of Theorem 7.1. By virtue of Proposition 7.4, we see that the isomorphism $i_r:\widetilde{E}_r\to E_r$ gives rise to the Steenrod operations which satisfy the conditions (1) and (2). The isomorphism $\rho\circ i_2:\widetilde{E}^{s,*}_2\xrightarrow{\cong}E^{s,*}_2\xrightarrow{\cong}H^s_{II}H_I(A)$ sends an element of the form $[z_s\oplus\cdots\oplus z_0]$ to $[z_s]$. Therefore the condition (4) is also satisfied.

The $\mathcal{E}$-algebra structure of A^s and the shuffle product define an $\mathcal{E}$-algebra structure of $\mathrm{Total}(A^\bullet)$ (see [12, 2.2 Lemma]). In particular, the product m of $\mathrm{Total}(A^\bullet)$ is given by

$$m(z_s\otimes z_{s'})=\sum_{\substack{(\nu,\mu)\\(-s,-s')\text{-shuffles}}}\theta(e_2\otimes s_\mu z_s\otimes s_\nu z_{s'})=\sum_{\substack{(\nu,\mu)\\(-s,-s')\text{-shuffles}}}m_{s+s'}(s_\mu z_s\otimes s_\nu z_{s'}),$$

where θ is the $\mathcal{E}$-algebra structure on $A^{s+s'}$ and e_2 is an element of $\mathcal{E}(2)^0$ which induces the product $m_{s+s'}$ on $A^{s+s'}$. It is immediate that $m(F^s\mathrm{Total}(A^\bullet)\otimes F^{s'}\mathrm{Total}(A^\bullet))$ is a submodule of $F^{s+s'}\mathrm{Total}(A^\bullet)$. Therefore the spectral sequence $\{E_r,d_r\}$ possesses the differential algebra structure induced by the product on $\mathrm{Total}(A^\bullet)$ and converges to $H(\mathrm{Total}(A^\bullet))$ as an algebra.

In order to obtain the Cartan formula, it suffices to prove that the formula holds in the E_2-term because the E_r-term inherits the algebra structure and the Steenrod operations from those of the E_{r-1}-term. On the E_2-term, we have

$$\beta^\varepsilon P^l m(z_s\otimes z_{s'})=\sum_{\substack{(\nu,\mu)\\(-s,-s')\text{-shuffles}}}(-1)^{\varepsilon(s+s')}\beta^\varepsilon P^l m_{s+s'}(s_\mu z_s\otimes s_\nu z_{s'})$$

for any $z_s\in i_*(H^q(NA_s))$ and $z_{s'}\in i_*(H^{q'}(NA_{s'}))$. Since the Cartan formula holds in $H(A_{s+s'})$ and degeneracy maps are $\mathcal{E}$-algebra maps, it follows that the right hand side is equal to

$$\sum_{\substack{(\nu,\mu)\\(-s,-s')\text{-shuffles}}}m_{s+s'}\Big(\sum_{i+j=l}(-1)^{\varepsilon s}s_\mu\beta^\varepsilon P^i(z_s)\otimes(-1)^{\varepsilon s'}s_\nu P^j(z_{s'})+ (-1)^{q\varepsilon}(-1)^{\varepsilon s}s_\mu P^i(z_s)\otimes(-1)^{\varepsilon s'}s_\nu\beta^\varepsilon P^j(z_{s'})\Big).$$

Thus we can get the Cartan formula on the E_2-term.

We have to prove the latter half of the assertion in Theorem 7.1. The isomorphism $H(\eta\circ\pi):F^sH\xrightarrow{\cong}F^sH(B,\partial)$ maps an element of the form $[z_s\oplus\cdots\oplus z_0]$

to $[\eta z_0]$. Lemma 7.5 implies that the element $[(-1)^{\varepsilon s}\beta^\varepsilon P^i z_s \oplus \cdots \oplus \beta^\varepsilon P^i z_0]$ is in $F^s H$. Since η is an $\mathcal{E}$-algebra map, it follows that

$$H(\eta \circ \pi)([(-1)^{\varepsilon s}\beta^\varepsilon P^i z_s \oplus \cdots \beta^\varepsilon P^i z_0]) = [\eta\beta^\varepsilon P^i z_0] = \beta^\varepsilon P^i[\eta z_0].$$

It is readily seen that the isomorphism $E_\infty^{*,*} \cong F^*H(B,\partial)/F^{*+1}H(B,\partial)$ is compatible with the Steenrod operations and respects the algebra structure. We have the result. □

Remark 7.6. When defining 'vertical' Steenrod operations on the spectral sequence, one may use the usual $\mathcal{E}$-algebra structure of $\mathrm{Total}(A_\bullet)$ which is induced by the shuffle product ([12, 2.2 Lemma]). However the attempt fails because the $\mathcal{E}$-algebra structure does not preserve the filtration of $\mathrm{Total}(NA)$.

As seen in the proof of Proposition 7.4, we define directly operations in the spectral sequence without considering those in the filtration F^*H of $\mathrm{Total}(NA)$. Unfortunately, the formula $\beta^\varepsilon P^i_{SS}(B^{s,*}_{r-1}) \subset Z^{s+1,*}_{r-1} + B^{s,*}_{r-1}$ in the proof of Proposition 7.4 means that the operations we define are not inherited to the filtration. Still the proof of Theorem 7.1 allows one to conclude that the spectral sequence arising from a simplicial $\mathcal{E}$-algebra A converges strongly to $H(\mathrm{Total}(A))$ as an algebra.

We conclude the section with some applications of Theorem 7.1.

Recall the simplicial resolution $\mathcal{E}(V_\bullet) \to \mathcal{E}(W)$ mentioned in Section 4. Then the simplicial $\mathcal{E}$-algebra $\mathcal{E}(V_\bullet) \oslash C_*(X)$ gives the spectral sequence in Theorem 1.4. Lemma 4.2 guarantees that the assumption in the second assertion of Theorem 7.1 is satisfied. Therefore Theorem 1.5 yields the following theorem.

Theorem 7.7. *The mod p BPS spectral sequence $\{E_r, d_r\}$ admits a well compatible DGA-$\mathcal{A}$-module structure. Moreover $\{E_r, d_r\}$ converges strongly to $H^*(\mathcal{F}(X,Y))$ as an algebra and an unstable $\mathcal{A}$-module.*

One can also define the action of the Steenrod operations on the bar type Eilenberg-Moore spectral sequence.

Theorem 7.8. *Let X, Y and Z be connected simplicial sets of finite p-type and assume that Z is simply connected. Let $X \to Z$ be a map and $Y \to Z$ a fibration. Then there exists a spectral sequence $\{E_r, d_r\}$ admitting a well compatible DGA-$\mathcal{A}$-module structure with*

$$E_2^{*,*} \cong \mathrm{Tor}_{H^*(Z;\mathbb{F}_p)}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)).$$

Moreover $\{E_r, d_r\}$ converges strongly to $H^*(X \times_Z Y;\mathbb{F}_p)$ as an algebra and an unstable $\mathcal{A}$-module

Proof. Let $\mathrm{Cobar}^\bullet(X,Z,Y)$ be the cobar construction which is a cosimplicial simplicial set (see [29, 2.3]). Observe that the inclusion $\iota : X \times_Z Y \to X \times Y$ induces a map $X \times_Z Y \to \mathrm{Cobar}^\bullet(X,Z,Y)$ of cosimplicial simplicial sets. The Eilenberg-Moore theorem asserts that the composition map

$$\begin{aligned} C^*(\mathrm{Cobar}^\bullet(X,Z,Y);\mathbb{F}_p) \xrightarrow{\pi} C^*(\mathrm{Cobar}^0(X,Z,Y);\mathbb{F}_p) &= C^*(X \times Y;\mathbb{F}_p) \\ &\xrightarrow{\iota} C^*(X \times_Z Y;\mathbb{F}_p) \end{aligned}$$

is a quasi-isomorphism, where π is the projection (see [31, Theorem 3.2]). Thus Theorem 7.1 allows us to obtain the result. Observe that the torsion product $\mathrm{Tor}_{H^*(Z;\mathbb{F}_p)}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p))$ possesses the $\mathcal{A}$-module structure. This implies that $\{E_r, d_r\}$ admits a well compatible DGA-$\mathcal{A}$-module structure and converges to $H^*(X \times_Z Y;\mathbb{F}_p)$ as an unstable $\mathcal{A}$-module. □

By applying Theorem 7.1 to the E_∞-model for a mapping space due to Chataur and Thomas [12], we have the following theorems (see Remark 7.6).

Theorem 7.9. *Let Y be connected space of $\mathbb{F}_p$-finite type and $X_\bullet$ a simplicial finite set. If the cosimplicial space $\mathcal{F}(X_\bullet, Y)$ is convergent, then there exists a spectral sequence $\{E_r, d_r\}$ admitting a well compatible DGA-$\mathcal{A}$-module structure with*

$$E_1^{-s,*} \cong H(Y;\mathbb{F}_p)^{\otimes \sharp X_s}.$$

Moreover $\{E_r, d_r\}$ converges strongly to $H^*(\mathcal{F}(|X_\bullet|, Y);\mathbb{F}_p)$ as an algebra.

Theorem 7.10. *Let Y be a simply-connected space. Then there exists a spectral sequence $\{E_r, d_r\}$ admitting a well compatible DGA-$\mathcal{A}$-module structure and converging strongly to $H^*(\mathcal{F}(S^1, Y);\mathbb{F}_p)$ as an algebra with*

$$E_2^{-s,*} \cong HH_s(H^*(Y;\mathbb{F}_p)).$$

Here $HH_s(H^*(Y;\mathbb{F}_p))$ denotes the Hochschild homology of $H^*(Y;\mathbb{F}_p)$.

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