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On the behavior of the Whitehead product $[\iota_{4n}, 4\nu_{4n}]$

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Abstract

In this note, we show that $[\iota_n, 4\nu_n]$ does not belong to $\nu_n \circ \pi_{2n+2}^{n+3}$ for the cases $n \equiv 0 \pmod 8 \geq 16$, $n \equiv 4 \pmod{16} \geq 20$ and $n \equiv 12 \pmod{32} \geq 44$, using the composition methods in homotopy groups of spheres.

1. Introduction

Let $\pi_{n+k}^n\,(k \neq n-1)$ denote the 2-primary component of the homotopy group $\pi_{n+k}(S^n)$ of the n-sphere S^n and let $\pi_{2n-1}^n = E^{-1}(\pi_{2n}^{n+1})$, where E is the suspension homomorphism. Let ι_n be the identity class of S^n and let $\nu_n \in \pi_{n+3}^n \cong Z_8\,(n \geq 5)$ be the generator. The first author proposed the following problem concerning his geometrical problem and expected an affirmative answer.

Problem. Let $n \equiv 0 \pmod 4$. Then, is it right that the Whitehead product $[\iota_n, 4\nu_n]$ does not belong to $\nu_n \circ \pi_{2n+2}^{n+3}$ unless it is zero ? (It is known that $[\iota_n, 4\nu_n] = 0$ if $n = 4, 12$.)

In this paper, we show the following results about the above problem using the composition methods [13]. Although the theorem is not complete as the answer, it covers seven-eighth of the whole cases.

Theorem 1 (i) *$[\iota_8, 4\nu_8]$ belongs to $\nu_8 \circ \pi_{18}^{11}$.*

In the following cases, $[\iota_n, 4\nu_n]$ does not belong to $\nu_n \circ \pi_{2n+2}^{n+3}$:

(ii) $n \equiv 0 \pmod 8$, $n \geq 16$.

(iii) $n \equiv 4 \pmod{16}$, $n \geq 20$.

(iv) $n \equiv 12 \pmod{32}$, $n \geq 44$.

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Note : By analysing the Adams spectral sequence of stable homotopy groups of spheres, M. Mahowald asserts and believes that the answer of the problem is affirmative except for the cases $n = 4, 8, 12$ ([8]). But, it seems that there exist similar difficulties also to have a complete proof according to the outline of [8].

Our main tool is the EHP exact sequence

$$\cdots \longrightarrow \pi_q^n \xrightarrow{E} \pi_{q+1}^{n+1} \xrightarrow{H} \pi_{q+1}^{2n+1} \xrightarrow{P} \pi_{q-1}^n \longrightarrow \cdots,$$

where $P(E^{n+1}\alpha) = [\iota_n, \alpha]$ for $\alpha \in \pi_{q-n}^n$. Henceforth, we follow the notations of [13].

2. Several facts and lemmas

To prove the theorem, the following is essentially used ([10], the second table).

Lemma 2 (i) *Let $n \equiv 0 \pmod 8$, $n \geq 16$. Then, there exists an element $\delta = \delta(\eta\epsilon) \in \pi_{2n-4}^{n-6}$ such that $E^6\delta = [\iota_n, 4\nu_n]$ and $H\delta = \eta_{2n-13}\varepsilon_{2n-12} \in \pi_{2n-4}^{2n-13}$.*

(ii) *Let $n \equiv 4 \pmod{16}$, $n \geq 20$. Then, there exists an element $\delta = \delta(\zeta) \in \pi_{2n-6}^{n-8}$ such that $E^8\delta = [\iota_n, 4\nu_n]$ and $H\delta = \zeta_{2n-17} \in \pi_{2n-6}^{2n-17}$.*

(iii) *Let $n \equiv 12 \pmod{32}$, $n \geq 44$. Then, there exists an element $\delta = \delta(\rho) \in \pi_{2n-10}^{n-12}$ such that $E^{12}\delta = [\iota_n, 4\nu_n]$ and $H\delta = \rho_{2n-25} \in \pi_{2n-10}^{2n-25}$.*

Lemma 3 (i) *Let $n \equiv 0 \pmod 4$, $n \geq 12$. Then, $\pi_{2n-4}^{n-3} \xrightarrow{E} \pi_{2n-3}^{n-2} \xrightarrow{E} \pi_{2n-2}^{n-1}$ are respectively surjective. Furthermore, if $n \geq 20$, then $\pi_{2n-7}^{n-6} \xrightarrow{E} \pi_{2n-6}^{n-5} \xrightarrow{E} \pi_{2n-5}^{n-4} \xrightarrow{E} \pi_{2n-4}^{n-3}$ are respectively surjective and $\pi_{2n-2}^{n-1} = E^5\pi_{2n-7}^{n-6}$.*

(ii) *$\nu_{10}\rho_{13}$ belongs to $\{[\iota_{10}, \mu_{10}]\} \subset \pi_{28}^{10}$.*

Proof. (i) In the EHP sequence

$$\pi_{2n-k-2}^{n-k-1} \xrightarrow{E} \pi_{2n-k-1}^{n-k} \xrightarrow{H} \pi_{2n-k-1}^{2n-2k-1} \xrightarrow{P} \pi_{2n-k-3}^{n-k-1},$$

$\pi_{2n-2}^{2n-3} = \{\eta_{2n-3}\} \cong Z_2$, $\pi_{2n-3}^{2n-5} = \{\eta_{2n-5}^2\} \cong Z_2$, for $k = 1, 2$, respectively. Since $\mathrm{Im}P = \{[\iota_{n-2}, \eta_{n-2}]\} \cong Z_2$ if $k = 1$ $(n \neq 4, 8)$ and $\mathrm{Im}P = \{[\iota_{n-3}, \eta_{n-3}^2]\} \cong Z_2$ if $k = 2$ $(n \neq 4, 8)$ by [4], [6], the map P is injective for $k = 1, 2$. Hence $\mathrm{Im}H = 0$ and E is surjective for $k = 1, 2$. Let $k = 3$. Then $\pi_{2n-4}^{2n-7} =$

$\{\nu_{2n-7}\} \cong Z_8$, and by [7], $\mathrm{Im}P = \{[\iota_{n-4}, \nu_{n-4}]\} \cong Z_8\,(n \neq 8, 16)$. Hence P is injective, and so E is surjective for $k = 3$ and $n \geq 20$. For $k = 4, 5$, since $\pi^{2n-9}_{2n-5} = 0\,(n \geq 8)$, $\pi^{2n-11}_{2n-6} = 0\,(n \geq 9)$, we have $H = 0$ and so E is surjective.

(ii) By (7) of Proposition (2.17) of [11], we have $\nu_{10}\rho_{13} \in \{2\sigma_{10}\zeta_{17}\}$. On the other hand, by (12.25) of [13], $2\sigma_{10}\zeta_{17} = P(\mu_{21}) = [\iota_{10}, \mu_{10}]$. So we have $\nu_{10}\rho_{13} \in \{[\iota_{10}, \mu_{10}]\}$. $\square$

Let O_n, SO_n be the n-th orthogonal and rotation groups, respectively. Let $i : SO_n \to SO_{n+1}$ be the inclusion map and let $p : SO_{n+1} \to S^n$ be the projection. We have the exact sequence associated with the fibre bundle $SO_n \to SO_{n+1} \to SO_{n+1}/SO_n = S^n$:

$$\cdots \to \pi_k(SO_n) \xrightarrow{i_*} \pi_k(SO_{n+1}) \xrightarrow{p_*} \pi_k(S^n) \xrightarrow{\Delta} \pi_{k-1}(SO_n) \to \cdots$$

Let $\omega_n(R) \in \pi_{n-1}(SO_n)$ be the characteristic element for the above bundle, and similarly let $\omega_n(C) \in \pi_{2n}(U_n)$ be the one for the bundle $U_n \to U_{n+1} \to U_{n+1}/U_n = S^{2n+1}$. Let $r : U_n \to SO_{2n}$ be the canonical inclusion map. We put $\tau'_{2n} = r_*\omega_n(C) \in \pi_{2n}(SO_{2n})$. The following facts are well-known ([12]):

(1) $\omega_n(R) = \Delta\iota_n$.
(2) $i_*\tau'_{2n} = \omega_{2n+1}(R)$.
(3) $p_*\tau'_{2n} = (n-1)\eta_{2n-1}$, $n \geq 2$.
(4) $\Delta\eta_{4n+2} = 2\tau'_{4n+2}$, $n \geq 2$.

Here, (4) follows from the exact sequence

$$\begin{array}{l} \pi_{4n+3}(S^{4n+2}) = \{\eta_{4n+2}\} \cong Z_2 \\ \qquad \Delta \downarrow \\ \pi_{4n+2}(SO_{4n+2}) \cong Z_4 \ \ ([5]) \\ \qquad i_* \downarrow \\ \pi_{4n+2}(SO_{4n+3}) = \{\omega_{4n+3}(R)\} \cong Z_2, \end{array}$$

where i_* is surjective by (2). Hence $2\tau'_{4n+2} \neq 0$, $i_*(2\tau'_{4n+2}) = 0$, and so we have $2\tau'_{4n+2} = \Delta\eta_{4n+2}$.

Let $J : \pi_k(SO_m) \to \pi_{k+m}(S^m)$ be the J homomorphism, and put $\tau_{2n} = J\tau'_{2n} \in \pi_{4n}(S^{2n})$. Then, we have the following.

Lemma 4 (i) $E\tau_{2n} = [\iota_{2n+1}, \iota_{2n+1}]$.

(ii) $2\tau_{4n+2} = [\iota_{4n+2}, \eta_{4n+2}]$.

(iii) $H\tau_{2n} = (n-1)\eta_{4n-1}$ $(n \geq 2)$.

(iv) *If* $\alpha \in \pi_{k+4n+2}(S^{4n+3})$ *is a suspension element and* $2\alpha = 0$, *then* $[\iota_{4n+2}, \eta_{4n+2}\alpha] = 0$.

Proof. (i)-(iii) can be proved by the homotopy exact sequence associated with the fibre bundle $SO_m \to SO_{m+1} \to S^m$ and the EHP sequence connected by the J homomorphism, using the above facts (1)-(4). (iv) is a straightforward calculation using (ii). $\square$

For a given element $\alpha \in \pi_k(S^n)$, if there exists an element $\beta \in \pi_k(SO_{n+1})$ satisfying $p_*\beta = \alpha$, then β is called a *lift* of α and is denoted by $[\alpha]$.

From (3), we have
Example. $\tau'_{4n} = [\eta_{4n-1}]$ and $\tau_{4n} = J[\eta_{4n-1}]$.

Lemma 5 (i) *If* $n \equiv 3 \pmod 4$, *then* $\Delta\eta_n = \Delta\varepsilon_n = 0$ *and* $[\iota_n, \eta_n] = [\iota_n, \varepsilon_n] = 0$.

(ii) *If* $n \equiv 4, 5, 7 \pmod 8$, *then* $\Delta\nu_n^2 = 0$ *and* $[\iota_n, \nu_n^2] = 0$.

(iii) *If* $n \equiv 0 \pmod 4$ $(n \geq 12)$, *then* $[\iota_n, 8\sigma_n] \in E^4\pi_{2n+2}^{n-4}$. *Furthermore, if* $n \equiv 0 \pmod 8$ $(n \geq 16)$, *then* $[\iota_n, 8\sigma_n] \in E^8\pi_{2n-2}^{n-8}$.

(iv) *If* $n \equiv 1 \pmod 8$ $(n \geq 9)$, *then there exists an element* $\delta_1 \in \pi_{2n-1}^{n-3}$ *such that* $[\iota_n, \nu_n] = E^3\delta_1$ *and* $H\delta_1 = \nu_{2n-7}^2$.

(v) *If* $n \equiv 2 \pmod 4$ $(n \geq 10)$, *then there exists an element* $\delta_2 \in \pi_{2n+6}^{n-2}$ *such that* $[\iota_n, \mu_n] = E^2\delta_2$ *and* $H\delta_2 = \pm 2\zeta_{2n-5}$.

Proof. (i) is obtained by Lemma 1.3 of [9].

(ii) is obtained by Theorem 2.2, Lemma 3.3 and (3.11) of [2].

(iii) is known from Lemma 2.9 (1)-(2) of [9], and similarly (iv) from Lemma 2.13 (1) of [9].

(v) is obtained from Proposition 11.11 (ii) of [13], verifying that the stable secondary composition $\langle \eta, 2\iota, \mu \rangle$ is equal to $\pm 2\zeta$. $\square$

3. Proof of Theorem 1

Proof of (i). We know $\pm P(\nu_{17}) = 2\sigma_8\nu_{15} - x\nu_8\sigma_{11}$ by (7.19) of [13], where x is an odd integer. So, we have $[\iota_8, 4\nu_8] = 4P(\nu_{17}) = 4x\nu_8\sigma_{11} = 4\nu_8\sigma_{11} = \nu_8(4\sigma_{11})$.

Proof of (ii). Let $n \equiv 0 \pmod 8$, $n \geq 16$ and assume that $[\iota_n, 4\nu_n]$ is in $\nu_n \circ \pi^{n+3}_{2n+2}$. Then, by Lemma 2(i), there exists $\delta = \delta(\eta\varepsilon) \in \pi^{n-6}_{2n-4}$ satisfying $E^6\delta = [\iota_n, 4\nu_n]$ and $H\delta = \eta_{2n-13}\varepsilon_{2n-12} \in \pi^{2n-13}_{2n-4}$. Since $\pi^{2n-1}_{2n+3} = \pi^{2n-3}_{2n+2} = 0$, by the EHP sequence, the suspensions $\pi^{n-2}_{2n} \to \pi^{n-1}_{2n+1} \to \pi^{n}_{2n+2}$ are injective. So, we have

(1) $E^4\delta \in \nu_{n-2} \circ \pi^{n+1}_{2n}$.

Since $\pi^n_{2n-1} = \{[\iota_n, \iota_n]\} \oplus E\pi^{n-1}_{2n-2}$ and $\pi^{n+1}_{2n} = E^2\pi^{n-1}_{2n-2}$ for even $n(\neq 2, 4, 8)$, we may put as $E^4\delta = \nu_{n-2}E^2\gamma$, $\gamma \in \pi^{n-1}_{2n-2}$. Then $E(E^3\delta - \nu_{n-3}E\gamma) = 0$, and so

$E^3\delta - \nu_{n-3}E\gamma \in \mathrm{Ker}E = P\pi^{2n-5}_{2n+1} = P(\{\nu^2_{2n-5}\}) = \{[\iota_{n-3}, \nu^2_{n-3}]\} = 0$,

by Lemma 5(ii). Hence we have

(2) $E^3\delta = \nu_{n-3}E\gamma \in \nu_{n-3} \circ E\pi^{n-1}_{2n-2}$.

Similarly, we know

$$E^2\delta \in \nu_{n-4} \circ \pi^{n-1}_{2n-2} \mod P\pi^{2n-7}_{2n},\ \pi^{2n-7}_{2n} = \{\sigma_{2n-7}\} \cong Z_{16},$$

$$P\pi^{2n-7}_{2n} = P\{\sigma_{2n-7}\} = \{[\iota_{n-4}, \sigma_{n-4}]\}.$$

Since $\pi^{n-1}_{2n-2} = E^2\pi^{n-3}_{2n-4}$ by Lemma 3(i), we have

(3) $E^2\delta \in E^2(\nu_{n-6} \circ \pi^{n-3}_{2n-4}) \mod \{[\iota_{n-4}, \sigma_{n-4}]\}$.

So, for an element $\alpha \in \nu_{n-6} \circ \pi^{n-3}_{2n-4}$, we have $E^2\delta - E^2\alpha = c[\iota_{n-4}, \sigma_{n-4}]$, where $-8 \leq c \leq 8$. Then, taking the Hopf invariants of both sides, we have $0 = cH[\iota_{n-4}, \sigma_{n-4}] = \pm 2c\sigma_{2n-9}$. Since σ_{2n-9} has the order 16, we may assume $c = \pm 8$ if $c \neq 0$. Thus, we have

(4) $E^2\delta \in E^2(\nu_{n-6} \circ \pi^{n-3}_{2n-4}) \mod \{[\iota_{n-4}, 8\sigma_{n-4}]\}$.

By Lemma 5(iii), there exists $\delta_0 \in \pi^{n-8}_{2n-6}$ such that $[\iota_{n-4}, 8\sigma_{n-4}] = E^4\delta_0$. Hence,

$$E\delta \in E(\nu_{n-6}\circ\pi^{n-3}_{2n-4}) + \{E^3\delta_0\} \mod P\pi^{2n-9}_{2n-1},\ \pi^{2n-9}_{2n-1} = \{\bar\nu_{2n-9}\} \oplus \{\varepsilon_{2n-9}\} \cong Z_2 \oplus Z_2.$$

By Lemma 6.4 of [13] and Lemma 5(i),

$$P(\bar\nu_{2n-9} + \varepsilon_{2n-9}) = P(\eta_{2n-9}\sigma_{2n-8}) = [\iota_{n-5}, \eta_{n-5}\sigma_{n-4}] = [\iota_{n-5}, \eta_{n-5}]\circ\sigma_{2n-10} = 0,$$

$$P(\varepsilon_{2n-9}) = [\iota_{n-5}, \varepsilon_{n-5}] = 0.$$

Therefore $P\pi_{2n-1}^{2n-9} = 0$, and so

(5) $E\delta \in E(\nu_{n-6} \circ \pi_{2n-4}^{n-3}) + \{E^3\delta_0\}$.

Thus, we have

$$\delta \in \nu_{n-6} \circ \pi_{2n-4}^{n-3} + \{E^2\delta_0\} \mod P\pi_{2n-2}^{2n-11}.$$

Now we assume that $n \neq 16$. Then, since $E : \pi_{2n-5}^{n-4} \to \pi_{2n-4}^{n-3}$ is surjective from Lemma 3(i), we have

(6) $\delta \in E(\nu_{n-7} \circ \pi_{2n-5}^{n-4}) + \{E^2\delta_0\} \mod P\pi_{2n-2}^{2n-11}$,
$\pi_{2n-2}^{2n-11} = \{\nu_{2n-11}^3\} \oplus \{\mu_{2n-11}\} \oplus \{\eta_{2n-11}\varepsilon_{2n-10}\} \cong Z_2 \oplus Z_2 \oplus Z_2$.

Then, taking the Hopf invariant of $P\pi_{2n-2}^{2n-11}$, we have

$$HP\nu_{2n-11}^3 = H[\iota_{n-6}, \nu_{n-6}^3] = 2\nu_{2n-13}^3 = 0,\ HP\mu_{2n-11} = 2\mu_{2n-13} = 0,$$

$$HP(\eta_{2n-11}\varepsilon_{2n-10}) = 2\eta_{2n-13}\varepsilon_{2n-12} = 0.$$

So we obtain $HP\pi_{2n-2}^{2n-11} = 0$. Thus, considering the Hopf invariants of both sides of (6), there arises a contradiction since $H\delta = \eta_{2n-13}\varepsilon_{2n-12} \neq 0$.

Let $n = 16$ and assume that $[\iota_{16}, 4\nu_{16}] \in \nu_{16} \circ \pi_{34}^{19}$. Here, $\pi_{34}^{19} = \{\rho_{19}\} \oplus \{\bar{\varepsilon}_{19}\} \cong Z_{32} \oplus Z_2$. Then, by Lemma 3(ii), $\nu_{16}\rho_{19} \in E^6\{[\iota_{10}, \mu_{10}]\} = 0$. Furthermore, since $\bar{\varepsilon}_n = \eta_n\kappa_{n+1}$ ($n \geq 6$) by (10.23) of [13], we have $\nu_{16}\bar{\varepsilon}_{19} = \nu_{16}(\eta_{19}\kappa_{20}) = (\nu_{16}\eta_{19})\kappa_{20}$, which belongs to $\pi_{20}^{16} \circ \kappa_{20} = 0$. Thus, $\nu_{16} \circ \pi_{34}^{19} = 0$. This contradicts the fact that $[\iota_{16}, 4\nu_{16}] \neq 0$. This completes the proof of (ii).

Proof of (iii). Let $n \equiv 4 \pmod{16}$, $n \geq 20$ and assume that $[\iota_n, 4\nu_n]$ is in $\nu_n \circ \pi_{2n+2}^{n+3}$. By Lemma 2(ii), there exists $\delta = \delta(\zeta) \in \pi_{2n-6}^{n-8}$ such that $E^8\delta = [\iota_n, 4\nu_n]$ and $H\delta = \zeta_{2n-17} \in \pi_{2n-6}^{2n-17}$. Hence, just as in (2) of the proof of (ii), we have

(1) $E^5\delta \in \nu_{n-3} \circ E\pi_{2n-2}^{n-1} \mod\{[\iota_{n-3}, \nu_{n-3}^2]\}$.

Now, $[\iota_{n-3}, \nu_{n-3}^2] = [\iota_{n-3}, \nu_{n-3}] \circ \nu_{2n-4}$, and by Lemma 5(iv), there exists $\delta_1 \in \pi_{2n-7}^{n-6}$ such that $[\iota_{n-3}, \nu_{n-3}] = E^3\delta_1$ and $H\delta_1 = \nu_{2n-13}^2$. Therefore, we have

$$[\iota_{n-3}, \nu_{n-3}^2] = (E^3\delta_1) \circ \nu_{2n-4} = E^3(\delta_1\nu_{2n-7}),$$

where we note that $E^3(\delta_1\nu_{2n-7})$ has the order 2. Hence, from (1) and by Lemma 3(i),

(2) $E^5\delta \in E^6(\nu_{n-9} \circ \pi_{2n-7}^{n-6}) + \{E^3(\delta_1\nu_{2n-7})\}$.

So, we have

$$E^4\delta \in E^5(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + \{E^2(\delta_1\nu_{2n-7})\} \bmod P\pi^{2n-7}_{2n},$$

$$P\pi^{2n-7}_{2n} = P\{\sigma_{2n-7}\} = \{[\iota_{n-4}, \sigma_{n-4}]\}.$$

Then, similarly as in (3) of the proof of (ii), we have

(3) $E^4\delta \in E^5(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + \{E^2(\delta_1\nu_{2n-7})\} \mod \{[\iota_{n-4}, 8\sigma_{n-4}]\}$.

By Lemma 5(iii), $[\iota_{n-4}, 8\sigma_{n-4}] \in E^8\pi^{n-12}_{2n-10}$ $(n \geq 20)$.

Let L be a subgroup of π^k_n and L' that of π^{k-m}_{n-m} such that $L = E^m L'$. Then, we write

$$L = o(m).$$

Then, from (3), we see that

$$E^3\delta \in E^4(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + \{E(\delta_1\nu_{2n-7})\} + o(7) \bmod P\pi^{2n-9}_{2n-1}.$$

Here $P\pi^{2n-9}_{2n-1} = 0$ as in (4) of the proof of (ii). So,

(4) $E^3\delta \in E^4(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + \{E(\delta_1\nu_{2n-7})\} + o(7)$.

Therefore,

$$E^2\delta \in E^3(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + \{\delta_1\nu_{2n-7}\} + o(6) \bmod P\pi^{2n-11}_{2n-2},$$

$$\pi^{2n-11}_{2n-2} = \{\nu^3_{2n-11}\} \oplus \{\mu_{2n-11}\} \oplus \{\eta_{2n-11}\varepsilon_{2n-10}\} \cong Z_2 \oplus Z_2 \oplus Z_2.$$

Then, we have $P\nu^3_{2n-11} = [\iota_{n-6}, \nu^3_{n-6}] = 0$ by Lemma 4(iv) since $\nu^3_{n-6} = \eta_{n-6}\bar{\nu}_{n-5}$ and $2\bar{\nu}_{n-5} = 0$, and similarly $P(\eta_{2n-11}\varepsilon_{2n-10}) = [\iota_{n-6}, \eta_{n-6}\varepsilon_{n-5}] = 0$ since $2\varepsilon_{n-5} = 0$. So, we know $P\pi^{2n-11}_{2n-2} = \{P\mu_{2n-11}\} = \{[\iota_{n-6}, \mu_{n-6}]\}$. Hence,

(5) $E^2\delta \in E^3(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + \{\delta_1\nu_{2n-7}\} + o(6) \mod\{[\iota_{n-6}, \mu_{n-6}]\}$.

Here, $H(\delta_1\nu_{2n-7}) = (H\delta_1)\nu_{2n-7} = \nu^2_{2n-13}\nu_{2n-7} = \nu^3_{2n-13}$, and $H[\iota_{n-6}, \mu_{n-6}] = 2\mu_{2n-13} = 0$.

Now, in (2), if $E^5\delta$ has the term $kE^3(\delta_1\nu_{2n-7})$, then in (5), $E^2\delta$ has the term $k(\delta_1\nu_{2n-7})$. So, considering the Hopf invariants of both sides, we have $0 = kH(\delta_1\nu_{2n-7}) = k\nu^3_{2n-13}$. Therefore, k must be even. Since $E^3(\delta_1\nu_{2n-7})$ has the order 2, $E^5\delta$ does not contain the term $E^3(\delta_1\nu_{2n-7})$ from the beginning. Thus,

(6) $E^2\delta \in E^3(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + o(6) \mod\{[\iota_{n-6}, \mu_{n-6}]\}$.

Futhermore, by Lemma 5(v), there exists $\delta_2 \in \pi^{n-8}_{2n-6}$ such that $[\iota_{n-6}, \mu_{n-6}] = E^2\delta_2$ and $H\delta_2 = \pm 2\zeta_{2n-17}$. So,

$$E\delta \in E^2(\nu_{n-9}\circ\pi^{n-6}_{2n-7}) + \{E\delta_2\} + o(5) \bmod P\pi^{2n-13}_{2n-3},$$

$$\pi^{2n-13}_{2n-3} = \{\eta_{2n-13}\mu_{2n-12}\} \cong Z_2.$$

By Lemma 4(i), we have

$$P(\eta_{2n-13}\mu_{2n-12}) = [\iota_{n-7}, \eta_{n-7}\mu_{n-6}] = E(\tau_{n-8}\eta_{2n-16}\mu_{2n-15}).$$

Hence,

(7) $E\delta \in E^2(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{E\delta_2\} + o(5) \mod \{E(\tau_{n-8}\eta_{2n-16}\mu_{2n-15})\}$.

Thus, we have

(8) $\delta \in E(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{\delta_2, \tau_{n-8}\eta_{2n-16}\mu_{2n-15}\} + o(4) \mod P\pi^{2n-15}_{2n-4}$,
$P\pi^{2n-15}_{2n-4} = P\{\zeta_{2n-15}\} = \{[\iota_{n-8}, \zeta_{n-8}]\}$.

We have the Hopf invariants

$$H\delta_2 = \pm 2\zeta_{2n-17},\ H[\iota_{n-8}, \zeta_{n-8}] = \pm 2\zeta_{2n-17}$$

and by Lemma 4(iii) ((7.14) of [13])

$$H(\tau_{n-8}\eta_{2n-16}\mu_{2n-15}) = H(\tau_{n-8})\eta_{2n-16}\mu_{2n-15} = \eta^2_{2n-17}\mu_{2n-15} = 4\zeta_{2n-17}.$$

Hence, there arises a relation $H\delta = \zeta_{2n-17} \in \{2\zeta_{2n-17}\}$, which is a contradiction. This completes the proof of (iii).

Proof of (iv). Let $n \equiv 12 \pmod{32}$, $n \geq 44$ and assume that $[\iota_n, 4\nu_n]$ is in $\nu_n \circ \pi^{n+3}_{2n+2}$. We continue the arguments parallel to the proofs of (ii), (iii). By Lemma 2(iii), there exists $\delta = \delta(\rho) \in \pi^{n-12}_{2n-10}$ such that $E^{12}\delta = [\iota_n, 4\nu_n]$ and $H\delta = \rho_{2n-25} \in \pi^{2n-25}_{2n-10}$. We have

$$E^9\delta \in \nu_{n-3} \circ E\pi^{n-1}_{2n-2} \bmod\{[\iota_{n-3}, \nu^2_{n-3}]\}.$$

Then, similarly to (1), (2) in the proof of (iii), by Lemma 5(iv) and Lemma 3(i), we have

(1) $E^9\delta \in E^6(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{E^3(\delta_1\nu_{2n-7})\}$,

where $E^3(\delta_1\nu_{2n-7})$ has the order 2. So, similarly to (2), (3) of (iii),

$$E^8\delta \in E^5(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{E^2(\delta_1\nu_{2n-7})\} \bmod\{[\iota_{n-4}, 8\sigma_{n-4}]\}.$$

Now, we need further informations due to ([10], the second table).

Lemma 6 *Let* $n \equiv 12 \pmod{32}$, $n \geq 44$. *Then* $[\iota_{n-4}, 8\sigma_{n-4}]$, $[\iota_{n-6}, \mu_{n-6}]$, $[\iota_{n-7}, \eta_{n-7}\mu_{n-6}]$, *and* $[\iota_{n-8}, 4\zeta_{n-8}]$ *desuspend* 9, 8, 9, *and* 12 *dimensions, respectively.*

Then, $[\iota_{n-4}, 8\sigma_{n-4}] \in E^9\pi^{n-13}_{2n-11}$ and from the above,
(2) $E^8\delta \in E^5(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{E^2(\delta_1\nu_{2n-7})\} + o(9)$.
Hence, we have

$$E^7\delta \in E^4(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{E(\delta_1\nu_{2n-7})\} + o(8),$$

where we note that $P\pi^{2n-9}_{2n-1} = 0$ as in the proof of (ii). Therefore,

$$E^6\delta \in E^3(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{\delta_1\nu_{2-7}\} + o(7) \bmod P\pi^{2n-11}_{2n-2}.$$

By Lemma 4(iv), we know $P\pi^{2n-11}_{2n-2} = \{[\iota_{n-6}, \mu_{n-6}]\}$ as in the proof of (iii). Since $[\iota_{n-6}, \mu_{n-6}] \in E^8\pi^{n-14}_{2n-12}$ by Lemma 6, we have
(3) $E^6\delta \in E^3(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + \{\delta_1\nu_{2n-7}\} + o(7)$.
Now, we can apply the argument similar to (5) in the proof of (iii). In (1), if $E^9\delta$ has the term $kE^3(\delta_1\nu_{2n-7})$, then in (3), $E^6\delta$ has the term $k(\delta_1\nu_{2n-7})$. But, k must be even, and so $E^9\delta$ does not contain the term $E^3(\delta_1\nu_{2n-7})$ from the beginning. Hence,

$$E^6\delta \in E^3(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + o(7)$$

and so,

$$E^5\delta \in E^2(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + o(6) \bmod P\pi^{2n-13}_{2n-3},$$
$$P\pi^{2n-13}_{2n-3} = P\{\eta_{2n-13}\mu_{2n-12}\} = \{[\iota_{n-7}, \eta_{n-7}\mu_{n-6}]\},$$

which belongs to $E^9\pi^{n-16}_{2n-14}$ by Lemma 6. Therefore,
(4) $E^5\delta \in E^2(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + o(6)$,
and so

$$E^4\delta \in E(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + o(5) \bmod P\pi^{2n-15}_{2n-4},$$
$$P\pi^{2n-15}_{2n-4} = P\{\zeta_{2n-15}\} = \{[\iota_{n-8}, \zeta_{n-8}]\}.$$

Since $H[\iota_{n-8}, \zeta_{n-8}] = \pm 2\zeta_{2n-17}$ and ζ_{2n-17} has the order 8, considering the Hopf invariants of both sides, we have

$$E^4\delta \in E(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + o(5) \bmod \{[\iota_{n-8}, 4\zeta_{n-8}]\}.$$

Then, $[\iota_{n-8}, 4\zeta_{n-8}] \in E^{12}\pi^{n-20}_{2n-18}$ by Lemma 6 again. Hence we have
(5) $E^4\delta \in E(\nu_{n-9} \circ \pi^{n-6}_{2n-7}) + o(5)$,
and so

(6) $E^3\delta \in \nu_{n-9} \circ \pi^{n-6}_{2n-7} + o(4)$,
where we note that $P\pi^{2n-17}_{2n-5} = 0$ since $\pi^S_{12}(S^0) = 0$.

Let $V_{n,k} = O_n/O_{n-k}$ be the Stiefel manifold. Denote by $i_{m,n} : SO_m \to SO_n$ for $m \leq n$ the canonical inclusion map. Then, we need the following.

Lemma 7 *Let $n \equiv 12 \pmod{32}$, $n \geq 44$. Then, there exists a lift $[\nu^2_{n-7}] \in \pi_{n-1}(SO_{n-6})$ with the order 4 such that $HJ[\nu^2_{n-7}] = \nu^2_{2n-13}$, $E^5 J[\nu^2_{n-7}] = 0$, and $\pi^{n-6}_{2n-7} = \{J[\nu^2_{n-7}]\} + E\pi^{n-7}_{2n-8}$.*

Proof. By [1] and [3], we have the following exact sequence :

$$\begin{array}{l}
\pi_{n-1}(SO_{n-7}) \cong \pi_{n-1}(SO) \oplus \pi_n(V_{n+2,9}) \cong Z \oplus Z_2 \oplus Z_2 \\
\qquad i_* \downarrow \\
\pi_{n-1}(SO_{n-6}) \cong \pi_{n-1}(SO) \oplus \pi_n(V_{n+2,8}) \cong Z \oplus Z_4 \\
\qquad p_* \downarrow \\
\pi_{n-1}(S^{n-7}) = \{\nu^2_{n-7}\} \cong Z_2. \\
\qquad \Delta \downarrow
\end{array}$$

Since $\Delta\nu^2_{n-7} = 0$ by Lemma 5(ii), there exists a lift $[\nu^2_{n-7}] \in \pi_{n-1}(SO_{n-6})$ which is the generator of the torsion part. We have

$$HJ[\nu^2_{n-7}] = E^{n-6}p_*[\nu^2_{n-7}] = \nu^2_{2n-13},\ E^5 J[\nu^2_{n-7}] = J(i_{n-6,n-1})_*[\nu^2_{n-7}] = 0,$$

where we note $\pi_{n-1}(SO_{n-1}) \cong Z$. The last assertion is known from the exact sequence

$$\pi^{n-7}_{2n-8} \xrightarrow{E} \pi^{n-6}_{2n-7} \xrightarrow{H} \pi^{2n-13}_{2n-7} = \{\nu^2_{2n-13}\} \cong Z_2,$$

where $HJ[\nu^2_{n-7}] = \nu^2_{2n-13}$. In fact, for any $x \in \pi^{n-6}_{2n-7}$, let $Hx = a\nu^2_{2n-13}$ $(a = 0, 1)$. Then, $y = x - aJ[\nu^2_{n-7}]$ belongs to $\mathrm{Ker}H = \mathrm{Im}E$. □

Now we continue the proof of (iv). From (6) and by Lemma 7,

$$E^3\delta \in \nu_{n-9} \circ (\{J[\nu^2_{n-7}]\} + E\pi^{n-7}_{2n-8}) + o(4).$$

Then, since $E^5 J[\nu^2_{n-7}] = 0$, we have

$$E^8\delta \in E^5(\nu_{n-9} \circ E\pi^{n-7}_{2n-8}) + o(9).$$

That is, in (2) we can replace π^{n-6}_{2n-7} by $E\pi^{n-7}_{2n-8}$. Thus, repeating the argument, we have

$$E^3\delta \in \nu_{n-9} \circ E\pi^{n-7}_{2n-8} + o(4),\ E^2\delta \in \nu_{n-10} \circ \pi^{n-7}_{2n-8} + o(3),$$

where we note that $P\pi^{2n-19}_{2n-6} = 0$ since $\pi^S_{13}(S^0) \cong Z_3$. Here, we know the fact $\pi^{n-7}_{2n-8} = E\pi^{n-8}_{2n-9}$ from the EHP sequence

$$\pi^{n-8}_{2n-9} \xrightarrow{E} \pi^{n-7}_{2n-8} \xrightarrow{H} \pi^{2n-15}_{2n-8} \xrightarrow{P} \pi^{n-8}_{2n-10},$$

where $\pi^{2n-15}_{2n-8} = \{\sigma_{2n-15}\} \cong Z_{16}$. Since $P\sigma_{2n-15} = [\iota_{n-8}, \sigma_{n-8}]$ has the order 16 by [7], P is injective and so E is surjective. Thus,

$$E\delta \in \nu_{n-11} \circ \pi^{n-8}_{2n-9} + o(2) \bmod P\pi^{2n-21}_{2n-7},\ \pi^{2n-21}_{2n-7} = \{\sigma^2_{2n-21}\} \oplus \{\kappa_{2n-21}\} \cong Z_2 \oplus Z_2.$$

By Lemma 4(**i**),

$$P\sigma^2_{2n-21} = [\iota_{n-11}, \iota_{n-11}]\sigma^2_{2n-23} = (E\tau_{n-12})\sigma^2_{2n-23},\ P\kappa_{2n-21} = (E\tau_{n-12})\kappa_{2n-23}.$$

So, we have

$$P\pi^{2n-21}_{2n-7} = E\{\tau_{n-12}\sigma^2_{2n-24}, \tau_{n-12}\kappa_{2n-24}\}.$$

Hence, we have

$$(7) \quad E\delta \in \nu_{n-11} \circ \pi^{n-8}_{2n-9} + o(2) \quad \bmod E\{\tau_{n-12}\sigma^2_{2n-24}, \tau_{n-12}\kappa_{2n-24}\}.$$

Lemma 8 *Let $n \equiv 12 \pmod{32}$, $n \geq 44$. Then, there exist lifts $[\eta_{n-9}] \in \pi_{n-8}(SO_{n-8})$ and $[\varepsilon_{n-9}] \in \pi_{n-1}(SO_{n-8})$ with the order 2 respectively such that*

$$H(J[\eta_{n-9}]\sigma_{2n-16}) = \bar{\nu}_{2n-17} + \varepsilon_{2n-17},\ H(J[\varepsilon_{n-9}]) = \varepsilon_{2n-17}$$

and

$$E^2(J[\eta_{n-9}]\sigma_{2n-16}) = E^7 J[\varepsilon_{n-9}] = 0.$$

Furthermore,

$$\pi^{n-8}_{2n-9} = \{J[\eta_{n-9}]\sigma_{2n-16}, J[\varepsilon_{n-9}]\} \oplus E\pi^{n-9}_{2n-10},$$

and so $E^7\pi^{n-8}_{2n-9} = E^8\pi^{n-9}_{2n-10}$.

Proof. We have $[\eta_{n-9}]$ already as an example of lifts, with the order 2 since $\pi_{n-8}(SO_{n-8}) \cong Z_2 \oplus Z_2$ by [5]. On the other hand, we have the following exact sequence by [1] and [3] :

$$\begin{array}{l}
\pi_{n-1}(SO_{n-9}) \cong \pi_{n-1}(SO) \oplus \pi_n(V_{n+2,11}) \cong Z \oplus Z_2 \oplus Z_2 \\
\qquad i_* \downarrow \\
\pi_{n-1}(SO_{n-8}) \cong \pi_{n-1}(SO) \oplus \pi_n(V_{n+2,10}) \cong Z \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \\
\qquad p_* \downarrow
\end{array}$$

$$\pi_{n-1}(S^{n-9}) = \{\bar{\nu}_{n-9}\} \oplus \{\varepsilon_{n-9}\} \cong Z_2 \oplus Z_2$$
$$\Delta \downarrow$$

Considering the orders of the torsion parts, p_* must be surjective. In fact, $\Delta\varepsilon_{n-9} = 0$ by Lemma 5(i). Hence, there exists a lift $[\varepsilon_{n-9}]$ with the order 2. Then, by Lemma 6.4 of [13],

$$H(J[\eta_{n-9}]\sigma_{2n-16}) = (E^{n-8}p_*[\eta_{n-9}])\sigma_{2n-16} = \eta_{2n-17}\sigma_{2n-16} = \bar{\nu}_{2n-17} + \varepsilon_{2n-17},$$

and similarly

$$H(J[\varepsilon_{n-9}]) = \varepsilon_{2n-17}.$$

Since $\pi_{n-8}(SO_{n-6}) = 0$ and $\pi_{n-1}(SO_{n-1}) \cong Z$ by [5], we have

$$E^2J[\eta_{n-9}] = J(i_{n-8,n-6})_*[\eta_{n-9}] = 0,\ E^7J[\varepsilon_{n-9}] = J(i_{n-8,n-1})_*[\varepsilon_{n-9}] = 0.$$

The last assertion is known from the exact sequence

$$\pi^{n-9}_{2n-10} \xrightarrow{E} \pi^{n-8}_{2n-9} \xrightarrow{H} \pi^{2n-17}_{2n-9} \longrightarrow 0,$$

where $\pi^{2n-17}_{2n-9} = \{\bar{\nu}_{2n-17} + \varepsilon_{2n-17}\} \oplus \{\varepsilon_{2n-17}\} \cong Z_2 \oplus Z_2$. So, the sequence splits from the above facts. $\square$

Note : Let $n \equiv 0 \mod 4$ and $n \geq 8$. Then, from the fact that $[\eta_n]$ has the order 2, a lift $[\varepsilon_n]$ is taken as a representative of the Toda bracket

$$\{[\eta_n], 2\iota_{n+1}, \nu^2_{n+1}\}.$$

Now, we continue the proof of (iv). From (7), since $E^7\pi^{n-8}_{2n-9} = E^8\pi^{n-9}_{2n-10}$ by Lemma 8, we have

$$E^8\delta \in E^7(\nu_{n-11} \circ E\pi^{n-9}_{2n-10}) + o(9) \bmod E^8\{\tau_{n-12}\sigma^2_{2n-24}, \tau_{n-12}\kappa_{2n-24}\}.$$

Then, repeating similar and necessary arguments from (2), we have

(8) $E\delta \in \nu_{n-11} \circ E\pi^{n-9}_{2n-10} + o(2) \quad \bmod E\{\tau_{n-12}\sigma^2_{2n-24}, \tau_{n-12}\kappa_{2n-24}\}.$

Thus, we obtain

(9) $\delta \in \nu_{n-12} \circ \pi^{n-9}_{2n-10} + \{\tau_{n-12}\sigma^2_{2n-24}, \tau_{n-12}\kappa_{2n-24}\} + o(1) \quad \bmod P\pi^{2n-23}_{2n-8},$

$$\pi^{2n-23}_{2n-8} = \{\rho_{2n-23}\} \oplus \{\eta_{2n-23}\kappa_{2n-22}\} \cong Z_{32} \oplus Z_2,$$
$$P\pi^{2n-23}_{2n-8} = \{[\iota_{n-12}, \rho_{n-12}], [\iota_{n-12}, \eta_{n-12}\kappa_{n-11}]\}.$$

Now, we consider the Hopf invariants of both sides of (9).

$$H(\nu_{n-12} \circ \pi^{n-9}_{2n-10}) = E(\nu_{n-13} \wedge \nu_{n-13}) \circ H\pi^{n-9}_{2n-10} \subset \nu^2_{2n-19} \circ \pi^{2n-19}_{2n-10} = 0,$$

where the last equality is known by Theorem 14.1 of [13] verifying the compositions of the generators to be zero. Similarly, by Lemma 4(iii),

$$H(\tau_{n-12}\sigma^2_{2n-24}) = \eta_{2n-25}\sigma^2_{2n-24} = 0, \ H(\tau_{n-12}\kappa_{2n-24}) = \eta_{2n-25}\kappa_{2n-24}.$$

Furthermore,

$$H[\iota_{n-12}, \rho_{n-12}] = \pm 2\rho_{2n-25}, \ H[\iota_{n-12}, \eta_{n-12}\kappa_{n-11}] = \pm 2\eta_{2n-25}\kappa_{2n-24} = 0.$$

Thus we have

$$H\delta = \rho_{2n-25} \in \{\eta_{2n-25}\kappa_{2n-24}, 2\rho_{2n-25}\}.$$

This is a contradiction and this completes the proof of (iv). Thus, the proof of Theorem 1 is complete.

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