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*The intersection of normal closed subsets of an association scheme is not always normal**

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Abstract

There has been a question that the intersection of normal closed subsets of an association scheme is also a normal closed subset. A. Hanaki proved the answer of this question is true for group-like association schemes [4]. We found counterexamples for this question by using GAP4 and a list of association schemes by A. Hanaki and I. Miyamoto [2] and [3].

1 Introduction

When we consider association schemes, we have three substructures, those are closed subsets, normal closed subsets and strongly normal closed subsets. If we consider finite groups as association schemes, closed subsets correspond to subgroups, and normal closed subsets and strongly normal closed subsets correspond to normal subgroups, respectively. We know closed subsets and strongly normal closed subsets are closed under taking the intersection like as subgroups and normal subgroups. However we do not know whether normal closed subsets are closed under taking the intersection or not. Even so we believe normal closed subsets are important and appropriate to call it normal since, for an association scheme (X, G) and each normal closed subset H of G, the element $n_H^{-1}\sum_{h\in H}\sigma_h$ of the adjacency algebra of (X, G) over the complex field is a central idempotent like as that of normal subgroups for finite groups [5].

When we consider only commutative association schemes, any closed subsets are normal. Thus the question makes sense for non-commutative association schemes. A. Hanaki defined a class of association schemes that are called group-like. The center of an adjacency algebra of it over the complex field has a basis described by a sum of the elements in a partition of G as well as the center of a group algebra over the

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complex field has a basis given by conjugacy classes of the group. Then he proved the answer is true for group-like association schemes [4]. Thus it remains to check for non-group-like association schemes.

2 Association Schemes

In this section, we recall the definition and some properties for association schemes (see [1] and [6] for more details). We will use notations of Zieschang [6].

Let X be a finite set and G a partition of $X \times X$. We call a pair $\mathfrak{X}=(X, G)$ an *association scheme* if the following conditions are satisfied:

(1) $1=\{(x, x)|x \in X\} \in G$.

(2) There exists $g' \in G$ such that $g'=\{(y, x)|(x, y) \in g\}$ for any $g \in G$.

(3) For every $g, h, k \in G$, there exists a non-negative integer p_{ghk} such that for all $y, z \in X$, $|\{x \in X|(y, x) \in g$ and $(x, z) \in h\}|=p_{ghk}$ if $(y, z) \in k$.

The elements of $\{p_{ghk}\}$ will be called *intersection numbers* of (X, G). We call a positive integer $p_{gg'1}$ the *valency* of g and denote it by n_g. We put $n_G=\sum_{g \in G} n_g$.

Let E and F be any subsets of G. We define the *complex product* EF of E and F by $EF:=\{g \in G|\sum_{e \in E}\sum_{f \in F} p_{efg} \neq 0\}$. We define $F':=\{f'|f \in F\}$ for each $F \subset G$. Then a non-empty subset F of G is said to be *closed* if $FF' \subset F$. A closed subset F is called *normal* if $\{g\}F=F\{g\}$ for $g \in G$. We put $n_F=\sum_{f \in F} n_f$ for a closed subset F of G.

Let $\mathfrak{X}=(X, G)$ be an association scheme. For each $g \in G$, we define a $|X| \times |X|$ matrix σ_g indexed by elements of X by

$$(\sigma_g)_{xy}:=\begin{cases} 1 & \text{if } (x, y) \in g, \\ 2 & \text{otherwise.} \end{cases}$$

Let J be the $|X| \times |X|$ all 1 matrix. Then clearly we have $\sum_{g \in G} \sigma_g=J$, and $\sigma_g \sigma_h=\sum_{k \in G} p_{ghk}\, \sigma_k$ for all $g, h \in G$.

We can naturally define an algebra from the above fact. For a commutative ring R with 1, we put $R\mathfrak{X}=\bigoplus_{g \in G} R\sigma_g$ as a matrix ring over R, and it will be called an *adjacency algebra* of $\mathfrak{X}$ over R. In particular, the adjacency algebra of an association scheme over a field of characteristic 0 is semisimple.

Let $\mathbb{C}\mathfrak{X}$ be the adjacency algebra of an association scheme $\mathfrak{X}$ over the complex number field. We denote the set of irreducible characters of $\mathbb{C}\mathfrak{X}$ by $Irr(\mathfrak{X})$. Since the adjacency algebra is defined as a matrix ring, we can consider a natural representation $\sigma_g \mapsto \sigma_g$. We call it a *standard representation*, and the character corresponding to it the *standard character*, and it will be denoted by $\gamma(\mathfrak{X})$. When $\gamma(\mathfrak{X})=\sum_{\chi \in Irr(\mathfrak{X})} m_\chi \chi$ be an irreducible decomposition of the standard character, we call m_χ the *multiplicity* of χ.

Let η be a character of an association scheme $\mathfrak{X}$. Put $K(\eta)=\{g \in G|\eta(\sigma_g)=n_g\eta(1)\}$, and $I(\eta)=\{\chi \in Irr(\mathfrak{X})|\chi(\sigma_g)=n_g\chi(1)$ for all $g \in K(\eta)\}$. Then the followings are known.

Lemma 1. [4, Lemma 3.1] *For a character η of G, we have $K(\eta)=\bigcap_\chi K(\chi)=K(\sum_\chi \chi)$, where χ runs over all irreducible constituents of η.*

Theorem 2. [4, Theorem 3.2] *Let η be a character of G. Then $K(\eta)$ is a closed subset of G.*

Theorem 3. [4, Theorem 3.4] *For a character η of G, $K(\eta)$ is a normal closed subset of G if and only if $\sum_{\chi\in I(\eta)} m_\chi \chi(1)=\frac{n_G}{n_{K(\eta)}}$.*

3 Main Result

Our major concern in the present paper is to give an answer to the following question :

Question 4. *Is the intersection of normal closed subsets again a normal closed subset ?*

We already know that the answer is true for group-like association schemes [4]. Therefore it is sufficient to check only non-group-like schemes. We worked out all normal closed subsets of all non-group-like schemes such that $|X|\leq 30$ in the list [2]. Then we found eight counterexamples of the question, which are illustrated hereinafter.

We use the symbol, like as16[186] for association schemes, which are symbols used in the list [2]. We denote $\{R_0, R_1, R_3\}$ by [0, 1, 3]. We will show in the case as16[186] only.

The relation matrix of as 16 [186] is the following ;

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 9 & 9 & 8 & 8 & 11 & 11 & 10 & 10 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 9 & 9 & 8 & 8 & 11 & 11 & 10 & 10 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 10 & 10 & 11 & 11 & 8 & 8 & 9 & 9 \\
5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 & 10 & 10 & 11 & 11 & 8 & 8 & 9 & 9 \\
6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 & 11 & 11 & 10 & 10 & 9 & 9 & 8 & 8 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 11 & 11 & 10 & 10 & 9 & 9 & 8 & 8 \\
8 & 8 & 9 & 9 & 11 & 11 & 10 & 10 & 0 & 1 & 2 & 3 & 6 & 7 & 4 & 5 \\
8 & 8 & 9 & 9 & 11 & 11 & 10 & 10 & 1 & 0 & 3 & 2 & 7 & 6 & 5 & 4 \\
9 & 9 & 8 & 8 & 10 & 10 & 11 & 11 & 2 & 3 & 0 & 1 & 4 & 5 & 6 & 7 \\
9 & 9 & 8 & 8 & 10 & 10 & 11 & 11 & 3 & 2 & 1 & 0 & 5 & 4 & 7 & 6 \\
11 & 11 & 10 & 10 & 8 & 8 & 9 & 9 & 6 & 7 & 4 & 5 & 0 & 1 & 2 & 3 \\
11 & 11 & 10 & 10 & 8 & 8 & 9 & 9 & 7 & 6 & 5 & 4 & 1 & 0 & 3 & 2 \\
10 & 10 & 11 & 11 & 9 & 9 & 8 & 8 & 4 & 5 & 6 & 7 & 2 & 3 & 0 & 1 \\
10 & 10 & 11 & 11 & 9 & 9 & 8 & 8 & 5 & 4 & 7 & 6 & 3 & 2 & 1 & 0
\end{pmatrix}.$$

And the character table is as follows;

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	m_x
χ_1	1	1	1	1	1	1	1	1	2	2	2	2	1
χ_2	1	1	1	1	1	1	1	1	-2	-2	-2	-2	1
χ_3	1	1	1	1	-1	-1	-1	-1	2	2	-2	-2	1
χ_4	1	1	1	1	-1	-1	-1	-1	-2	-2	2	2	1
χ_5	2	2	-2	-2	0	0	0	0	0	0	0	0	2
χ_6	1	-1	1	-1	1	-1	1	-1	0	0	0	0	2
χ_7	1	-1	1	-1	-1	1	-1	1	0	0	0	0	2
χ_8	1	-1	-1	1	1	-1	-1	1	0	0	0	0	2
χ_9	1	-1	-1	1	-1	1	1	-1	0	0	0	0	2

Once we obtain a character table of an association scheme, we are apprised of all normal closed subsets by Theorem 3. For example, $[0, 2, 4, 6]=K(\chi_6)$. Then since $I(\chi_6)=\{\chi_1, \chi_2, \chi_6\}$ and $m_{\chi_1}\chi_1(1)+m_{\chi_2}\chi_2(1)+m_{\chi_6}\chi_6(1)=4=16/4=n_G/n_{K(\chi 6)}$, $[0, 2, 4, 6]=K(\chi_6)$ is a normal closed subset.

All normal closed subsets are [0], [0, 1], [0, 2], [0, 3], [0, 1, 2, 3], [0, 2, 4, 6], [0, 2, 5, 7], [0, 3, 4, 7], [0, 3, 5, 6], [0, 1, 2, 3, 8, 9], [0, 1, 2, 3, 10, 11], [0, 1, 2, 3, 4, 5, 6, 7] and [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. However, $[0, 2, 4, 6]\cap[0, 3, 4, 7]=[0, 4]$, $[0, 2, 4, 6]\cap[0, 3, 5, 6]=[0, 6]$, $[0, 2, 5, 7]\cap[0, 3, 4, 7]=[0, 7]$, and $[0, 2, 5, 7]\cap[0, 3, 5, 6]=[0, 5]$, which means they are not normal closed subsets.

In a similar way, we can also show in the case as18[79], as24[452], [546], [647], [702], as30[175], and [225].

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References

[1] E. Bannai and T. Ito, Algebraic Combinatorics. I. Association Schemes, Benjamin-Cummings, Menlo Park, CA, 1984.

[2] A. Hanaki and I. Miyamoto, http://kissme.shinshu-u.ac.jp/as/.

[3] A. Hanaki and I. Miyamoto, "Classification of Association Schemes with 16 and 17 Vertices", Kyushu J. Math., vol. **52**, 1998, 383-395.

[4] A. Hanaki, "Characters of association schemes and normal closed subsets", Graphs Comb., **19** (3), 363-369, 2003.

[5] A. Hanaki, "Representations of association schemes and their factor schemes", Graphs Combin. **19** (2), 2003, 195-201.

[6] P. -H. Zieschang, An Algebraic Approach to Association Schemes (Lecture Notes in Math. **1628**), Springer, Berlin-Heidelberg-New York, 1996.