J. FAC. SCI., SHINSHU UNIVERSITY, Vol. 40, pp.27-36 2005

*The modular adjacency algebras of non-symmetric imprimitive association schemes of class 3**

By Masayoshi YOSHIKAWA

Department of Mathematical Sciences, Interdisciplinary Graduate School of Science and Technology, Shinshu University, Matsumoto 390-8621, Japan

(Received November 14, 2005)

Abstract

We continue to determine the structure of modular adjacency algebras of association schemes from [8] and [5]. In this paper we will determine the structure of modular adjacency algebras of non-symmetric imprimitive association schemes of class 3.

1 Introduction

To each association scheme (X,G) and to each field F, there is associated naturally an associative algebra, so-called the adjacency algebra FG of (X,G) over F. It is well-known that FG is semisimple if F has characteristic 0. However, little is known if F has a positive characteristic. In the present paper, we focus on this case.

We want to characterize adjacency algebras of association schemes algebraically. For example, we can consider a finite group as an association scheme and its adjacency algebra is isomorphic to the group algebra of the original group. Then it is known that the group algebra of the finite group over any field is a Frobenius algebra. However we can find an association scheme easily such that its adjacency algebra is not a Frobenius algebra. We have determined the structure of modular adjacency algebras of some association schemes ([8] and [5]). Especially, in [5], we determined the structure of modular adjacency algebras of association schemes of class 2. So in the present paper, we will determine the structure of modular adjacency algebras of non-symmetric imprimitive association schemes of class three.

It is known that association schemes of class $d \leq 4$ are commutative and So are their adjacency algebras.

*Communicated by A. Hanaki

2 Association Schemes

In this section, we will recall a definition and some properties for association schemes (see [2] and [9] for more details). We will almost use notations of Bannai and Ito [2].

Let X be a finite set and $\{R_i\}_{0\le i\le d}$ a partition of $X\times X$. We call a pair $\mathfrak{X}=(X, \{R_i\}_{0\le i\le d})$ an *association scheme* if the following conditions are satisfied:

(1) $R_0=\{(x,x)|x\in X\}$.

(2) There exists $i'\in\{0,1,...,d\}$ such that $R_{i'}=\{(y,x)|(x,y)\in R_i\}$ for any $i\in\{0,1,...,d\}$.

(3) For all $i,j,k\in\{0,1,...,d\}$, there exists a non-negative integer p_{ijk} such that for all $y,x\in X|\{x\in X|(y,x)\in R_i$ and $(x,z)\in R_j\}|=p_{ijk}$ if $(y,z)\in R_k$.

Moreover $\mathfrak{X}$ is called *commutative* if $p_{ijk}=p_{jik}$ for any $i, j, k\in\{0,...,d\}$ and *symmetric* if $R_i=R_{i'}$ for any $i\in\{0,...,d\}$. The elements of $\{p_{ijk}\}$ will be called *intersection numbers* of (X,G).

Let $\mathfrak{X}=(X, \{R_i\}_{0\le i\le d})$ be an association scheme. Then the cardinality $|X|$ of X is called the *order* of the association scheme $\mathfrak{X}$ and d the *class* of $\mathfrak{X}$. We call the positive integer $p_{ii'0}$ the *valency* of R_i and denote it by v_i. The following properties for intersection numbers are known.

Proposition 1. ([9, Lemma 1.1.3 and 1.1.4]) *Let $\mathfrak{X}=(X, \{R_i\}_{0\le i\le d})$ be an association scheme and $\{p_{ijk}|0\le i, j, k\le d\}$ the intersection numbers of $\mathfrak{X}$. Then*

(1) $\sum_{b=0}^{d}p_{aeb}p_{bfg}=\sum_{c=0}^{d}p_{acg}p_{efc}$.

(2) $p_{abc}=p_{b'a'c'}$.

(3) $p_{abc}v_c=p_{cb'a}v_a$.

(4) $\sum_{a=0}^{d}p_{abc}=v_{b'}$.

(5) $\sum_{b=0}^{d}p_{abc}=v_a$.

(6) $\sum_{c=0}^{d}p_{abc}v_c=v_av_b$.

Let E and F be any subsets of $\{R_i\}_{0\le i\le d}$. We define the *complex product* EF of E and F by

$$EF:=\{R_g|\sum_{R_e\in E}\sum_{R_f\in F}p_{efg}\neq 0\}.$$

We define $F':=\{R_{f'}|R_f\in F\}$ for each $F\subset\{R_i\}_{0\le i\le d}$. Then a non-empty subset F of $\{R_i\}_{0\le i\le d}$ is said to be *closed* if $FF'\subset F$.

An association scheme is called *primitive* if it has no closed subsets except $\{R_0\}$ and $\{R_i\}_{0\le i\le d}$ (they are called *trivial* closed subsets), and *imprimitive* otherwise.

3 Adjacency algebras of association schemes

Let $\mathfrak{X}=(X, \{R_i\}_{0\le i\le d})$ be an association scheme. For each R_i, we define the $|X|\times|X|$ matrix A_i indexed by the elements of X by

$$(A_i)_{xy} := \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let J be the $|X| \times |X|$ all 1 matrix. Then, by the definition, it follows that $\sum_{i=0}^{d} A_i = J$. It also follows that $A_i A_j = \sum_{k=0}^{d} p_{ijk} A_k$. We can naturally define an algebra from this fact. For the commutative ring R with 1, we put $R\mathfrak{X} = \bigoplus_{i=0}^{d} RA_i$ as a matrix ring over R, which will be called the *adjacency algebra* of $\mathfrak{X}$ over R. In particular, the adjacency algebra of an association scheme over a field of characteristic 0 is semisimple [9, Theorem 4.1.3].

Let $\mathbb{C}\mathfrak{X}$ be the adjacency algebra of an association scheme $\mathfrak{X}$ over the complex number field. We denote the set of irreducible characters of $\mathbb{C}\mathfrak{X}$ by $Irr(\mathfrak{X})$. In particular, if $\mathfrak{X}$ is commutative, then $|Irr(\mathfrak{X})| = d+1$.

Since the adjacency algebra is defined as a matrix ring, we can consider the natural representation $A_i \mapsto A_i$. We call it the *standard representation* and its character the *standard character*. We denote the standard character by $\gamma(\mathfrak{X})$. Let $\gamma(\mathfrak{X}) = \sum_{\chi \in Irr(\mathfrak{X})} m_\chi \chi$ be the irreducible decomposition of the standard character. We call m_χ the *multiplicity* of χ.

We introduce some results for the modular representation of association schemes. Here modular means that we consider adjacency algebras of association schemes over a field of a positive characteristic p. In the present paper, we consider only commutative association schemes since association schemes of class 3 are commutative.

Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be an association scheme of order n. Let (K, R, F) be a splitting p-modular system for the adjacency algebra, and (π) the maximal ideal of R. We denote the image of the canonical epimorphism $R \to F$ by $*$. For the fundamental results of the modular representation of the association schemes, see [5].

Proposition 2. ([1, Theorem 1.1] *and* [3, Theorem 4.2]) *Let* $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ *be an association scheme of order* n. *We set that* $Irr(\mathfrak{X}) = \{\chi_0, \dots, \chi_d\}$. *Let* m_i *be the multiplicity of* χ_i, *and* p *be a prime number. Put*

$$\mathfrak{F}(\mathfrak{X}) = n^{d+1} \frac{\prod_{i=0}^{d} v_i}{\prod_{i=0}^{d} m_i}.$$

Then the adjacency algebra of the association scheme $\mathfrak{X}$ *over a field of characteristic* p *is semisimple, if and only if* $p \nmid \mathfrak{F}(\mathfrak{X})$.

Let $\mathfrak{X}$ be a commutative association scheme. Then $R\mathfrak{X}/\pi R\mathfrak{X} \cong F\mathfrak{X}$ and $\pi R\mathfrak{X} \subseteq rad(R\mathfrak{X})$ [6, Theorem I.14.1], so idempotents of $F\mathfrak{X}$ are liftable to idempotents of $R\mathfrak{X}$ [6, Theorem I.14.2]. Consider the primitive idempotent decomposition of $1_{F\mathfrak{X}}$ in $F\mathfrak{X}$:

$$1_{F\mathfrak{X}} = f_0 + \cdots + f_s \in F\mathfrak{X},$$

then we have the primitive idempotent decomposition of $1_{R\mathfrak{X}}$ in $R\mathfrak{X}$:

$$1_{R\mathfrak{X}}=e_{B_0}+\cdots+e_{B_s}\in R\mathfrak{X},$$

where $e^*_{B_i}=f_i$. This decomposition yields the decomposition of algebras. We call this $e_{B_i}(e^*_{B_i})$ a *block idempotent* of $R\mathfrak{X}(F\mathfrak{X})$, and we write $B_i=e_{B_i}R\mathfrak{X}$ and $B^*_i=e^*_{B_i}F\mathfrak{X}$.

Let $e_0,\dots,e_d$ be the set of primitive idempotents in $K\mathfrak{X}$. Then there is a partition $\{0,\dots,d\}=\cup_{j=0}^{s}T_j$ such that $e_{B_i}=\sum_{j\in T_i}e_j$. When $e_j\in T_i$, we say that e_j belongs to the block B_i.

Let χ_j be the (one-dimensional) irreducible representation of $K\mathfrak{X}$ corresponding to e_j. Then e_j belongs to B_i if and only if $\chi_j(e_{B_i})=1$. Since B^*_i has the unique idempotent $e^*_{B_i}$, so we have $B^*_i/rad(B^*_i)\cong F$. If χ_i and χ_j belong to the same block, then $\chi^*_i=\chi^*_j$.

Proposition 3. ([5, Lemma 1]) *Irreducible characters χ_i and χ_j of $K\mathfrak{X}$ belong to the same block if and only if $\chi_i(A_r)\equiv\chi_j(A_r)$ (mod (π)) for all $r=0,\dots,d$.*

Proposition 4. ([5, Lemma 2]) *The dimension of B^*_i is equal to the number of χ_j belonging to B_i.*

4 Non-symmetric imprimitive association schemes of class three

Let $\mathfrak{X}=(X,\{R_i\}_{0\le i\le 3})$ be a non-symmetric imprimitive association scheme of class 3, and $A_i(0\le i\le 3)$ its adjacency matrices. Put $n=|X|$ and ν_i the valency of R_i. We assume that $A_2={}^tA_3$. We set that $\nu_1=k_1$, and $\nu_2=\nu_3=k_2$. By Proposition 1 (2) and (3), we set non-negative integers a, b, c, d, e and f that

$$\begin{aligned}
&a=p_{111},\\
&b=p_{112}=p_{113}=p_{121}k_1/k_2=p_{131}k_1/k_2=p_{211}k_1/k_2=p_{311}k_1/k_2,\\
&c=p_{122}=p_{133}=p_{212}=p_{231}k_1/k_2=p_{313}=p_{321}k_1/k_2,\\
&d=p_{123}=p_{132}=p_{213}=p_{221}k_1/k_2=p_{312}=p_{331}k_1/k_2,\\
&e=p_{222}=p_{232}=p_{233}=p_{322}=p_{323}=p_{333},\\
&f=p_{223}=p_{332}.
\end{aligned}$$

Then it follows that from Proposition 1 (1), (4), (5) and (6),

$$\begin{aligned}
&a=k_1-1-2k_2+\frac{2k_2(c+d)}{k_1},\\
&b=k_1-c-d,\\
&e=\frac{k_2-1-c}{2},\\
&f=\frac{k_2-2d+1+c}{2},\\
&d^2k_2+d^2k_1+ck_1k_2=c^2k_2+cdk_1+dk_1+dk_1k_2.
\end{aligned}$$

Since we consider only imprimitive case in the present paper, we may assume that $b=0$ or $c=d=0$. Then there are the following three cases.

(1) $a=k_1-1,\ b=0,\ c=k_1,\ d=0,\ e=\frac{k_2-k_1-1}{2},\ f=\frac{k_2+k_1+1}{2}$,

(2) $a=k_1-1,\ b=0,\ c=\frac{k_1-1}{2},\ d=\frac{k_1+1}{2},\ e=f=\frac{2k_2-k_1-1}{4}$,

(3) $a=k_1-2k_2-1,\ b=k_1,\ c=0,\ d=0,\ e=\frac{k_2-1}{2},\ f=\frac{k_2+1}{2}$.

We can check that there are some association schemes for each type.

In the next section, we will determine the structure of modular adjacency algebras of association schemes of each case using character tables calculated by S. Y. Song (See [7]).

5 Modular Adjacency Algebras

In this section, we set $\mathfrak{X}=(X, \{R_i\}_{0\leq i\leq d})$ which is a non-symmetric imprimitive association scheme of class three and non-negative integers a, b, c, d, e, f, k_1 and k_2 are as above. Let F be a field of characteristic p such that F is a splitting field of $F\mathfrak{X}$.

5.1. Case 1. The multiplication table is shown below:

	A_0	A_1	A_2	A_3
A_0	A_0	A_1	A_2	A_3
A_1	A_1	$k_1A_0+(k_1-1)A_1$	k_1A_2	k_1A_3
A_2	A_2	k_1A_2	eA_2+fA_3	$k_2A_0+k_2A_1+eA_2+eA_3$
A_3	A_3	k_1A_3	$k_2A_0+k_2A_1+eA_2+eA_3$	fA_2+eA_3

And the character table is as follows (P_σ in [7]):

	A_0	A_1	A_2	A_3	m_χ
χ_0	1	k_1	k_2	k_2	1
χ_1	1	k_1	$\frac{-(k_1+1)+\sqrt{n(k_1+1)}\,i}{2}$	$\frac{-(k_1+1)-\sqrt{n(k_1+1)}\,i}{2}$	$\frac{k_2}{k_1+1}$
χ_2	1	k_1	$\frac{-(k_1+1)-\sqrt{n(k_1+1)}\,i}{2}$	$\frac{-(k_1+1)+\sqrt{n(k_1+1)}\,i}{2}$	$\frac{k_2}{k_1+1}$
χ_3	1	-1	0	0	$\frac{nk_1}{k_1+1}$

It follows that the Frame number is $\mathfrak{F}(\mathfrak{X})=n^3(k_1+1)^3$. Since the multiplicities are positive integers, we know that $k_1+1|k_2$ and $k_1+1|n$.

Lemma 5. *Let p be a prime such that $p|k_1+1$, then $p|e$.*

Proof. In the case $p\neq 2$, we have obviously $p|e=\frac{k_2-k_1-1}{2}$.

Let us assume that $p=2$. Let s and t be integers such that $k_1+1=2s$, $k_2=(k_1+1)t$. Then $e=s(t-1)$. When we consider the factor scheme $\mathfrak{Y}$ of $\mathfrak{X}$ by $\{R_0,R_1\}$, $\mathfrak{Y}$ is a non-symmetric class 2 association scheme whose valencies are 1, t, t. Each valency of a nonsymmetric class 2 association scheme is odd [5]. So t is also odd. Thus $2|e$. □

Theorem 6. *The structure of the modular adjacency algebra of an association scheme of case 1 is determined as follows.*

$$F\mathfrak{X}\cong\begin{cases}F[x,y,z]/(x^2,\ y^2,\ z^2,\ xy,\ yz,\ zx) & \textit{if } p|k_1+1,\\ F[x]/(x^3)\oplus F & \textit{if } p|n,\ p\nmid k_1+1,\\ F\oplus F\oplus F\oplus F & \textit{otherwise,}\end{cases}$$

where the first block from the left of the right hand sides is the principal block.

Proof. Let J be the all-one matrix. If $F\mathfrak{X}$ is not semisimple, $J^*\in rad(F\mathfrak{X})$. Let us assume that $p|k_1+1$. Then, from the character table above, it follows that $F\mathfrak{X}$ is a 4-dimensional local ring. We remark that $e\equiv f \pmod p$.

Since $p|e$ from Lemma 5, it follows that

$$(A_2^*)^2=eA_2^*+fA_3^*=0,$$
$$(A_2^*+A_3^*)^2=(f+3e)(A_2^*+A_3^*)=0,$$
$$\dim_F(A_2^*)F\mathfrak{X}=\dim_F(A_2^*+A_3^*)F\mathfrak{X}=1.$$

Therefore we obtain that $F\mathfrak{X}\cong F[x,\ y,\ z]/(x^2,\ y^2,\ z^2,\ xy,\ yz,\ zx)$.

Next, let us assume that $p|n$ and $p\nmid k_1+1$. Then it follows that $F\mathfrak{X}$ is the direct sum of a 3-dimension local ring and a simple ring from the character table. We remark that $p\nmid k_2$ and $p\neq 2$ because $n=2k_2+k_1+1$. Since

$$(A_2^*-A_3^*)^2=-2k_2J^*,$$
$$(J^*)F\mathfrak{X}\subset(A_2^*-A_3^*)F\mathfrak{X},$$

we obtain that $F\mathfrak{X}\cong F[x]/(x^3)\oplus F$. □

There exist association schemes for every case treated in this theorem. For example, let us consider an association scheme as06[6] in the list [4]. In this case, we can see that its modular adjacency algebra $F\mathfrak{X}$ is isomorphic to $F[x]/(x^3)$ if $p=3$, and $F[x,\ y,\ z]/(x^2,\ y^2,\ z^2,\ xy,\ yz,\ zx)$ if $p=2$.

5.2. Case 2. The multiplication table is as follows:

	A_0	A_1	A_2	A_3
A_0	A_0	A_1	A_2	A_3
A_1	A_1	$k_1A_0+(k_1-1)A_1$	$\frac{k_1-1}{2}A_2+\frac{k_1+1}{2}A_3$	$\frac{k_1+1}{2}A_2+\frac{k_1-1}{2}A_3$
A_2	A_2	$\frac{k_1-1}{2}A_2+\frac{k_1+1}{2}A_3$	$\frac{k_2d}{k_1}A_1+eA_2+eA_3$	$k_2A_0+\frac{k_2c}{k_1}A_1+eA_2+eA_3$
A_3	A_3	$\frac{k_1+1}{2}A_2+\frac{k_1-1}{2}A_3$	$k_2A_0+\frac{k_2c}{k_1}A_1+eA_2+eA_3$	$\frac{k_2d}{k_1}A_1+eA_2+eA_3$

And the character table is (P_σ in [7]):

	A_0	A_1	A_2	A_3	m_χ
χ_1	1	k_1	k_2	k_2	1
χ_2	1	-1	$\sqrt{\frac{k_2(k_1+1)}{2k_1}}\,i$	$-\sqrt{\frac{k_2(k_1+1)}{2k_1}}\,i$	$\frac{nk_1}{2(k_1+1)}$
χ_3	1	-1	$-\sqrt{\frac{k_2(k_1+1)}{2k_1}}\,i$	$\sqrt{\frac{k_2(k_1+1)}{2k_1}}\,i$	$\frac{nk_1}{2(k_1+1)}$
χ_4	1	k_1	$-\frac{k_1+1}{2}$	$-\frac{k_1+1}{2}$	$\frac{2k_2}{k_1+1}$

It follows that the Frame number is $\mathfrak{F}(\mathfrak{X})=\frac{2n^2k_2(k_1+1)^3}{k_1}$. Here we recall that intersection numbers are integers. Since $\frac{k_2d}{k_1}=\frac{k_2(k_1+1)}{2k_1}$ is an integer, $k_1|k_2$. Thus we may set that $k_2=mk_1$. Since $k_1+1=2d$, we have that $\mathfrak{F}(\mathfrak{X})=16n^2md^3$.

Then we can show the following lemma for parameters.

Lemma 7. *For the parameters of association scheme of case 2, we obtain that*

(1) $4|n$, $d|m$ *and* $d|n$,

(2) $k_1+1|2k_2$ *and* $k_1+1|n$,

(3) $d\equiv m(\mathrm{mod}\ 2)$,

(4) $p\nmid n$ *if* $p\neq 2$, $p\nmid d$ *and* $p|m$.

Proof. It follows that $4|n$ because e is an integer and $e=\frac{2k_2-k_1-1}{4}=\frac{4k_2-n}{4}$. And we know that $d|m$ since $m_{\chi 4}$ is an integer and thus $k_1+1|2k_2$. Since $n=2k_2+k_1+1=4md+2d-2m$, $d|n$. We obtain the second property because the multiplicities are integers. The third property holds since $4|n(=4md+2d-2m)$. The last property holds because $n=2k_2+k_1+1=2mk_1+2d$. □

Theorem 8. *The structure of the modular adjacency algebra of the association schemes of case 2 is determined as follows.*

$$F\mathfrak{X}\cong\begin{cases} F[x]/(x^4) & \text{if } p=2,\ p\nmid d,\ p|e,\\ F[x,y]/(x^2,y^2) & \text{if } p=2,\ p\nmid d,\ p\nmid e,\\ F[x,y,z]/(x^2,y^2,z^2,xy,yz,zx) & \text{if } p|d,\\ F\oplus F[x]/(x^2)\oplus F & \text{if } p\neq 2,\ p\nmid d,\ p|m,\\ F[x]/(x^2)\oplus F\oplus F & \text{if } p\neq 2,\ p\nmid d,\ p\nmid m,\ p|n,\\ F\oplus F\oplus F\oplus F & \text{otherwise,}\end{cases}$$

where the first block from the left on the right hand sides is the principal block.

Proof. From Lemma 7, the Frame number and the character table, it follows that

$$F\mathfrak{X} \cong \begin{cases} \text{the 4-dimension local ring} & \text{if } p=2 \text{ or } p \mid d, \\ F \oplus F[x]/(x^2) \oplus F & \text{if } p \neq 2,\ p \nmid d,\ p \mid m, \\ F[x]/(x^2) \oplus F \oplus F & \text{if } p \neq 2,\ p \nmid d,\ p \nmid m,\ p \mid n, \\ F \oplus F \oplus F \oplus F & \text{otherwise.} \end{cases}$$

Firstly, let us assume that $p=2$. Then it is obvious that $J^* \in rad(F\mathfrak{X})$. And we know that $(A_0^*+A_1^*)^2=0$, $(A_0^*+A_1^*)A_1^*=A_0^*+A_1^*$, $(A_0^*+A_1^*)A_2^*=d(A_2^*+A_3^*)$, and $(A_0^*+A_1^*)A_3^* = d(A_2^*+A_3^*)$. Therefore $\dim_F(A_0^*+A_1^*)F\mathfrak{X}=1$ if $d\equiv 0 \pmod 2$, and $FJ^* \subset (A_0^*+A_1^*)F\mathfrak{X}$ if $d \not\equiv 0 \pmod 2$.

In the case that $d\equiv 0 \pmod 2$, we have $(A_2^*)^2=e(A_2^*+A_3^*)=e(J^*-(A_0^*+A_1^*))$, $A_2^*A_1^*=A_2^*$, $A_2^*A_2^*=e(A_2^*+A_3^*)$, and $A_2^*A_3^*=e(A_2^*+A_3^*)$. Then from the possibility of the algebra structure, $(A_2^*)F\mathfrak{X}$ satisfies one of $FJ^* \subset (A_2^*)F\mathfrak{X}$, $F(A_0^*+A_1^*) \subset (A_2^*)F\mathfrak{X}$, or $\dim_F(A_2^*)F\mathfrak{X}=1$. Thus we obtain that $e\equiv 0 \pmod 2$. Therefore it follows that $F\mathfrak{X} \cong F[x, y, z]/(x^2, y^2, z^2, xy, yz, zx)$.

In the case that $d \not\equiv 0 \pmod 2$. We know that

$$(A_1^*+A_2^*)^2=\begin{cases} J^* & \text{if } e\equiv 1 \pmod 2, \\ A_0^*+A_1^* & \text{if } e\equiv 0 \pmod 2. \end{cases}$$

And it follows that $(A_1^*+A_2^*)A_1^*=A_0^*+A_3^*$, $(A_1^*+A_2^*)A_2^*=A_1^*+eA_2^*+(e+1)A_3^*$, and $(A_1^*+A_2^*)A_3^*=A_0^*+(e+1)A_2^*+eA_3^*$. Therefore we obtain that

$$F\mathfrak{X} \cong \begin{cases} F[x]/(x^4) & \text{if } e\equiv 0 \pmod 2, \\ F[x, y]/(x^2, y^2) & \text{if } e\equiv 1 \pmod 2. \end{cases}$$

From the above argument, it follow that, if $p=2$,

$$F\mathfrak{X} \cong \begin{cases} F[x]/(x^4) & \text{if } e\equiv 0,\ d\equiv 1 \pmod 2, \\ F[x, y]/(x^2, y^2) & \text{if } e\equiv 1,\ d\equiv 1 \pmod 2, \\ F[x, y, z]/(x^2, y^2, z^2, xy, yz, zx) & \text{if } d\equiv 0 \pmod 2. \end{cases}$$

Secondly, let us consider the case that $p\neq 2$ and $p \mid d$. Then, since $e\equiv 0 \pmod p$, it follows that $(J^*)^2=(A_2^*)^2=(A_3^*)^2=0$. Furthermore we have that $\dim F\mathfrak{X}J^*=\dim F\mathfrak{X}A_2^* =\dim F\mathfrak{X}A_3^*=1$. Therefore we obtain that $F\mathfrak{X} \cong F[x, y, z]/(x^2, y^2, z^2, xy, yz, zx)$. □

Note that there exist association schemes for every case treated in the above theorem. Let us consider the cyclic group C_4 of order 4. In this case, its group algebra $FC_4 \cong F[x]/(x^4)$ if $p=2$. Next, let us consider an association scheme as08[6] in the list [4]. Then its modular adjacency algebra $F\mathfrak{X} \cong F \oplus F[x]/(x^2) \oplus F$ if $p=3$. In the case of an association scheme as16[18], its modular adjacency algebra $F\mathfrak{X} \cong F[x, y, z]/(x^2, y^2, z^2, xy, yz, zx)$ if $p=2$. And last, let us consider an association scheme as24[14]. Then its modular adjacency algebra $F\mathfrak{X}$ is isomorphic to $F[x, y]/(x^2, y^2)$ if $p=2$ and $F[x]/(x^2) \oplus F \oplus F$ if $p=3$.

5.3. **Case 3**. The multiplication table is as follows:

	A_0	A_1	A_2	A_3
A_0	A_0	A_1	A_2	A_3
A_1	A_1	$k_1A_0+(k_1-1-2k_2)A_1+k_1A_2+k_1A_3$	k_2A_1	k_2A_1
A_2	A_2	k_2A_1	eA_2+fA_3	$k_2A_0+eA_2+eA_3$
A_3	A_3	k_2A_1	$k_2A_0+eA_2+eA_3$	fA_2+eA_3

And the character table is (P_r in [7]):

	A_0	A_1	A_2	A_3	m_χ
χ_1	1	k_1	k_2	k_2	1
χ_2	1	0	$\frac{-1+\sqrt{n-k_1}\,i}{2}$	$\frac{-1-\sqrt{n-k_1}\,i}{2}$	$\frac{nk_2}{n-k_1}$
χ_3	1	0	$\frac{-1-\sqrt{n-k_1}\,i}{2}$	$\frac{-1+\sqrt{n-k_1}\,i}{2}$	$\frac{nk_2}{n-k_1}$
χ_4	1	k_1-n	k_2	k_2	$\frac{k_1}{n-k_1}$

The Frame number is $\mathfrak{F}(\mathfrak{X})=n^2(n-k_1)^3$. Since $\{R_0, R_2, R_3\}$ is the closed subset, we note that $n-k_1|n$, and $n-k_1|k_1$. Then we can show the following theorem holds.

Theorem 9. *The structure of the modular adjacency algebra of an association scheme of case 3 is determined as follows.*

$$F\mathfrak{X}\cong\begin{cases}F[x, y]/(x^3, xy, y^2) & \text{if } p|n-k_1,\\ F[x]/(x^2)\oplus F\oplus F & \text{if } p|n,\ p\nmid n-k_1,\\ F\oplus F\oplus F\oplus F & \text{otherwise.}\end{cases}$$

where the first block from the left of the right hand sides is the principal block.

Proof. From the Frame number and the character table, it is enough to show the theorem for the case $p|n-k_1$ only. It follows that $F\mathfrak{X}$ is a 4-dimension local ring from the character table since $2\nmid 2k_2+1=n-k_1$. It is obvious that $J^*\in rad(F\mathfrak{X})$ and

$$(A_0^*+A_2^*+A_3^*)^2=(2k_2+1)(A_0^*+A_2^*+A_3^*)=0,$$
$$(A_2^*-A_3^*)^2=A_0^*+A_2^*+A_3^*.$$

Furthermore it follows that

$$\dim_F(A_0^*+A_2^*+A_3^*)F\mathfrak{X}=1,$$
$$(A_2^*-A_3^*)A_1=0,$$
$$(A_2^*-A_3^*)A_2=k_2(A_2^*-A_3^*)-k_2(A_0^*+A_2^*+A_3^*),$$
$$(A_2^*-A_3^*)A_3=k_2(A_0^*+A_2^*+A_3^*)-k_2(A_2^*-A_3^*).$$

Here $p\nmid k_2$ since $p|n-k_1=2k_2+1$. Hence $F\mathfrak{X}\cong F[x, y]/(x^3, xy, y^2)$. □

There exist association schemes for every case handled in the above theorem. Let us consider an association scheme as06[4] in the list [4]. Then we see, from the theorem, its modular adjacency algebra $F\mathfrak{X}$ is isomorphic to $F[x, y]/(x^3, xy, y^2)$ if $p=3$, and $F[x]/(x^2)\oplus F\oplus F$ if $p=2$.

Acknowledgement

The author is thankful to Akihide Hanaki for valuable suggestions and comments and to the referee for lots of helpful remarks and suggestions.

References

[1] Z. Arad, E. Fisman, and M. Muzychuk, "Generalized table algebras," Israel J. Math. **144**, 29–60, (1999).

[2] E. Bannai and T. Ito, Algebraic Combinatorics. I. Association Schemes, Benjamin-Cummings, Menlo Park, CA, 1984.

[3] A. Hanaki, "Semisimplicity of adjacency algebras of association schemes," J. Alg. **225**, 124–129, (2000).

[4] A. Hanaki and I. Miyamoto, http://kissme.shinshu-u.ac.jp/as/.

[5] A. Hanaki, M. Yoshikawa, "On modular standard modules of association schemes," J. Alg. Comb, **21** (3), 269–279, (2005).

[6] H. Nagao and Y. Tsushima, Representations of Finite Groups, Academic Press, San Diego CA, 1987.

[7] S. Y. Song, "Class 3 association schemes whose symmetrizations have two classes," Journal of Combinatorial Theory, Series A 70 (1995) No.1, 1–29.

[8] M. Yoshikawa, "Modular adjacency algebras of Hamming schemes," J. Alg. Comb. **20** (3), 331–340, (2004).

[9] P. -H. Zieschang, An Algebraic Approach to Association Schemes (Lecture Notes in Math. **1628**), Springer, Berlin-Heidelberg-New York, 1996.