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On a theorem of MacCluer and Shapiro

Yasuo MATSUGU*, Satoru NAGAI** and Sei-ichiro UEKI***

*Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
**Matsumoto Preparatory School, Matsumoto 390-0814, Japan
***sueki@camel.plala.or.jp

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Abstract

Let u be a holomorphic function in the unit ball B of $\mathbf{C}^n$ and φ be a univalent holomorphic self-map of B. We give some sufficient conditions for u and φ that the weighted composition operator uC_φ is bounded or compact on the Hardy spaces $H^p(B)$ and the weighted Bergman spaces $A^p(\nu_\alpha)$ $(0<p<\infty,\ -1<\alpha<\infty)$. This our result is a generalization of a theorem of B. D. MacCluer and J. H. Shapiro[9] concerning the composition operator C_φ. And we also give similar sufficient conditions for such operator to be metrically bounded or metrically compact on the Privalov spaces $N^p(B)$ $(1<p<\infty)$ and the weighted Bergman-Privalov spaces $(AN)^p(\nu_\alpha)$ $(1\leq p<\infty,\ -1<\alpha<\infty)$.

1 Introduction

Let $n>1$ be a fixed integer. Let $B\equiv B_n$ and $S\equiv\partial B$ denote the unit ball and the unit sphere of the complex n-dimensional Euclidean space $\mathbf{C}^n$, respectively. Let ν and σ denote the normalized Lebesgue measure on B and on S, respectively. For each $\alpha\in(-1,\infty)$, we set $c_\alpha=\Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$ and $d\nu_\alpha(z)=c_\alpha(1-|z|^2)^\alpha d\nu(z)$ $(z\in B)$. Note that $\nu_\alpha(B)=1$. Let $H(B)$ denote the space of all holomorphic functions in B. For each $p\in(0,\infty)$ and $\alpha\in(-1,\infty)$, the *Hardy space* $H^p(B)$ and the *weighted Bergman space* $A^p(\nu_\alpha)$ are as usual defined by

$$H^p(B)=\left\{f\in H(B):\|f\|_{H^p(B)}^p\equiv\sup_{0<r<1}\int_S|f_r|^p d\sigma<\infty\right\},$$

$$A^p(\nu_\alpha)=\left\{f\in H(B):\|f\|_{A^p(\nu_\alpha)}^p\equiv\int_B|f|^p d\nu_\alpha<\infty\right\},$$

where $f_r(z)=f(rz)$ for $r\in(0,1)$, $z\in\mathbf{C}^n$ with $rz\in B$. As in [17], the *Privalov space* $N^p(B)$ $(1<p<\infty)$ is defined by

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$$N^p(B)=\left\{f\in H(B):\|f\|_{N^p(B)}^p\equiv\sup_{0<r<1}\int_S\{\log(1+|f_r|)\}^p d\sigma<\infty\right\}.$$

The *Nevanlinna space* $N(B)$ is as usual defined:

$$N(B)=\left\{f\in H(B):\|f\|_{N(B)}\equiv\sup_{0<r<1}\int_S\log(1+|f_r|)d\sigma<\infty\right\}.$$

For the sake of convenience, the symbol $N^1(B)$ as well as $N(B)$ is used to denote the Nevanlinna space. For each $p\in[1,\infty)$ and $\alpha\in(-1,\infty)$, we define the *weighted Bergman-Privalov space* $(AN)^p(\nu_\alpha)$ by

$$(AN)^p(\nu_\alpha)=\left\{f\in H(B):\|f\|_{(AN)^p(\nu_\alpha)}^p\equiv\int_B\{\log(1+|f|)\}^p d\nu_\alpha<\infty\right\}.$$

Let $ST_0^2(\mathbf{R})$ denote the class of those nondecreasing convex functions $\chi:\mathbf{R}\to[0,\infty)$ which are twice differentiable. Moreover, we define $ST^2(\mathbf{R})=\{\chi\in ST_0^2(\mathbf{R}):\lim_{t\to\infty}\frac{\chi(t)}{t}=\infty\}$. For $\alpha\in[-1,\infty)$ and $\chi\in ST_0^2(\mathbf{R})$, we define $\|\cdot\|_{\chi,\alpha}$ as follows:

$$\|f\|_{\chi,\alpha}=\begin{cases}\sup\limits_{0<r<1}\int_S\chi(\log|f_r|)d\sigma & \text{if } \alpha=-1,\\ \int_B\chi(\log|f|)d\nu_\alpha & \text{if } \alpha>-1,\end{cases}$$

for $f\in H(B)$. If $\chi(t)=e^{pt}$ ($t\in\mathbf{R}$, $0<p<\infty$), then $\|f\|_{\chi,\alpha}=\|f\|_{A^p(\nu_\alpha)}^p$ for $\alpha\in(-1,\infty)$ and $\|f\|_{\chi,-1}=\|f\|_{H^p(B)}^p$. If $\chi(t)=\{\log(1+e^t)\}^p$ ($t\in\mathbf{R}$, $1\le p<\infty$), then $\|f\|_{\chi,\alpha}=\|f\|_{(AN)^p(\nu_\alpha)}^p$ for $\alpha\in(-1,\infty)$ and $\|f\|_{\chi,-1}=\|f\|_{N^p(B)}^p$. For the sake of convenience, we define $A^p(\nu_{-1})\equiv H^p(B)$ ($0<p<\infty$) and $(AN)^p(\nu_{-1})\equiv N^p(B)$ ($1\le p<\infty$).

If $u\in H(B)$ and φ is a holomorphic self-map of B, then u and φ induce a linear operator uC_φ on $H(B)$ by means of the equation $uC_\varphi f=u\cdot(f\circ\varphi)$. This uC_φ is called the weighted composition operator induced by u and φ. In the case $u\equiv1$ in B, uC_φ is the composition operator C_φ. In the present paper we study the operator uC_φ on the above function spaces.

In 1986, B. D. MacCluer and J. H. Shapiro got the following result about the boundedness and the compactness of C_φ on $H^p(B)$ and on $A^p(\nu_\alpha)$:

Theorem 1.1 ([9], B. D. MacCluer-J. H. Shapiro). *Suppose that $\varphi: B\to B$ is a univalent holomorphic map, and that the Fréchet derivative of φ^{-1} is bounded on $\varphi(B)$. Then*

(a) *For each $p\in(0,\infty)$ and $\alpha\in(-1,\infty)$, C_φ is bounded on $H^p(B)$ and on $A^p(\nu_\alpha)$.*

(b) *For $p\in(0,\infty)$ and $\alpha\in(-1,\infty)$, C_φ is compact on $H^p(B)$ and on $A^p(\nu_\alpha)$ if and only if*

$$\lim_{|z|\uparrow1}\frac{1-|z|^2}{1-|\varphi(z)|^2}=0.$$

Recently, C_φ on $N^p(B_1)$ have been studied by J. S. Choa and H. O. Kim [2, 3, 4]. And C_φ on $(AN)^1(\nu)$ (in the case $n=1$) have been studied by J. Jarchow and J. Xiao [7, 18]. Following [4], we say that a linear operator T is *metrically bounded* on $(AN)^p(\nu_\alpha)$ if there exists a positive constant L such that $\|Tf\|_{(AN)^p(\nu_\alpha)}\le L\|f\|_{(AN)^p(\nu_\alpha)}$ for all $f\in$

$(AN)^p(\nu_\alpha)$. And we say that T is *metrically compact* on $(AN)^p(\nu_\alpha)$ if T maps every closed ball $B_R=\{f\in(AN)^p(\nu_\alpha):\|f\|_{(AN)^p(\nu_\alpha)}\le R\}$ $(0<R<\infty)$ into a relatively compact set in $(AN)^p(\nu_\alpha)$.

This paper is organized as follows. In Section 2, we enumerate 13 lemmas that will be used afterward. In Section 3, we give some sufficient conditions for u and φ that uC_φ is bounded or compact on $H^p(B)$ and on $A^p(\nu_\alpha)$. This result is a generalization of Theorem 1.1. Finally, in Section 4 we also give similar sufficient conditions for uC_φ to be metrically bounded or metrically compact on $N^p(B)$ and on $(AN)^p(\nu_\alpha)$. As a corollary of this, we obtain an analogous version of Theorem 1.1 with respect to $N^p(B)$ and $(AN)^p(\nu_\alpha)$.

2 Preliminaries

For each $\alpha\in[-1,\infty)$, we define the nonnegative decreasing function K_α by

$$K_\alpha(t)=\begin{cases}2nc_\alpha\displaystyle\int_t^1\rho^{2n-1}(1-\rho^2)^\alpha\log\frac{\rho}{t}d\rho & \text{if } \alpha>-1,\\ \log\dfrac{1}{t} & \text{if } \alpha=-1,\end{cases}$$

for all $t\in(0,1]$. It is obvious that $K_\alpha(1)=0$ and $K_\alpha(t)>0$ for all $t\in(0,1)$. The following lemma is easily verified ([1, Proposition 2.3]).

Lemma 2.1. (*a*) $K_{-1}(t)<1-t^2$ *if* $\frac{1}{2}\le t<1$.

(b) *For each* $\alpha\in(-1,\infty)$,

$$\lim_{t\uparrow1}\frac{K_\alpha(t)}{(1-t^2)^{\alpha+2}}=\frac{nc_\alpha}{2(\alpha+1)(\alpha+2)}.$$

Lemma 2.2. *Let* $-1\le\alpha<\infty$, $0<r<1$ *and* $z\in r\bar{B}\setminus\{0\}$. *Then it holds that*

$$K_\alpha(|z|)\le\log\frac{1}{r}+K_\alpha\Big(\frac{|z|}{r}\Big).$$

Proof. See [1, pages 48-49]. □

Lemma 2.3. *Suppose* $-1\le\alpha<\infty$, $\chi\in ST^2(\mathbf{R})$, $0<r\le1$ *and* $f\in H(B)\setminus\{0\}$. *Then*

$$\|f_r\|_{\chi,\alpha}=\chi(\log|f(0)|)+\frac{1}{2n}\int_{rB}\chi''(\log|f(z)|)\frac{|Rf(z)|^2}{|f(z)|^2|z|^2}|z|^{-2(n-1)}K_\alpha\Big(\frac{|z|}{r}\Big)d\nu(z),$$

where $(Rf)(z)=\Sigma_{j=1}^n z_j\frac{\partial f}{\partial z_j}(z)$ *is the radial derivative of* f.

Proof. In the case $r=1$, this lemma is just [10, Lemma 3.10]. If $0<r<1$ and $\alpha=-1$, it is just [10, Lemma 3.7]. If $0<r<1$ and $-1<\alpha<\infty$, it follows from [10, Lemma 3. 10] with a simple change of variables. □

Lemma 2.4. (a) *Let* $1\le p<\infty$ *and* $-1<\alpha<\infty$. *Every* $f\in H(B)\setminus\{0\}$ *satisfies the following inequalities :*

$$\frac{a_n\Gamma(n+\alpha+1)}{2^{\alpha+\alpha^+}(n+\alpha+1)\Gamma(\alpha+2)}\int_B\tilde{\Delta}(\{\log(1+|f|)\}^p)(z)(1-|z|^2)^\alpha d\nu(z)$$
$$+\{\log(1+|f(0)|)\}^p$$
$$\leq\|f\|^p_{(AN)^p(\nu_\alpha)}$$
$$\leq\frac{b_n2^{\alpha+\alpha^+}\Gamma(n+\alpha+1)}{(n+\alpha+1)\Gamma(\alpha+1)}\int_B\tilde{\Delta}(\{\log(1+|f|)\}^p)(z)(1-|z|^2)^\alpha d\nu(z)$$
$$+\int_S\{\log(1+|f_{\frac{1}{2}}|)\}^p d\sigma,$$

where $\tilde{\Delta}$ *is the Laplacian with respect to the Bergman metric on* B, $a_n=\dfrac{n+1}{2^{n+2}\Gamma(n+1)}$, $b_n=\dfrac{2^{3n-1}(n+1)}{\Gamma(n+1)}$ and $\alpha^+=\max\{0,\alpha\}$.

(b) *Let* $1<p<\infty$. *Every* $f\in H(B)\setminus\{0\}$ *satisfies the following inequalities :*

$$\frac{2a_n\Gamma(n)}{n}\int_B\tilde{\Delta}(\{\log(1+|f|)\}^p)(z)\frac{d\nu(z)}{1-|z|^2}+\{\log(1+|f(0)|)\}^p$$
$$\leq\|f\|^p_{N^p(B)}$$
$$\leq\frac{b_n\Gamma(n)}{2n}\int_B\tilde{\Delta}(\{\log(1+|f|)\}^p)(z)\frac{d\nu(z)}{1-|z|^2}+\int_S\{\log(1+|f_{\frac{1}{2}}|)\}^p d\sigma.$$

Proof. (a) is just [11, Theorem 1(a)]. By letting $\alpha\downarrow-1$ in (a), we obtain (b) (cf. [11, the proof of Theorem 2]). $\square$

Considering the weighted Bergman spaces $A^p(\nu_\alpha)$ instead of the weighted Bergman-Privalov spaces $(AN)^p(\nu_\alpha)$, we have the following lemma of which proof is essentially the same as that of Lemma 2.4.

Lemma 2.5. *Let* $0<p<\infty$ *and* $-1\leq\alpha<\infty$. *Every* $f\in H(B)\setminus\{0\}$ *satisfies the following inequalities :*

$$a_{n,\alpha}\int_B\tilde{\Delta}(|f|^p)(z)(1-|z|^2)^\alpha d\nu(z)+|f(0)|^p$$
$$\leq\|f\|^p_{A^p(\nu_\alpha)}$$
$$\leq b_{n,\alpha}\int_B\tilde{\Delta}(|f|^p)(z)(1-|z|^2)^\alpha d\nu(z)+\int_S|f_{\frac{1}{2}}|^p d\sigma,$$

where

$$a_{n,\alpha}=\begin{cases}\dfrac{a_n\Gamma(n+\alpha+1)}{2^{\alpha+\alpha^+}(n+\alpha+1)\Gamma(\alpha+2)} & \textit{if } \alpha>-1,\\ \dfrac{2a_n\Gamma(n)}{n} & \textit{if } \alpha=-1,\end{cases}$$

$$b_{n,\alpha}=\begin{cases}\dfrac{b_n2^{\alpha+\alpha^+}\Gamma(n+\alpha+1)}{(n+\alpha+1)\Gamma(\alpha+1)} & \textit{if } \alpha>-1,\\ \dfrac{b_n\Gamma(n)}{2n} & \textit{if } \alpha=-1.\end{cases}$$

Let $\tilde{\nabla}$ denote the gradient with respect to the Bergman metric on B ([16, p.27]). Then as in [16, p.30], for $f\in H(B)$ and $z\in B$,

$$|(\tilde{\nabla}f)(z)|^2=\frac{2}{n+1}(1-|z|^2)\Big[|(\nabla f)(z)|^2-|(Rf)(z)|^2\Big]$$
$$\leq\frac{2}{n+1}(1-|z|^2)|(\nabla f)(z)|^2$$

where $|(\nabla f)(z)|^2=\Sigma_{j=1}^{n}|\frac{\partial f}{\partial z_j}(z)|^2$. We note that

$$|(Rf)(z)|\leq|z||(\nabla f)(z)| \tag{2.1}$$

and

$$|(\tilde{\nabla}f)(z)|^2\geq\frac{2}{n+1}(1-|z|^2)^2|(\nabla f)(z)|^2. \tag{2.2}$$

Moreover, we have by a simple computation

$$\tilde{\Delta}(\chi(\log|f|))(z)=\frac{1}{2}\chi''(\log|f|)(z)\frac{|(\tilde{\nabla}f)(z)|^2}{|f(z)|^2} \tag{2.3}$$

for $f\in H(B)$ and $z\in B\backslash Z(f)$, where $Z(f)=\{w\in B: f(w)=0\}$.

Lemma 2.6. *Let* $1\leq p<\infty$ *and* $-1\leq\alpha<\infty$. *Suppose* $f\in H(B)$ *and* $z\in B$. *Then*

$$\log(1+|f(z)|)\leq\left(\frac{1+|z|}{1-|z|}\right)^{\frac{n+\alpha+1}{p}}\|f\|_{(AN)^p(\nu_\alpha)}.$$

Proof. See [12, Lemma 1], [15, Proposition 3.3] and [17, p.233]. □

By a simple computation with some change of variables in the case $-1<\alpha<\infty$, or with Lemma 2.6 in the case $\alpha=-1$, we can easily prove the following lemma.

Lemma 2.7. *Suppose* $\alpha\in[-1,\infty)$, $p\in[1,\infty)$ *and* φ *is a biholomorphic map of* B *onto* B. *Then the composition operator* C_φ *induced by* φ *is metrically bounded on* $(AN)^p(\nu_\alpha)$:

$$\|C_\varphi f\|_{(AN)^p(\nu_\alpha)}\leq\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{n+1+\alpha}{p}}\|f\|_{(AN)^p(\nu_\alpha)}$$

for $f\in(AN)^p(\nu_\alpha)$.

Let $\varphi: B\to B$ be a univalent holomorphic map. For $z\in B$, We define

$$\Omega_\varphi(z)=\frac{\|\varphi'(z)\|^2}{|J_\varphi(z)|^2}$$

where $\varphi'(z)$ is the derivative of φ at z, $\|\varphi'(z)\|$ denotes its norm as a linear transformation on $\mathbf{C}^n$ and $J_\varphi(z)$ is the complex Jacobian of φ at z. We can easily see the next lemma (see [5], p.171).

Lemma 2.8. *Suppose that a univalent holomorphic map* $\varphi: B\to B$ *satisfies*

$$\sup_{w\in\varphi(B)}\|(\varphi^{-1})'(w)\|<\infty.$$

Then Ω_φ *is bounded in* B.

Lemma 2.9. *Let* $1<p<\infty$. *Suppose that* $\varphi: B\to B$ *is a univalent holomorphic map and* Ω_φ *is bounded in* B. *Then* C_φ *is metrically bounded on* $N^p(B)$. *More precisely, there exists a positive constant* L *depending only on* n *and* φ *such that for all* $f\in H(B)$

$$\|C_\varphi f\|_{N^p(B)}^p \leq L\|f\|_{N^p(B)}^p.$$

Proof. Set $a=\varphi(0)$ and $\psi=\varphi_a\circ\varphi$, where φ_a is the involution described in [14, p.25]. Then ψ is a univalent holomorphic self-map of B and $\psi(0)=0$. First, we show that C_ψ is metrically bounded on $N^p(B)$. Take $f\in H(B)\setminus\{0\}$. Define $\chi(t)=\{\log(1+e^t)\}^p(t\in\mathbf{R})$. Since $\Omega_\psi(z)\leq\Omega_{\varphi_a}(\varphi(z))\Omega_\varphi(z)$ for any $z\in B$ and $(\Omega_{\varphi_a}\circ\varphi)\cdot\Omega_\varphi$ is bounded in B, we have

$$M\equiv\sup_{z\in B}\Omega_\psi(z)<\infty.$$

Note that M is a positive constant depending only on φ. It follows from the chain rule that for any $z\in B$

$$|\nabla(f\circ\psi)(z)|^2\leq|((\nabla f)\circ\psi)(z)|^2\|\psi'(z)\|^2\leq M|((\nabla f)\circ\psi)(z)|^2|J_\psi(z)|^2. \tag{2.4}$$

Since $\psi(0)=0$, Schwarz's lemma gives

$$|\psi(z)|\leq|z| \tag{2.5}$$

for any $z\in B$. Using (2.1), Lemma 2.1(a), (2.4), (2.5), a change of variables, (2.2), (2.3) and Lemma 2.4(b) one after another, we have

$$\begin{aligned}
&\int_{B\setminus\frac{1}{2}B}\chi''(\log|(f\circ\psi)(z)|)\frac{|R(f\circ\psi)(z)|^2}{|(f\circ\psi)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\
&\leq 2^{2(n-1)}\int_{B\setminus\frac{1}{2}B}\chi''(\log|(f\circ\psi)(z)|)\frac{|\nabla(f\circ\psi)(z)|^2}{|(f\circ\psi)(z)|^2}(1-|z|^2)d\nu(z)\\
&\leq 2^{2(n-1)}M\\
&\quad\times\int_{B\setminus\frac{1}{2}B}\chi''(\log|f(\psi(z))|)\frac{|(\nabla f)(\psi(z))|^2}{|f(\psi(z))|^2}(1-|\psi(z)|^2)|J_\psi(z)|^2d\nu(z)\\
&=2^{2(n-1)}M\int_{\psi(B\setminus\frac{1}{2}B)}\chi''(\log|f(w)|)\frac{|(\nabla f)(w)|^2}{|f(w)|^2}(1-|w|^2)d\nu(w)\\
&\leq\frac{2^{2(n-1)}(n+1)M}{2}\int_B\chi''(\log|f(w)|)\frac{|(\tilde{\nabla}f)(w)|^2}{|f(w)|^2}\frac{d\nu(w)}{1-|w|^2}\\
&\leq\frac{2^{2(n-1)}n(n+1)M}{2a_n\Gamma(n)}\|f\|_{N^p(B)}^p.
\end{aligned} \tag{2.6}$$

On the other hand, since $\frac{\log3-\log2}{\log2}\log\frac{1}{t}\leq\log\frac{3}{4t}$ for all $t\in(0,\frac{1}{2}]$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\int_{\frac{1}{2}B}\chi''(\log|(f\circ\psi)(z)|)\frac{|R(f\circ\psi)(z)|^2}{|(f\circ\psi)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\
&\leq\frac{\log2}{\log3-\log2}\\
&\quad\times\int_{\frac{1}{2}B}\chi''(\log|(f\circ\psi)(z)|)\frac{|R(f\circ\psi)(z)|^2}{|(f\circ\psi)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{3}{4|z|}d\nu(z)\\
&\leq\frac{2n\log2}{\log3-\log2}\|(f\circ\psi)_{\frac{3}{4}}\|_{N^p(B)}^p.
\end{aligned} \tag{2.7}$$

By Lemma 2.6, we have

$$\begin{aligned}\|(f\circ\psi)_{\frac{3}{4}}\|^p_{N^p(B)}&=\int_S\{\log(1+|(f\circ\psi)_{\frac{3}{4}}|)\}^p d\sigma\\ &\leq\max_{w\in\psi(\frac{3}{4}S)}\{\log(1+|f(w)|)\}^p\\ &\leq 2^n\max_{w\in\psi(\frac{3}{4}S)}(1-|w|)^{-n}\|f\|^p_{N^p(B)}.\end{aligned}\tag{2.8}$$

(2.7) and (2.8) show that

$$\begin{aligned}&\int_{\frac{1}{2}B}\chi''(\log|(f\circ\psi)(z)|)\frac{|R(f\circ\psi)(z)|^2}{|(f\circ\psi)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\ &\leq\frac{2^{n+1}n\log 2}{\log 3-\log 2}\max_{w\in\psi(\frac{3}{4}S)}(1-|w|)^{-n}\|f\|^p_{N^p(B)}.\end{aligned}\tag{2.9}$$

Since $\psi(0)=0$, it follows from Lemma 2.6 that

$$\{\log(1+|(f\circ\psi)(0)|)\}^p\leq\|f\|^p_{N^p(B)}.\tag{2.10}$$

By (2.6), (2.9), (2.10) and Lemma 2.3, we have

$$\|f\circ\psi\|^p_{N^p(B)}\leq L_1\|f\|^p_{N^p(B)}$$

where

$$L_1=\frac{2^{2n-2}(n+1)M}{4a_n\Gamma(n)}+\frac{2^n\log 2}{\log 3-\log 2}\max_{w\in\psi(\frac{3}{4}S)}(1-|w|)^{-n}+1.$$

Hence C_ψ is metrically bounded on $N^p(B)$. Since φ_a is a biholomorphic map of B onto B, Lemma 2.7 implies that C_{φ_a} is also metrically bounded on $N^p(B)$. It holds that $C_\varphi=C_\psi\circ C_{\varphi_a}$, because $\varphi=\varphi_a\circ\psi$. Thus we conclude that C_φ is metrically bounded on $N^p(B)$:

$$\|C_\varphi f\|^p_{N^p(B)}\leq L\|f\|^p_{N^p(B)}$$

for all $f\in H(B)$ where

$$L=L_1\left(\frac{1+|\varphi_a(0)|}{1-|\varphi_a(0)|}\right)^n=L_1\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^n.$$

Note that L is a positive constant depending only on n and φ. □

For each $\alpha\in(-1,\infty)$ and $\chi\in ST_0^2(\mathbf{R})$, we define the *weighted Bergman-Orlicz space* $A_\chi(\nu_\alpha)$ by

$$A_\chi(\nu_\alpha)=\{f\in H(B):\|f\|_{\chi,\alpha}<\infty\}.$$

Lemma 2.10. *It holds that*

$$(AN)^1(\nu_\alpha)=\bigcup_{\chi\in ST^2(\mathbf{R})}A_\chi(\nu_\alpha).$$

for any $\alpha\in(-1,\infty)$.

Proof. It is easily seen that

$$(AN)^1(\nu_\alpha) \supset \bigcup_{\chi \in ST^2(\mathbf{R})} A_\chi(\nu_\alpha).$$

Take $f \in (AN)^1(\nu_\alpha)$. By the subharmonicity of the function $\log(1+|f|)$ in B, we have

$$\lim_{r \uparrow 1} \int_B \log(1+|f_r|)\, d\nu_\alpha = \sup_{0<r<1} \int_B \log(1+|f_r|)\, d\nu_\alpha \leq \int_B \log(1+|f|)\, d\nu_\alpha < \infty. \tag{2.11}$$

On the other hand, Fatou's lemma gives

$$\begin{aligned}\int_B \log(1+|f|)\, d\nu_\alpha &= \int_B \lim_{r \uparrow 1} \log(1+|f_r|)\, d\nu_\alpha \\ &\leq \liminf_{r \uparrow 1} \int_B \log(1+|f_r|)\, d\nu_\alpha.\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we obtain

$$\lim_{r \uparrow 1} \int_B \log(1+|f_r|)\, d\nu_\alpha = \int_B \log(1+|f|)\, d\nu_\alpha < \infty.$$

It follows from [6, Chap. V, Lemma 1.4] and [6, Chap. V, Theorem 1.3] that the family $\{\log(1+|f_r|)\}_{0<r<1}$ is uniformly integrable with respect to the measure ν_α. The de la Vallée Poussin's theorem ([13, Theorem 3.10]) therefore implies that

$$\int_B \chi(\log(1+|f|))\, d\nu_\alpha = \sup_{0<r<1} \int_B \chi(\log(1+|f_r|))\, d\nu_\alpha < \infty$$

for some $\chi \in ST^2(\mathbf{R})$. Thus $f \in A_\chi(\nu_\alpha)$. This completes the proof. □

Lemma 2.11. *Let $1 \leq p < \infty$ and $-1 < \alpha < \infty$. Suppose that $\varphi: B \to B$ is a univalent holomorphic map and Ω_φ is bounded in B. Then C_φ is metrically bounded on $(AN)^p$ (ν_α). More precisely, there exists a positive constant L_α depending only on n, α and φ such that*

$$\|C_\varphi f\|^p_{(AN)^p(\nu_\alpha)} \leq L_\alpha \|f\|^p_{(AN)^p(\nu_\alpha)}$$

for all $f \in (AN)^p(\nu_\alpha)$.

Proof. First we consider the case $1 < p < \infty$. Define $\chi(t) = \{\log(1+e^t)\}^p (t \in \mathbf{R})$. Take $f \in H(B) \setminus \{0\}$. As in the proof of Lemma 2.9, we set $a = \varphi(0)$ and $\psi = \varphi_a \circ \varphi$. Like (2.4) and (2.5), we have

$$|\nabla(f \circ \psi)(z)|^2 \leq M |((\nabla f) \circ \psi)(z)|^2 |J_\psi(z)|^2, \quad |\psi(z)| \leq |z| \tag{2.13}$$

for all $z \in B$, where $M = \sup_{z \in B} \Omega_\psi(z) < \infty$. By Lemma 2.1(b), there are a positive constant $d_{\alpha,n}$ depending only α and n, and $r_0 \in (\frac{1}{2}, 1)$ such that

$$K_\alpha(|z|) \leq d_{\alpha,n} (1-|z|^2)^{\alpha+2} \tag{2.14}$$

for all $z \in B \setminus r_0 \bar{B}$. By using (2.1)~(2.3), (2.13), (2.14) and Lemma 2.4(a), we have

$$\int_{B \setminus r_0 \bar{B}} \chi''(\log|(f \circ \psi)(z)|) \frac{|R(f \circ \psi)(z)|^2}{|(f \circ \psi)(z)|^2 |z|^2} |z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z)$$

$$
\begin{aligned}
&\leq 2^{2(n-1)} d_{\alpha,n} M \\
&\quad \times \int_{B\setminus r_0\bar{B}} \chi''(\log|f(\psi(z))|) \frac{|(\nabla f)(\psi(z))|^2}{|f(\psi(z))|^2}(1-|\psi(z)|^2)^{\alpha+2}|J_\psi(z)|^2 d\nu(z) \\
&= 2^{2(n-1)} d_{\alpha,n} M \int_{\psi(B\setminus r_0\bar{B})} \chi''(\log|f(w)|) \frac{|(\nabla f)(w)|^2}{|f(w)|^2}(1-|w|^2)^{\alpha+2} d\nu(w) \\
&\leq \frac{2^{2(n-1)} d_{\alpha,n}(n+1)M}{2} \int_B \chi''(\log|f(w)|) \frac{|(\tilde{\nabla} f)(w)|^2}{|f(w)|^2}(1-|w|^2)^{\alpha} d\nu(w) \\
&= 2^{2(n-1)} d_{\alpha,n}(n+1)M \int_B \tilde{\Delta}(\{\log(1+|f|)\}^p)(w)(1-|w|^2)^{\alpha} d\nu(w) \\
&\leq \frac{2^{2(n-1)} d_{\alpha,n}(n+1)M}{A_{n,\alpha}} \|f\|^p_{(AN)^p(\nu_\alpha)},
\end{aligned}
\tag{2.15}
$$

where $A_{n,\alpha}=\{a_n\Gamma(n+\alpha+1)\}/\{2^{\alpha+\alpha^+}(n+\alpha+1)\Gamma(\alpha+2)\}$.

On the other hand, by Lemma 2.2 we can easily see that for all $z\in r_0\bar{B}\setminus\{0\}$

$$
\begin{aligned}
K_\alpha(|z|) &\leq \left[(\log\frac{1}{r_0})\left\{K_\alpha(\frac{r_0}{r_1})\right\}^{-1}+1\right]K_\alpha(\frac{|z|}{r_1}) \\
&= C_\alpha K_\alpha(\frac{|z|}{r_1}),
\end{aligned}
\tag{2.16}
$$

where $r_1=\dfrac{1+r_0}{2}$ and $C_\alpha=(\log\dfrac{1}{r_0})\{K_\alpha(\dfrac{r_0}{r_1})\}^{-1}+1$. It follows from (2.16) and Lemma 2.3 that

$$
\begin{aligned}
&\int_{r_0\bar{B}} \chi''(\log|(f\circ\psi)(z)|) \frac{|R(f\circ\psi)(z)|^2}{|(f\circ\psi)(z)|^2|z|^2}|z|^{-2(n-1)}K_\alpha(|z|)d\nu(z) \\
&\leq C_\alpha \int_{r_1 B} \chi''(\log|(f\circ\psi)(z)|) \frac{|R(f\circ\psi)(z)|^2}{|(f\circ\psi)(z)|^2|z|^2}|z|^{-2(n-1)}K_\alpha(\frac{|z|}{r_1})d\nu(z) \\
&\leq 2nC_\alpha \|(C_\psi f)_{r_1}\|^p_{(AN)^p(\nu_\alpha)}.
\end{aligned}
\tag{2.17}
$$

Moreover, by Lemma 2.6, we have

$$
\|(C_\psi f)_{r_1}\|^p_{(AN)^p(\nu_\alpha)} \leq \max_{w\in\psi(r_1\bar{B})}\left\{\frac{1+|w|}{1-|w|}\right\}^{n+\alpha+1}\|f\|^p_{(AN)^p(\nu_\alpha)},
\tag{2.18}
$$

$$
\{\log(1+|f(0)|)\}^p \leq \|f\|^p_{(AN)^p(\nu_\alpha)}.
\tag{2.19}
$$

It follows from (2.15), (2.17)~(2.19) and Lemma 2.3 that C_ψ is metrically bounded on $(AN)^p(\nu_\alpha)$:

$$
\|C_\psi f\|^p_{(AN)^p(\nu_\alpha)} \leq D_\alpha \|f\|^p_{(AN)^p(\nu_\alpha)}
$$

for any $p\in(1, \infty)$ and any $f\in H(B)$, where

$$
D_\alpha = \frac{2^{2(n-1)}d_{\alpha,n}(n+1)M}{2nA_{n,\alpha}} + C_\alpha \max_{w\in\psi(r_1\bar{B})}\left\{\frac{1+|w|}{1-|w|}\right\}^{n+\alpha+1}+1.
$$

Since $C_\varphi=C_\psi\circ C_{\varphi_a}$, it follows from Lemma 2.7 that

$$
\|C_\varphi f\|^p_{(AN)^p(\nu_\alpha)} \leq L_\alpha \|f\|^p_{(AN)^p(\nu_\alpha)}
\tag{2.20}
$$

for any $p\in(1, \infty)$ and any $f\in H(B)$, where

$$L_\alpha = D_\alpha \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right)^n .$$

Note that L_α is a positive constant depending only on n, α and φ.

Now we consider the case $p=1$. Take $f \in (AN)^1(\nu_\alpha)$. For any $r \in (0, 1)$ and any $p \in (1, \infty)$, we have by (2.20)

$$\int_B \{\log(1+|f(r\varphi(z))|)\}^p d\nu_\alpha(z) \leq L_\alpha \int_B \{\log(1+|f(rz)|)\}^p d\nu_\alpha(z).$$

By taking the limit as $p \downarrow 1$ above, it holds that

$$\int_B \log(1+|f(r\varphi(z))|) d\nu_\alpha(z) \leq L_\alpha \int_B \log(1+|f(rz)|) d\nu_\alpha(z). \tag{2.21}$$

By Lemma 2.10, the family $\{\log(1+|f_r|)\}_{0<r<1}$ is uniformly integrable with respect to the measure ν_α. Hence

$$\lim_{r \uparrow 1} \int_B \log(1+|f_r|) d\nu_\alpha = \int_B \log(1+|f|) d\nu_\alpha. \tag{2.22}$$

On the other hand, Fatou's lemma gives

$$\int_B \log(1+|f(\varphi(z))|) d\nu_\alpha(z) \leq \liminf_{r \uparrow 1} \int_B \log(1+|f(r\varphi(z))|) d\nu_\alpha(z). \tag{2.23}$$

(2.21)$\sim$(2.23) show that

$$\|C_\varphi f\|_{(AN)^1(\nu_\alpha)} \leq L_\alpha \|f\|_{(AN)^1(\nu_\alpha)}.$$

This completes the proof. □

The next lemma is a characterization of the compactness of uC_φ on $H^p(B)$ and on $A^p(\nu_\alpha)$ in terms of sequential convergence. Its proof, which we omit, is based on the fact that bounded subsets of $H^p(B)$ (respectively $A^p(\nu_\alpha)$) are normal families. (cf. [5, Lemma 3.11])

Lemma 2.12. *Let $0<p<\infty$ and $-1 \leq \alpha < \infty$. Suppose that $u \in H(B)$ and a holomorphic self-map φ of B satisfy $(uC_\varphi)(A^p(\nu_\alpha)) \subset A^p(\nu_\alpha)$. Then uC_φ is compact on $A^p(\nu_\alpha)$ if and only if for every bounded sequence $\{f_j\}$ in $A^p(\nu_\alpha)$ which converges to 0 uniformly on compact subsets of B, we have $\lim_{j\to\infty} \|uC_\varphi f_j\|_{A^p(\nu_\alpha)} = 0$.*

For the metrically compactness of uC_φ on $N^p(B)$ (respectively on $(AN)^p(\nu_\alpha)$), an analogous result of Lemma 2.12 holds:

Lemma 2.13. *Let $1 \leq p < \infty$* and *$-1 \leq \alpha < \infty$. Suppose that $u \in H(B)$ and a holomorphic self-map φ of B satisfy $(uC_\varphi)((AN)^p(\nu_\alpha)) \subset (AN)^p(\nu_\alpha)$. Then uC_φ is metrically compact on $(AN)^p(\nu_\alpha)$ if and only if for every bounded sequence $\{f_j\}$* in *$(AN)^p(\nu_\alpha)$ which converges to 0 uniformly on compact subsets of B, we have $\lim_{j\to\infty} \|uC_\varphi f_j\|_{(AN)^p(\nu_\alpha)} = 0$.*

3 uC_φ on $H^p(B)$ and $A^p(\nu_\alpha)$

Theorem 3.1. *Let $0<p<\infty$, $u \in H(B) \setminus \{0\}$ and let $\varphi : B \to B$ be a univalent holomorphic map such that Ω_φ is bounded in B.*

(a) Suppose u and φ satisfy the following conditions :

$$\limsup_{|z|\uparrow 1}|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)<\infty, \tag{3.1}$$

$$\limsup_{|z|\uparrow 1}\frac{|u(z)|^p(1-|z|^2)}{1-|\varphi(z)|^2}<\infty. \tag{3.2}$$

Then uC_φ is bounded on $H^p(B)$.

(b) Suppose u and φ satisfy the following conditions :

$$\lim_{|z|\uparrow 1}|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)=0, \tag{3.3}$$

$$\lim_{|z|\uparrow 1}\frac{|u(z)|^p(1-|z|^2)}{1-|\varphi(z)|^2}=0. \tag{3.4}$$

Then uC_φ is compact on $H^p(B)$.

Proof. Put $Tf=uC_\varphi f$ for $f\in H(B)$. If we show that $Tf\in H^p(B)$ whenever $f\in H^p(B)$, the closed graph theorem will give that T is bounded on $H^p(B)$. Since φ is a univalent holomorphic self-map of B and Ω_φ is bounded in B, by Theorem 1.1(a), C_φ is bounded on $H^p(B)$ (see [5, Theorem 3.41]). Thus $\|C_\varphi f\|_{H^p(B)}\leq\|C_\varphi\|\|f\|_{H^p(B)}$ for all $f\in H^p(B)$. And so,

$$\int_B|C_\varphi f|^p d\nu\leq\|C_\varphi f\|^p_{H^p(B)}\leq\|C_\varphi\|^p\|f\|^p_{H^p(B)}. \tag{3.5}$$

It follows from the chain rule that for any $z\in B$

$$\begin{aligned}|\nabla(Tf)(z)|^2&\leq 2|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2\\&\quad+2|u(z)|^2|(\nabla f)(\varphi(z))|^2\|\varphi'(z)\|^2\\&\leq 2|(\nabla u)(z)|^2|f(\varphi(z))|^2\\&\quad+2M|u(z)|^2|(\nabla f)(\varphi(z))|^2|J_\varphi(z)|^2,\end{aligned} \tag{3.6}$$

where $M\equiv\sup_{z\in B}\Omega_\varphi(z)<\infty$. By (3.1) and (3.2), there are two positive constants ϵ_1, ϵ_2 and $r_0\in(\frac{1}{2}, 1)$ such that

$$|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)<\epsilon_1, \tag{3.7}$$

$$\frac{|u(z)|^p(1-|z|^2)}{1-|\varphi(z)|^2}<\epsilon_2, \tag{3.8}$$

for any $z\in B\backslash r_0\bar{B}$. Take $f\in H^p(B)\backslash\{0\}$. By (2.1)~(2.2), (3.5)~(3.8), Lemma 2.1(a) and a change of variables, we have

$$\begin{aligned}&\int_{B\backslash r_0\bar{B}}\frac{|R(Tf)(z)|^2}{|z|^2}|(Tf)(z)|^{p-2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\&\leq 2^{2n-1}\Big[\int_{B\backslash r_0\bar{B}}|(\nabla u)(z)|^2|u(z)|^{p-2}|f(\varphi(z))|^p(1-|z|^2)d\nu(z)\\&\quad+M\int_{B\backslash r_0\bar{B}}|u(z)|^p|(\nabla f)(\varphi(z))|^2|f(\varphi(z))|^{p-2}(1-|z|^2)|J_\varphi(z)|^2d\nu(z)\Big]\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{2n-1}\Big[\epsilon_1\int_{B\setminus r_0\bar{B}}|(C_\varphi f)(z)|^p d\nu(z)\\
&\quad + M\epsilon_2\int_{B\setminus r_0\bar{B}}|(\nabla f)(\varphi(z))|^2|f(\varphi(z))|^{p-2}(1-|\varphi(z)|^2)|J_\varphi(z)|^2 d\nu(z)\Big]\\
&\leq 2^{2n-1}\Big[\epsilon_1\int_B|(C_\varphi f)(z)|^p d\nu(z)\\
&\quad + M\epsilon_2\int_B|(\nabla f)(w)|^2|f(w)|^{p-2}(1-|w|^2)d\nu(w)\Big]\\
&\leq 2^{2n-1}\epsilon_1\|C_\varphi\|^p\|f\|^p_{H^p(B)}\\
&\quad + 2^{2n-2}(n+1)M\epsilon_2\int_B|(\tilde{\nabla}f)(w)|^2|f(w)|^{p-2}\frac{d\nu(w)}{1-|w|^2}.
\end{aligned}
\tag{3.9}
$$

By (2.3) and Lemma 2.5, we have

$$
\int_B|(\tilde{\nabla}f)(w)|^2|f(w)|^{p-2}\frac{d\nu(w)}{1-|w|^2}\leq\frac{2}{a_{n,-1}p^2}\|f\|^p_{H^p(B)}. \tag{3.10}
$$

It follows from (3.9) and (3.10) that

$$
\begin{aligned}
&\int_{B\setminus r_0\bar{B}}\frac{|R(Tf)(z)|^2}{|z|^2}|(Tf)(z)|^{p-2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\
&\qquad\leq 2^{2n-1}\Big[\epsilon_1\|C_\varphi\|^p+\frac{(n+1)M\epsilon_2}{a_{n,-1}p^2}\Big]\|f\|^p_{H^p(B)}<\infty.
\end{aligned}
\tag{3.11}
$$

On the other hand, we set $r_1=\dfrac{1+r_0}{2}$ and $\delta=1-\dfrac{\log r_1}{\log r_0}$. We can easily see that $0<\delta<1$ and for all $t\in(0, r_0]$

$$
\delta\log\frac{1}{t}\leq\log\frac{r_1}{t}. \tag{3.12}
$$

By using (3.12) and Lemma 2.3, we obtain

$$
\begin{aligned}
&\int_{r_0\bar{B}}\frac{|R(Tf)(z)|^2}{|z|^2}|(Tf)(z)|^{p-2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\
&\quad\leq\frac{1}{\delta}\int_{r_0\bar{B}}\frac{|R(Tf)(z)|^2}{|z|^2}|(Tf)(z)|^{p-2}|z|^{-2(n-1)}\log\frac{r_1}{|z|}d\nu(z)\\
&\quad\leq\frac{2n}{\delta p^2}\|(uC_\varphi f)_{r_1}\|^p_{H^p(B)}\\
&\quad\leq\frac{2n}{\delta p^2}\max_{z\in r_1S}|u(z)|^p\max_{z\in\varphi(r_1S)}|f(z)|^p<\infty.
\end{aligned}
\tag{3.13}
$$

(3.11), (3.13) and Lemma 2.3 show $\|Tf\|^p_{H^p(B)}<\infty$. Hence $uC_\varphi f\in H^p(B)$. This completes the proof of (a).

To prove (b), suppose that $\{f_j\}$ is a sequence in $H^p(B)$ which converges to zero uniformly on compact subsets of B and $\|f_j\|^p_{H^p(B)}\leq L<\infty$ for all $j\in\mathbf{N}$. Let $\epsilon>0$ be given. By (3.3) and (3.4), we can choose $r_0\in(\frac{1}{2}, 1)$ such that

$$
|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)<\epsilon, \tag{3.14}
$$

$$
\frac{|u(z)|^p(1-|z|^2)}{1-|\varphi(z)|^2}<\epsilon \tag{3.15}
$$

for any $z\in B\backslash r_0\bar{B}$. By (a), it holds that $T(H^p(B))\subset H^p(B)$. If we show that $\lim_{j\to\infty}\|Tf_j\|_{H^p(B)}=0$, then Lemma 2.12 will give that $T\equiv uC_\varphi$ is compact on $H^p(B)$. In the same way as in the proof of (a), by using (3.14), (3.15) and Lemma 2.5, we have

$$\int_{B\backslash r_0\bar{B}}\frac{|R(Tf_j)(z)|^2}{|z|^2}|(Tf_j)(z)|^{p-2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)$$
$$\leq 2^{2n-1}L\left[\|C_\varphi\|^p+\frac{(n+1)M}{a_{n,-1}p^2}\right]\epsilon \tag{3.16}$$

for all $j\in\mathbf{N}$.

On the other hand, as in (3.13), we obtain for all $j\in\mathbf{N}$

$$\int_{r_0\bar{B}}\frac{|R(Tf_j)(z)|^2}{|z|^2}|(Tf_j)(z)|^{p-2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)$$
$$\leq\frac{2n}{\delta p^2}\max_{z\in r_1S}|u(z)|^p\max_{z\in\varphi(r_1S)}|f_j(z)|^p. \tag{3.17}$$

where $r_1=\frac{1+r_0}{2}$ and $\delta=1-\frac{\log r_1}{\log r_0}$. Since $\{f_j\}$ converges to zero uniformly on compact subsets of B,

$$\lim_{j\to\infty}\left[\max_{z\in\varphi(r_1S)}|f_j(z)|^p\right]=0 \tag{3.18}$$

(3.17) and (3.18) show that

$$\lim_{j\to\infty}\int_{r_0\bar{B}}\frac{|R(Tf_j)(z)|^2}{|z|^2}|(Tf_j)(z)|^{p-2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)=0. \tag{3.19}$$

(3.16), (3.19) and Lemma 2.3 imply that $\lim_{j\to\infty}\|Tf_j\|_{H^p(B)}=0$. This completes the proof of (b). □

Theorem 3.2. *Let $0<p<\infty$ and $-1<\alpha<\infty$. Let $u\in H(B)$ and $\varphi: B\to B$ be a univalent holomorphic map such that Ω_φ is bounded in B.*

(a) Suppose u and φ satisfy the following conditions :

$$\limsup_{|z|\uparrow 1}|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)^2<\infty, \tag{3.20}$$

$$\limsup_{|z|\uparrow 1}|u(z)|^p\left\{\frac{1-|z|^2}{1-|\varphi(z)|^2}\right\}^{\alpha+2}<\infty. \tag{3.21}$$

Then uC_φ is bounded on $A^p(\nu_\alpha)$.

(b) Suppose u and φ satisfy the following conditions :

$$\lim_{|z|\uparrow 1}|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)^2=0, \tag{3.22}$$

$$\lim_{|z|\uparrow 1}|u(z)|^p\left\{\frac{1-|z|^2}{1-|\varphi(z)|^2}\right\}^{\alpha+2}=0. \tag{3.23}$$

Then uC_φ is compact on $A^p(\nu_\alpha)$.

Proof. Take $f\in A^p(\nu_\alpha)\backslash\{0\}$. As in the proof of Theorem 3.1,

$$|\nabla(uC_\varphi f)(z)|^2 \leq 2|(\nabla u)(z)|^2|f(\varphi(z))|^2 + 2M|u(z)|^2|(\nabla f)(\varphi(z))|^2|J_\varphi(z)|^2 \tag{3.24}$$

for any $z\in B$, where $M=\sup_{z\in B}\Omega_\varphi(z)<\infty$. By Theorem 1.1(a), C_φ is bounded on $A^p(\nu_\alpha)$. By (3.20) and (3.21), there exist positive constants ϵ_1, ϵ_2 and $r_0\in(\frac{1}{2}, 1)$ such that

$$|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)^2<\epsilon_1, \tag{3.25}$$

$$|u(z)|^p\left\{\frac{1-|z|^2}{1-|\varphi(z)|^2}\right\}^{\alpha+2}<\epsilon_2 \tag{3.26}$$

for any $z\in B\backslash r_0\bar{B}$. Furthermore, we have by Lemma 2.1(b)

$$K_\alpha(|z|)\leq d_{n,\alpha}(1-|z|^2)^{\alpha+2} \tag{3.27}$$

for any $z\in B\backslash r_0\bar{B}$, where $d_{n,\alpha}$ is a positive constant depending only on n and α. By (3.24)~(3.27), Lemma 2.5 and the same argument as in the proof of Theorem 3.1(a), we have

$$\begin{aligned}\int_{B\backslash r_0\bar{B}} &\frac{|R(uC_\varphi f)(z)|^2}{|z|^2}|(uC_\varphi f)(z)|^{p-2}|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z)\\ &\leq\left[\frac{2^{2n-1}d_{n,\alpha}\epsilon_1\|C_\varphi\|^p}{c_\alpha}\right.\\ &\qquad\left.+\frac{2^{2n-1}(n+1)d_{n,\alpha}\epsilon_2 M}{a_{n,\alpha}p^2}\right]\|f\|^p_{A^p(\nu_\alpha)}<\infty.\end{aligned} \tag{3.28}$$

On the other hand, by (2.16) and Lemma 2.3, we obtain

$$\begin{aligned}\int_{r_0\bar{B}} &\frac{|R(uC_\varphi f)(z)|^2}{|z|^2}|(uC_\varphi f)(z)|^{p-2}|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z)\\ &\leq C_\alpha\int_{r_0\bar{B}}\frac{|R(uC_\varphi f)(z)|^2}{|z|^2}|(uC_\varphi f)(z)|^{p-2}|z|^{-2(n-1)}K_\alpha\left(\frac{|z|}{r_1}\right)d\nu(z)\\ &\leq\frac{2nC_\alpha}{p^2}\|(uC_\varphi f)_{r_1}\|^p_{A^p(\nu_\alpha)}<\infty,\end{aligned} \tag{3.29}$$

where $r_1=\frac{1+r_0}{2}$ and $C_\alpha=(\log\frac{1}{r_0})\{K_\alpha(\frac{r_0}{r_1})\}^{-1}+1$. (3.28), (3.29) and Lemma 2.3 give that $\|uC_\varphi f\|_{A^p(\nu_\alpha)}<\infty$, that is, $uC_\varphi f\in A^p(\nu_\alpha)$. This completes the proof of (a).

In order to prove (b), suppose that $\{f_j\}$ is a bounded sequence in $A^p(\nu_\alpha)$ which converges to zero uniformly on compact subsets of B. As in the proof of Theorem 3.1(b), we can show that $\lim_{j\to\infty}\|uC_\varphi f_j\|_{A^p(\nu_\alpha)}=0$. It follows from Lemma 2.12 that uC_φ is compact on $A^p(\nu_\alpha)$. The proof is now complete. □

4 uC_φ on $N^p(B)$ and $(AN)^p(\nu_\alpha)$

Theorem 4.1. *Let $1<p<\infty$, $u\in H(B)\backslash\{0\}$ and let $\varphi: B\to B$ be a univalent holomorphic map such that Ω_φ is bounded in B.*

(a) Suppose u and φ satisfy the following conditions :

(i) When $1<p\le 2$,

$$\limsup_{|z|\uparrow 1}[\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)]<\infty, \tag{4.1}$$

$$\limsup_{|z|\uparrow 1}\left[\frac{\max\{|u(z)|^{p-3},|u(z)|^2\}(1-|z|^2)}{1-|\varphi(z)|^2}\right]<\infty. \tag{4.2}$$

(ii) When $2<p<\infty$,

$$\limsup_{|z|\uparrow 1}[\max\{|u(z)|^{-2},|u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)]<\infty, \tag{4.3}$$

$$\limsup_{|z|\uparrow 1}\left[\frac{\max\{|u(z)|^{-1},|u(z)|^{p}\}(1-|z|^2)}{1-|\varphi(z)|^2}\right]<\infty. \tag{4.4}$$

Then uC_φ *is metrically bounded on* $N^p(B)$.

(b) Suppose u *and* φ *satisfy the following conditions :*

(i) When $1<p\le 2$,

$$\lim_{|z|\uparrow 1}[\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)]=0, \tag{4.5}$$

$$\lim_{|z|\uparrow 1}\left[\frac{\max\{|u(z)|^{p-3},|u(z)|^2\}(1-|z|^2)}{1-|\varphi(z)|^2}\right]=0. \tag{4.6}$$

(ii) When $2<p<\infty$,

$$\lim_{|z|\uparrow 1}[\max\{|u(z)|^{-2},|u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)]=0, \tag{4.7}$$

$$\lim_{|z|\uparrow 1}\left[\frac{\max\{|u(z)|^{-1},|u(z)|^{p}\}(1-|z|^2)}{1-|\varphi(z)|^2}\right]=0. \tag{4.8}$$

Then uC_φ *is metrically compact on* $N^p(B)$.

Proof. Take $f\in N^p(B)\setminus\{0\}$. Since φ is a univalent holomorphic self-map of B and Ω_φ is bounded in B, by Lemma 2.9, C_φ is metrically bounded on $N^p(B)$, that is, $\|C_\varphi f\|^p_{N^p(B)}\le L\|f\|^p_{N^p(B)}$ where L is a positive constant depending only on n and φ. And so,

$$\int_B\{\log(1+|C_\varphi f|)\}^p d\nu\le\|C_\varphi f\|^p_{N^p(B)}\le L\|f\|^p_{N^p(B)}. \tag{4.9}$$

By (4.1)~(4.4), there are positive constants ϵ_1, ϵ_2 and $r_0\in(\frac{1}{2}, 1)$ such that when $1<p\le 2$,

$$\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)<\epsilon_1, \tag{4.10}$$

$$\frac{\max\{|u(z)|^{p-3},|u(z)|^2\}(1-|z|^2)}{1-|\varphi(z)|^2}<\epsilon_2, \tag{4.11}$$

when $2<p<\infty$,

$$\max\{|u(z)|^{-2},|u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)<\epsilon_1, \tag{4.12}$$

$$\frac{\max\{|u(z)|^{-1},|u(z)|^{p}\}(1-|z|^2)}{1-|\varphi(z)|^2}<\epsilon_2 \tag{4.13}$$

for any $z\in B\backslash r_0\bar{B}$. Define

$$E_1=\{z\in B : |u(z)|\leq 1\},\quad E_1'=E_1\cap(B\backslash r_0\bar{B}),$$
$$E_2=\{z\in B : |u(z)|>1\},\quad E_2'=E_2\cap(B\backslash r_0\bar{B}).$$

It is obvious that the following inequalities hold:

$$(p-1)|uC_\varphi f|+\log(1+|uC_\varphi f|)\leq(p-1)|C_\varphi f|+\log(1+|C_\varphi f|), \tag{4.14}$$

$$\frac{1}{(1+|uC_\varphi f|)^2}\leq\frac{|u|^{-2}}{(1+|C_\varphi f|)^2} \tag{4.15}$$

in E_1, and

$$(p-1)+\frac{\log(1+|uC_\varphi f|)}{|uC_\varphi f|}\leq(p-1)+\frac{\log(1+|C_\varphi f|)}{|C_\varphi f|} \tag{4.16}$$

in E_2. We can also easily see that if $1<p\leq 2$,

$$\{\log(1+|uC_\varphi f|)\}^{p-2}\leq|u|^{p-2}\{\log(1+|C_\varphi f|)\}^{p-2}\quad\text{in } E_1, \tag{4.17}$$

$$\frac{\{\log(1+|uC_\varphi f|)\}^{p-2}}{(1+|uC_\varphi f|)^2}\leq\frac{\{\log(1+|C_\varphi f|)\}^{p-2}}{(1+|C_\varphi f|)^2}\quad\text{in } E_2. \tag{4.18}$$

When $2<p<\infty$,

$$\{\log(1+|uC_\varphi f|)\}^{p-2}\leq\{\log(1+|C_\varphi f|)\}^{p-2}\quad\text{in } E_1, \tag{4.19}$$

$$\{\log(1+|uC_\varphi f|)\}^{p-2}\leq|u|^{p-2}\{\log(1+|C_\varphi f|)\}^{p-2}\quad\text{in } E_2. \tag{4.20}$$

Put $\chi(t)=\{\log(1+e^t)\}^p$ $(t\in\mathbf{R})$. By a simple computation, we have

$$\begin{aligned}\chi''(\log|uC_\varphi f|)&=p\frac{\{\log(1+|uC_\varphi f|)\}^{p-2}}{(1+|uC_\varphi f|)^2}\\&\quad\times\{(p-1)|uC_\varphi f|+\log(1+|uC_\varphi f|)\}|uC_\varphi f|\end{aligned} \tag{4.21}$$

in B. By (2.1), Lemma 2.1(a) and (3.6), we have

$$\begin{aligned}&\int_{B\backslash r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|R(uC_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\&\leq 2^{2n-1}\Big[\int_{B\backslash r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2}(1-|z|^2)d\nu(z)\\&\quad+M\int_{B\backslash r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|u(z)|^2|(\nabla f)(\varphi(z))|^2}{|(uC_\varphi f)(z)|^2}(1-|z|^2)|J_\varphi(z)|^2d\nu(z)\Big]\end{aligned} \tag{4.22}$$

where $M\equiv\sup_{z\in B}\Omega_\varphi(z)<\infty$. Let V_1 and V_2 be defined by

$$V_1=\int_{B\backslash r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2}(1-|z|^2)d\nu(z), \tag{4.23}$$

$$V_2=\int_{B\backslash r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|u(z)|^2|(\nabla f)(\varphi(z))|^2}{|(uC_\varphi f)(z)|^2}(1-|z|^2)|J_\varphi(z)|^2d\nu(z). \tag{4.24}$$

By (4.21) and (4.23), we have

$$V_1=p\int_{B\setminus r_0\bar{B}}\Big[\frac{\{\log(1+|(uC_\varphi f)(z)|)\}^{p-2}}{(1+|(uC_\varphi f)(z)|)^2}\Big\{(p-1)+\frac{\log(1+|(uC_\varphi f)(z)|)}{|(uC_\varphi f)(z)|}\Big\}$$
$$\times|(uC_\varphi f)(z)|^2\frac{|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2}(1-|z|^2)\Big]d\nu(z)$$
$$\leq p^2\int_{B\setminus r_0\bar{B}}\frac{\{\log(1+|(uC_\varphi f)(z)|)\}^{p-2}}{(1+|(uC_\varphi f)(z)|)^2}|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2(1-|z|^2)d\nu(z). \tag{4.25}$$

Let $V_{1,k}$ $(k=1, 2)$ be defined by

$$V_{1,k}=\int_{E_k'}\frac{\{\log(1+|(uC_\varphi f)(z)|)\}^{p-2}}{(1+|(uC_\varphi f)(z)|)^2}|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2(1-|z|^2)d\nu(z). \tag{4.26}$$

Note that the following elementary inequality holds:

$$\frac{t}{1+t}\leq\log(1+\mathrm{t})\quad(t\geq 0). \tag{4.27}$$

Let $1<p\leq 2$. By (4.15), (4.17), (4.26) and (4.27), we have

$$V_{1,1}\leq\int_{E_1'}\frac{|u(z)|^{p-2}\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{|u(z)|^2(1+|(C_\varphi f)(z)|)^2}|\nabla u(z)|^2|(C_\varphi f)(z)|^2(1-|z|^2)d\nu(z)$$
$$=\int_{E_1'}|u(z)|^{p-4}\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}$$
$$\times\Big\{\frac{|(C_\varphi f)(z)|}{1+|(C_\varphi f)(z)|}\Big\}^2|(\nabla u)(z)|^2(1-|z|^2)d\nu(z)$$
$$\leq\int_{E_1'}|u(z)|^{p-4}\{\log(1+|(C_\varphi f)(z)|)\}^{p}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z). \tag{4.28}$$

And we have by (4.18), (4.26) and (4.27),

$$V_{1,2}\leq\int_{E_2'}\frac{\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{(1+|(C_\varphi f)(z)|)^2}|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2(1-|z|^2)d\nu(z)$$
$$\leq\int_{E_2'}\{\log(1+|(C_\varphi f)(z)|)\}^{p}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z). \tag{4.29}$$

It follows from (4.25), (4.26), (4.28) and (4.29) that

$$V_1\leq p^2\int_{E_1'}|u(z)|^{p-4}\{\log(1+|(C_\varphi f)(z)|)\}^{p}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z)$$
$$+p^2\int_{E_2'}\{\log(1+|(C_\varphi f)(z)|)\}^{p}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z)$$
$$=p^2\int_{B\setminus r_0\bar{B}}\{\log(1+|(C_\varphi f)(z)|)\}^{p}\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z). \tag{4.30}$$

By (4.9), (4.10) and (4.30), we obtain

$$V_1\leq\epsilon_1 p^2\int_{B\setminus r_0\bar{B}}\{\log(1+|C_\varphi f|)\}^p d\nu\leq\epsilon_1 p^2 L\|f\|^p_{N^p(B)}. \tag{4.31}$$

When $2<p<\infty$, by (4.15), (4.19), (4.26) and (4.27), we have

$$V_{1,1}\leq\int_{E_1'}\frac{\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{|u(z)|^2(1+|(C_\varphi f)(z)|)^2}|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2(1-|z|^2)d\nu(z)$$
$$\leq\int_{E_1'}\{\log(1+|(C_\varphi f)(z)|)\}^{p}|u(z)|^{-2}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z). \tag{4.32}$$

And we have by (4.20), (4.26) and (4.27),

$$V_{1,2}\leq\int_{E_2'}\frac{|u(z)|^{p-2}\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{(1+|(C_\varphi f)(z)|)^2}|(\nabla u)(z)|^2|(C_\varphi f)(z)|^2(1-|z|^2)d\nu(z)$$
$$\leq\int_{E_2'}\{\log(1+|(C_\varphi f)(z)|)\}^p|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z). \tag{4.33}$$

By (4.25), (4.26), (4.32) and (4.33),

$$\begin{aligned}V_1&\leq p^2\int_{E_1'}\{\log(1+|(C_\varphi f)(z)|)\}^p|u(z)|^{-2}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z)\\&\quad+p^2\int_{E_2'}\{\log(1+|(C_\varphi f)(z)|)\}^p|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2)d\nu(z)\\&=p^2\int_{B\setminus r_0\bar{B}}\{\log(1+|(C_\varphi f)(z)|)\}^p\max\{|u(z)|^{-2},|u(z)|^{p-2}\}\\&\qquad\times|(\nabla u)(z)|^2(1-|z|^2)d\nu(z).\end{aligned} \tag{4.34}$$

(4.9), (4.12) and (4.34) give

$$V_1\leq\epsilon_1 p^2L\|f\|^p_{N^p(B)}. \tag{4.35}$$

On the other hand, by (4.21) and (4.24), we have

$$\begin{aligned}V_2=p\int_{B\setminus r_0\bar{B}}\Big[&\frac{\{\log(1+|(uC_\varphi f)(z)|)\}^{p-2}}{(1+|(uC_\varphi f)(z)|)^2}\Big\{(p-1)+\frac{\log(1+|(uC_\varphi f)(z)|)}{|(uC_\varphi f)(z)|}\Big\}\\&\times|u(z)|^2|(\nabla f)(\varphi(z))|^2(1-|z|^2)|J_\varphi(z)|^2\Big]d\nu(z).\end{aligned} \tag{4.36}$$

Let $V_{2,k}$ ($k=1, 2$) be defined by

$$\begin{aligned}V_{2,k}=\int_{E_k'}\Big[&\frac{\{\log(1+|(uC_\varphi f)(z)|)\}^{p-2}}{(1+|(uC_\varphi f)(z)|)^2}\Big\{(p-1)+\frac{\log(1+|(uC_\varphi f)(z)|)}{|(uC_\varphi f)(z)|}\Big\}\\&\times|u(z)|^2|(\nabla f)(\varphi(z))|^2(1-|z|^2)|J_\varphi(z)|^2\Big]d\nu(z).\end{aligned} \tag{4.37}$$

Let $1<p\leq2$. By (4.14), (4.15), (4.17), (4.21) and (4.37), we have

$$\begin{aligned}V_{2,1}&\leq\int_{E_1'}\Big[\frac{|u(z)|^{p-2}\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{|u(z)|^2(1+|(C_\varphi f)(z)|)^2}\\&\qquad\times\{(p-1)|(C_\varphi f)(z)|+\log(1+|(C_\varphi f)(z)|)\}\\&\qquad\times\frac{|u(z)|}{|(C_\varphi f)(z)|}|(\nabla f)(\varphi(z))|^2(1-|z|^2)|J_\varphi(z)|^2\Big]d\nu(z)\\&=\int_{E_1'}\Big[\frac{\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{(1+|(C_\varphi f)(z)|)^2}\Big\{(p-1)+\frac{\log(1+|(C_\varphi f)(z)|)}{|(C_\varphi f)(z)|}\Big\}|(C_\varphi f)(z)|^2\\&\qquad\times\frac{|u(z)|^{p-3}}{|(C_\varphi f)(z)|^2}|(\nabla f)(\varphi(z))|^2(1-|z|^2)|J_\varphi(z)|^2\Big]d\nu(z)\\&=\frac{1}{p}\int_{E_1'}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^{p-3}(1-|z|^2)|J_\varphi(z)|^2d\nu(z).\end{aligned} \tag{4.38}$$

And we have by (4.16), (4.18), (4.21) and (4.37),

$$V_{2,2}\le\int_{E_2'}\Big[\frac{\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{(1+|(C_\varphi f)(z)|)^2}\Big\{(p-1)+\frac{\log(1+|(C_\varphi f)(z)|)}{|(C_\varphi f)(z)|}\Big\}$$
$$\times|u(z)|^2|(\nabla f)(\varphi(z))|^2(1-|z|^2)|J_\varphi(z)|^2\Big]d\nu(z)$$
$$=\frac{1}{p}\int_{E_2'}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^2(1-|z|^2)|J_\varphi(z)|^2d\nu(z). \tag{4.39}$$

(4.11) and (4.36)~(4.39) give

$$V_2\le\int_{E_1'}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^{p-3}(1-|z|^2)|J_\varphi(z)|^2d\nu(z)$$
$$+\int_{E_2'}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^2(1-|z|^2)|J_\varphi(z)|^2d\nu(z)$$
$$=\int_{B\setminus r_0\bar{B}}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}$$
$$\times\max\{|u(z)|^{p-3},|u(z)|^2\}(1-|z|^2)|J_\varphi(z)|^2d\nu(z)$$
$$\le\epsilon_2\int_{B\setminus r_0\bar{B}}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}(1-|\varphi(z)|^2)|J_\varphi(z)|^2d\nu(z). \tag{4.40}$$

From (2.2), (2.3), (4.40), Lemma 2.4(b) and a change of variables, it follows that

$$V_2\le\epsilon_2\int_{\varphi(B\setminus r_0\bar{B})}\chi''(\log|f(z)|)\frac{|(\nabla f)(z)|^2}{|f(z)|^2}(1-|z|^2)d\nu(z)$$
$$\le\frac{\epsilon_2(n+1)}{2}\int_B\chi''(\log|f(z)|)\frac{|(\tilde\nabla f)(z)|^2}{|f(z)|^2}\frac{d\nu(z)}{1-|z|^2}$$
$$=\epsilon_2(n+1)\int_B\tilde\Delta(\{\log(1+|f|)\}^p)(z)\frac{d\nu(z)}{1-|z|^2}$$
$$\le\frac{\epsilon_2 n(n+1)}{2a_n\Gamma(n)}\|f\|^p_{N^p(B)}. \tag{4.41}$$

When $2<p<\infty$, by (4.14), (4.15), (4.19), (4.21) and (4.37) we have

$$V_{2,1}\le\int_{E_1'}\Big[\frac{\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{|u(z)|^2(1+|(C_\varphi f)(z)|)^2}$$
$$\times\{(p-1)|(C_\varphi f)(z)|+\log(1+|(C_\varphi f)(z)|)\}$$
$$\times\frac{|u(z)|}{|(C_\varphi f)(z)|}|(\nabla f)(\varphi(z))|^2(1-|z|^2)|J_\varphi(z)|^2\Big]d\nu(z)$$
$$=\frac{1}{p}\int_{E_1'}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^{-1}(1-|z|^2)|J_\varphi(z)|^2d\nu(z). \tag{4.42}$$

And we have by (4.16), (4.20), (4.21) and (4.37),

$$V_{2,2}\le\int_{E_2'}\Big[\frac{|u(z)|^{p-2}\{\log(1+|(C_\varphi f)(z)|)\}^{p-2}}{(1+|(C_\varphi f)(z)|)^2}\Big\{(p-1)+\frac{\log(1+|(C_\varphi f)(z)|)}{|(C_\varphi f)(z)|}\Big\}$$
$$\times|u(z)|^2|(\nabla f)(\varphi(z))|^2(1-|z|^2)|J_\varphi(z)|^2\Big]d\nu(z)$$
$$=\frac{1}{p}\int_{E_2'}\chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^p(1-|z|^2)|J_\varphi(z)|^2d\nu(z). \tag{4.43}$$

By (4.13), (4.36), (4.37), (4.42) and (4.43), we obtain

$$
\begin{aligned}
V_2 \le& \int_{E_1'} \chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^{-1}(1-|z|^2)|J_\varphi(z)|^2 d\nu(z)\\
&+\int_{E_2'} \chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}|u(z)|^{p}(1-|z|^2)|J_\varphi(z)|^2 d\nu(z)\\
=& \int_{B\setminus r_0\bar{B}} \chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}\\
&\times \max\{|u(z)|^{-1},|u(z)|^p\}(1-|z|^2)|J_\varphi(z)|^2 d\nu(z)\\
\le& \epsilon_2\int_{B\setminus r_0\bar{B}} \chi''(\log|f(\varphi(z))|)\frac{|(\nabla f)(\varphi(z))|^2}{|f(\varphi(z))|^2}(1-|\varphi(z)|^2)|J_\varphi(z)|^2 d\nu(z).
\end{aligned}
\tag{4.44}
$$

As in (4.41), we have

$$
V_2 \le \frac{\epsilon_2 n(n+1)}{2a_n\Gamma(n)}\|f\|^p_{N^p(B)}. \tag{4.45}
$$

By (4.22)∼(4.24), (4.31), (4.35), (4.41) and (4.45), we obtain

$$
\begin{aligned}
&\int_{B\setminus r_0\bar{B}} \chi''(\log|(uC_\varphi f)(z)|)\frac{|R(uC_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\
&\le 2^{2n-1}\left[\epsilon_1 p^2 L+\frac{\epsilon_2 n(n+1)M}{2a_n\Gamma(n)}\right]\|f\|^p_{N^p(B)}.
\end{aligned}
\tag{4.46}
$$

By (3.12), Lemma 2.3 and Lemma 2.6, we have

$$
\begin{aligned}
&\int_{r_0\bar{B}} \chi''(\log|(uC_\varphi f)(z)|)\frac{|R(uC_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\
&\le \frac{1}{\delta}\int_{r_0\bar{B}} \chi''(\log|(uC_\varphi f)(z)|)\frac{|R(uC_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{r_1}{|z|}d\nu(z)\\
&\le \frac{2n}{\delta}\|(uC_\varphi f)_{r_1}\|^p_{N^p(B)}\\
&\le \frac{2n}{\delta}\int_S [\max\{1,|u_{r_1}|\}]^p\{\log(1+|(C_\varphi f)_{r_1}|)\}^p d\sigma\\
&\le \frac{2n}{\delta}\max_{z\in r_1S}\{1+|u(z)|\}^p\max_{z\in\varphi(r_1S)}\{\log(1+|f(z)|)\}^p\\
&\le \frac{2n}{\delta}\max_{z\in r_1S}\{1+|u(z)|\}^p\max_{z\in\varphi(r_1S)}\left\{\frac{1+|z|}{1-|z|}\right\}^n\|f\|^p_{N^p(B)}
\end{aligned}
\tag{4.47}
$$

where $r_1=\dfrac{1+r_0}{2}$ and $\delta=1-\dfrac{\log r_1}{\log r_0}$. Moreover, we have by Lemma 2.6

$$
\{\log(1+|u(0)f(\varphi(0))|)\}^p \le [\max\{1,|u(0)|\}]^p\left\{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}^n\|f\|^p_{N^p(B)}. \tag{4.48}
$$

By (4.46)∼(4.48) and Lemma 2.3, we obtain

$$
\begin{aligned}
\|uC_\varphi f\|^p_{N^p(B)} \le& \left[\frac{2^{2n-1}}{2n}\left\{\epsilon_1 p^2 L+\frac{\epsilon_2 n(n+1)M}{2a_n\Gamma(n)}\right\}\right.\\
&+\frac{1}{\delta}\max_{z\in r_1S}\{1+|u(z)|\}^p\max_{z\in\varphi(r_1S)}\left\{\frac{1+|z|}{1-|z|}\right\}^n\\
&\left.+[\max\{1,|u(0)|\}]^p\left\{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}^n\right]\|f\|^p_{N^p(B)}.
\end{aligned}
\tag{4.49}
$$

Hence uC_φ is metrically bounded on $N^p(B)$. This completes the proof of (a).

To prove (b), suppose that $\{f_j\}$ is a sequence in $N^p(B)$ which converges to zero uniformly on compact subsets of B and $\|f_j\|^p_{N^p(B)}\leq\gamma<\infty$ for all $j\in\mathbf{N}$. Let $\epsilon>0$ be given. By (4.5)~(4.8), we can choose $r_0\in(\frac{1}{2}, 1)$ such that if $1<p\leq2$,

$$\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)<\epsilon, \tag{4.50}$$

$$\frac{\max\{|u(z)|^{p-3},|u(z)|^2\}(1-|z|^2)}{1-|\varphi(z)|^2}<\epsilon, \tag{4.51}$$

if $2<p<\infty$,

$$\max\{|u(z)|^{-2},|u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)<\epsilon, \tag{4.52}$$

$$\frac{\max\{|u(z)|^{-1},|u(z)|^p\}(1-|z|^2)}{1-|\varphi(z)|^2}<\epsilon \tag{4.53}$$

for any $z\in B\backslash r_0\bar{B}$. By (a), it holds that $(uC_\varphi)(N^p(B))\subset N^p(B)$. By the same argument as in the proof of (a), we have instead of (4.46)

$$\begin{aligned}&\int_{B\backslash r_0\bar{B}}\chi''(\log|(uC_\varphi f_j)(z)|)\frac{|R(uC_\varphi f_j)(z)|^2}{|(uC_\varphi f_j)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\&\leq2^{2n-1}\gamma\left[p^2L+\frac{n(n+1)M}{2a_n\Gamma(n)}\right]\epsilon\end{aligned} \tag{4.54}$$

for all $j\in\mathbf{N}$, by virtue of (4.50)~(4.53).

On the other hand, as in (4.47) we obtain for all $j\in\mathbf{N}$

$$\begin{aligned}&\int_{r_0\bar{B}}\chi''(\log|(uC_\varphi f_j)(z)|)\frac{|R(uC_\varphi f_j)(z)|^2}{|(uC_\varphi f_j)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)\\&\leq\frac{2n}{\delta}\max_{z\in r_1S}\{1+|u(z)|\}^p\max_{z\in\varphi(r_1S)}\{\log(1+|f_j(z)|)\}^p\end{aligned} \tag{4.55}$$

where $r_1=\dfrac{1+r_0}{2}$ and $\delta=1-\dfrac{\log r_1}{\log r_0}$. Since $\{f_j\}$ converges to zero uniformly on compact subsets of B,

$$\lim_{j\to\infty}\left[\max_{z\in\varphi(r_1S)}\{\log(1+|f_j(z)|)\}^p\right]=0. \tag{4.56}$$

By (4.55) and (4.56), we have

$$\lim_{j\to\infty}\int_{r_0\bar{B}}\chi''(\log|(uC_\varphi f_j)(z)|)\frac{|R(uC_\varphi f_j)(z)|^2}{|(uC_\varphi f_j)(z)|^2|z|^2}|z|^{-2(n-1)}\log\frac{1}{|z|}d\nu(z)=0. \tag{4.57}$$

By (4.54), (4.57) and Lemma 2.3, we can conclude that $\lim_{j\to\infty}\|uC_\varphi f_j\|_{N^p(B)}=0$. Thus Lemma 2.13 implies that uC_φ is metrically compact on $N^p(B)$. This completes the proof.

Corollary 1. *Let $1<p<\infty$. Suppose that φ is a univalent holomorphic self-map of B such that Ω_φ is bounded in B. Then C_φ is metrically compact on $N^p(B)$ if and only if*

$$\lim_{|z|\uparrow1}\frac{1-|z|^2}{1-|\varphi(z)|^2}=0.$$

Proof. Sufficiency is the case $u\equiv1$ of Theorem 4.1(b). Suppose that C_φ is metrically compact on $N^p(B)$. Since φ is a holomorphic self-map of B, φ has a radial limit $\varphi^*(\zeta) \equiv \lim_{r\uparrow 1}\varphi(r\zeta)\in\bar{B}$ for almost every $\zeta\in S$. First we show that the pull-back measure $\mu_\varphi = \sigma\circ\varphi^{*-1}$ satisfies the following condition:

$$\mu_\varphi(S(\zeta,h))=o(h^n) \qquad \text{as } h\downarrow 0 \tag{4.58}$$

uniformly in $\zeta\in S$, where $S(\zeta,h)=\{z\in\bar{B} : |1-\langle z,\zeta\rangle|<h\}$. We assume that it does not hold that $\mu_\varphi(S(\zeta,h))=o(h^n)$ uniformly in $\zeta\in S$. Then there are $\{\zeta_j\}\subset S$, $\{h_j\}\in(0, 1)$ with $h_j\downarrow 0$ $(j\to\infty)$ and $\epsilon_0>0$ such that

$$\mu_\varphi(S(\zeta_j,h_j))\geq\epsilon_0 h_j^n \tag{4.59}$$

for all $j\in\mathbf{N}$. Put $a_j=(1-h_j)\zeta_j\,(j\in\mathbf{N})$. Define

$$f_j(z)=(1-|a_j|)\exp\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n}{p}} \tag{4.60}$$

for $z\in B$ and $j\in\mathbf{N}$. We can easily see that f_j is in the ball algebra $A(B)$ and

$$\|f_j\|^p_{N^p(B)}\leq 2^{p-1}\{(\log 2)^p+1\}$$

for all $j\in\mathbf{N}$. Moreover, by (4.60) we see that $\{f_j\}$ converges to 0 uniformly on compact subsets of B. Since C_φ is metrically compact, we have

$$\lim_{j\to\infty}\|C_\varphi f_j\|_{N^p(B)}=0, \tag{4.61}$$

by Lemma 2.13.

On the other hand, by using the continuity of the function $F(v)=\mathrm{Re}(1+v)^{-\frac{2n}{p}}\,(v\in\mathbf{C})$ at the origin in $\mathbf{C}$, we can choose $j_0\in\mathbf{N}$ such that

$$\mathrm{Re}\left\{1+\frac{|a_j|(1-\langle z,\zeta_j\rangle)}{1-|a_j|}\right\}^{-\frac{2n}{p}}>\frac{1}{2} \tag{4.62}$$

for any $j\in\mathbf{N}$ with $j\geq j_0$ and $z\in S(\zeta_j,h_j)$. Moreover, we have by (4.62)

$$\mathrm{Re}\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n}{p}}>\frac{1}{2h_j^{\frac{n}{p}}}$$

for any $j\in\mathbf{N}$ with $j\geq j_0$ and $z\in S(\zeta_j,h_j)$. Thus, for any $j\in\mathbf{N}$ with $j\geq j_0$ and $z\in S(\zeta_j,h_j)$, we have

$$\begin{aligned}\log^+|f_j(z)| &=\log^+\left[(1-|a_j|)\exp\left(\mathrm{Re}\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n}{p}}\right)\right]\\ &\geq\log^+\left[(1-|a_j|)\exp\left(\frac{1}{2h_j^{\frac{n}{p}}}\right)\right]\\ &=\log^+\left[h_j\exp\left(\frac{1}{2h_j^{\frac{n}{p}}}\right)\right].\end{aligned} \tag{4.63}$$

Hence, by Fatou's lemma and (4.63) we obtain for any $j\in\mathbf{N}$ with $j\geq j_0$

$$
\begin{aligned}
&\left\{\log^+\left(h_j\exp\left(\frac{1}{2h_j^{\frac{n}{p}}}\right)\right)\right\}^p \mu_\varphi(S(\zeta_j,h_j)) \\
&\leq \int_{S(\zeta_j,h_j)} \{\log^+|f_j|\}^p d\mu_\varphi \\
&\leq \int_S \{\log(1+|f_j\circ\varphi^*|)\}^p d\sigma \\
&= \int_S \lim_{r\uparrow 1}\{\log(1+|(f_j\circ\varphi)_r|)\}^p d\sigma \\
&\leq \liminf_{r\uparrow 1}\int_S \{\log(1+|(f_j\circ\varphi)_r|)\}^p d\sigma \\
&= \| C_\varphi f_j\|^p_{N^p(B)}.
\end{aligned} \tag{4.64}
$$

It follows from (4.61) and (4.64) that

$$
\lim_{j\to\infty}\left\{\log^+\left(h_j\exp\left(\frac{1}{2h_j^{\frac{n}{p}}}\right)\right)\right\}^p \mu_\varphi(S(\zeta_j,h_j))=0. \tag{4.65}
$$

Since

$$
\lim_{j\to\infty} h_j^n\left\{\log^+\left(h_j\exp\left(\frac{1}{2h_j^{\frac{n}{p}}}\right)\right)\right\}^p=\frac{1}{2^p},
$$

(4.65) implies that

$$
\lim_{j\to\infty}\frac{\mu_\varphi(S(\zeta_j,h_j))}{h_j^n}=0.
$$

This contradicts (4.59). Thus we establish that (4.58) holds. By MacCluer's Carleson-measure criterion([8]), C_φ is compact on $H^2(B)$. It follows from Theorem 1.1(b) that

$$
\lim_{|z|\uparrow 1}\frac{1-|z|^2}{1-|\varphi(z)|^2}=0.
$$

This completes the proof. □

Theorem 4.2. *Let* $1\leq p<\infty$ *and* $-1<\alpha<\infty$. *Let* $u\in H(B)\backslash\{0\}$ and φ: $B\to B$ *be a univalent holomorphic map such that* Ω_φ *is bounded in* B.

(a) *Suppose* u *and* φ *satisfy the following conditions :*

(i) *When* $1\leq p\leq 2$,

$$
\limsup_{|z|\uparrow 1}[\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)^2]<\infty, \tag{4.66}
$$

$$
\limsup_{|z|\uparrow 1}\left[\max\{|u(z)|^{p-3},|u(z)|^2\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right]<\infty. \tag{4.67}
$$

(ii) *When* $2<p<\infty$,

$$
\limsup_{|z|\uparrow 1}[\max\{|u(z)|^{-2},|u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)^2]<\infty, \tag{4.68}
$$

$$
\limsup_{|z|\uparrow 1}\left[\max\{|u(z)|^{-1},|u(z)|^{p}\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right]<\infty. \tag{4.69}
$$

Then uC_φ *is metrically bounded on* $(AN)^p(\nu_\alpha)$.

(b) *Suppose* u *and* φ *satisfy the following conditions :*

(i) When $1\leq p\leq 2$,

$$\lim_{|z|\uparrow 1}[\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)^2]=0, \tag{4.70}$$

$$\lim_{|z|\uparrow 1}\left[\max\{|u(z)|^{p-3},|u(z)|^2\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right]=0. \tag{4.71}$$

(ii) When $2<p<\infty$,

$$\lim_{|z|\uparrow 1}[\max\{|u(z)|^{-2},|u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)^2]=0, \tag{4.72}$$

$$\lim_{|z|\uparrow 1}\left[\max\{|u(z)|^{-1},|u(z)|^{p}\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right]=0. \tag{4.73}$$

Then uC_φ *is metrically compact on* $(AN)^p(\nu_\alpha)$.

Proof. Take $f\in(AN)^p(\nu_\alpha)\backslash\{0\}$. By the hypothesis of the present theorem and Lemma 2.11, C_φ is metrically bounded on $(AN)^p(\nu_\alpha)$, that is, $\|C_\varphi f\|^p_{(AN)^p(\nu_\alpha)}\leq L\|f\|^p_{(AN)^p(\nu_\alpha)}$ where L is a positive constant depending only on n, α and φ. By (4.66)~(4.69), there are positive constants ϵ_1, ϵ_2 and $r_0\in(\frac{1}{2},1)$ such that when $1<p\leq 2$,

$$\max\{|u(z)|^{p-4},1\}|(\nabla u)(z)|^2(1-|z|^2)^2<\epsilon_1, \tag{4.74}$$

$$\max\{|u(z)|^{p-3},|u(z)|^2\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}<\epsilon_2, \tag{4.75}$$

when $2<p<\infty$,

$$\max\{|u(z)|^{-2},|u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)^2<\epsilon_1, \tag{4.76}$$

$$\max\{|u(z)|^{-1},|u(z)|^{p}\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}<\epsilon_2 \tag{4.77}$$

for any $z\in B\backslash r_0\bar{B}$. By (3.27), (4.74)~(4.78), Lemma 2.4(a) and the same argument as in the proof of Theorem 4.1(a), we obtain

$$\begin{aligned}&\int_{B\backslash r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|R(uC_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2|z|^2}|z|^{-2(n-1)}K_\alpha(|z|)d\nu(z)\\ &\leq 2^{2n-1}d_{n,\alpha}\left[\frac{\epsilon_1 p^2 L}{C_\alpha}+\epsilon_2(n+1)M\right.\\ &\qquad\left.\times\frac{2^{\alpha+\alpha^+}(n+\alpha+1)\Gamma(\alpha+2)}{a_n\Gamma(n+\alpha+1)}\right]\|f\|^p_{(AN)^p(\nu_\alpha)}\end{aligned} \tag{4.78}$$

where $M=\sup_{z\in B}\Omega_\varphi(z)<\infty$. (cf. (4.46))

On the other hand, by (2.16), Lemma 2.3 and Lemma 2.6, we obtain

$$\begin{aligned}&\int_{r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|R(uC_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2|z|^2}|z|^{-2(n-1)}K_\alpha(|z|)d\nu(z)\\ &\leq C_\alpha\int_{r_0\bar{B}}\chi''(\log|(uC_\varphi f)(z)|)\frac{|R(uC_\varphi f)(z)|^2}{|(uC_\varphi f)(z)|^2|z|^2}|z|^{-2(n-1)}K_\alpha\left(\frac{|z|}{r_1}\right)d\nu(z)\\ &\leq 2nC_\alpha\|(uC_\varphi f)_{r_1}\|^p_{(AN)^p(\nu_\alpha)}\end{aligned}$$

$$\leq 2nC_\alpha \max_{z\in r_1\bar{B}}\{1+|u(z)|\}^p \max_{z\in\varphi(r_1\bar{B})}\{\log(1+|f(z)|)\}^p$$

$$\leq 2nC_\alpha \max_{z\in r_1\bar{B}}(1+|u(z)|)^p \max_{z\in\varphi(r_1\bar{B})}\left\{\frac{1+|z|}{1-|z|}\right\}^{n+1+\alpha}\|f\|^p_{(AN)^p(\nu_\alpha)}. \tag{4.79}$$

where $r_1=\frac{1+r_0}{2}$ and $C_\alpha=(\log\frac{1}{r_0})\{K_\alpha(\frac{r_0}{r_1})\}^{-1}+1$. (cf. (4.47)) Moreover, we have by Lemma 2.6

$$\{\log(1+|(uC_\varphi f)(0)|)\}^p\leq[\max\{1,|u(0)|\}]^p\left\{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}^{n+1+\alpha}\|f\|^p_{(AN)^p(\nu_\alpha)}. \tag{4.80}$$

(4.78)~(4.80) and Lemma 2.3 show that uC_φ is metrically bounded on $(AN)^p(\nu_\alpha)$. This completes the proof of (a).

To prove (b), suppose that $\{f_j\}$ is a bounded sequence in $(AN)^p(\nu_\alpha)$ which converges to zero uniformly on compact subsets of B. Then we can show that $\lim_{j\to\infty}\|uC_\varphi f_j\|_{(AN)^p(\nu_\alpha)}=0$, by the same way as that in the proof of Theorem 4.1(b). Hence, by Lemma 2.13, we conclude that uC_φ is metrically compact on $(AN)^p(\nu_\alpha)$. The proof is complete. □

Corollary 2. *Let* $1\leq p<\infty$ *and* $-1<\alpha<\infty$. *Suppose that* φ *is a univalent holomorphic self-map of* B *such that* Ω_φ *is bounded in* B. *Then* C_φ *is metrically compact on* $(AN)^p(\nu_\alpha)$ *if and only if*

$$\lim_{|z|\uparrow 1}\frac{1-|z|^2}{1-|\varphi(z)|^2}=0$$

Proof. The proof is entirely similar to that of Corollary 1 except that we choose functions

$$f_j(z)=(1-|a_j|)\exp\left\{\frac{1-|a_j|^2}{(1-\langle z,a_j\rangle)^2}\right\}^{\frac{n+1+\alpha}{p}} \qquad (z\in B,\ j\in\mathbf{N})$$

instead of (4.60). □

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