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Statistics in piling block games

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Abstract

Using a computer, we have computed exactly the probability to describe the statistical properties of a pile of blocks in the case that when a child puts a block on the topmost block its center of mass is shifted to the right or left by the unit length with equal probability. The block length is restricted into positive integers. Two tables are presented relating to the respective probability of entire falling and partial falling of the pile of blocks.

Probably everyone has experiences of playing piling block games in his/her childhood. In a previous paper [1], Iwasaki and one of the authors (K.H) have clarified that this familiar game shows a certain kind of the self-organized criticality [2] and has properties of scaling and universality [3].

Suppose that, after you have accumulated $(n-1)$ blocks, as soon as the nth block is placed atop the pile, a part of the pile or the entire pile of blocks falls. We call these events nth partial falling and nth entire falling, respectively. Of course the entire falling is included in the corresponding partial falling. The problem studied here is to calculate the probability, $P(n;L)$, of the nth partial falling and, $Q(n;L)$, of the nth entire falling for the blocks of length L. The average height of piles is given as $\sum_{n=1}^{\infty} n P(n+1;L)$. The probability of piling blocks up to nth step is given by $1-\sum_{m=1}^{n} P(m;L) = \sum_{m=n+1}^{\infty} P(m;L)$.

As shown in [1], the probability $P(n;L)$ has a scaling form, as an asymptotic form for large n and L,

$$P(n;L)=L^{-2}f(\frac{n}{L^2}), \tag{1}$$

with a scaling function $f(x)$. After our proposition, Blanchard and Hongler [4] have suggested, noting the analogy with the fitting problem of random walkers at moving boundaries, that $f(x)$ is the inverse Gaussian distribution [5], but there exist some discrepancies between their suggestion and numerical data [4, 6]. Therefore the problem remains to be open. In this paper, as an attempt to obtain the asymptotic form of $P(n;L)$, we try to calculate $P(n;L)$ and $Q(n;L)$ for positive integers of L. If we can derive some relations among $P(n;L)$, they should be useful to obtain the general term in an explicit form and from it an asymptotic form of $P(n;L)$ for large n and L will be easily given. We present two tables of $P(n;L)$ and $Q(n;L)$ computed numerically. In these we find unpredicted facts in the piling block game itself and propose interesting problems in relation to the combinatorics.

We define the problem as follows: Suppose we have blocks of length L which are placed one another upward. Here L is restricted into positive integers, as mentioned above. When piling, a child cannot place a block exactly atop the topmost block, thus, the center of mass is shifted to the right or left by the unit length with equal probability. It is noted that, if we use blocks of fixed length, the shift distance varies in propotional to the inverse of L. Let y_k be the one-dimensional coordinate of the center of mass of the block in the kth step and ξ_k be a random variable taking the values of $+1$ or -1 with equal probability, then

$$y_k = y_{k-1} + \xi_k \tag{2}$$

for $k \geq 1$. We assume $y_0 = 0$ and $\xi_1 = +1$ without loss of generality.

In order that the nth partial falling does not occur, the nth block has to be placed atop the $(n-1)$th block, that is, $y_{n-1} - L/2 \leq y_n \leq y_{n-1} + L/2$. In addition, the center of mass of the nth block and the $(n-1)$th block, that is, $(1/2)(y_n + y_{n-1})$, should be placed between $y_{n-2} - L/2$ and $y_{n-2} + L/2$. Similarly, the center of mass of three blocks, $(1/3)(y_n + y_{n-1} + y_{n-2})$, has to be in $[y_{n-3} - L/2,\ y_{n-3} + L/2]$, and so forth. Lastly, $y_0 - L/2 \leq (1/n)(y_n + y_{n-1} + y_{n-2} + \cdots + y_1) \leq y_0 + L/2$ must be satisfied. These conditions are summarized for $m = 1, 2, \cdots, n$ as

$$\left|(1/m)\sum_{k=0}^{m-1} y_{n-k} - y_{n-m}\right| \leq L/2. \tag{3}$$

Even if one of the n inequalities above is broken, a part of the pile falls. On the other hand, breaking only the inequality for $m = n$ corresponds to the entire falling.

Let us first study the case of the entire falling. Substituting eq. (2) into eqs. (3), we have

$$|\xi_k| \leq L/2, \qquad (k = 1, 2, \cdots, n), \tag{4}$$

$$|2\xi_k + \xi_{k+1}| \leq L, \qquad (k = 1, 2, \cdots, n-1), \tag{5}$$

$$|3\xi_k + 2\xi_{k+1} + \xi_{k+2}| \leq 3L/2, \qquad (k = 1, 2, \cdots, n-2), \tag{6}$$

....

$$|(n-1)\xi_k + (n-2)\xi_{k+1} + \cdots + 2\xi_{k+n-3} + \xi_{k+n-2}| \leq (n-1)L/2, \qquad (k = 1, 2). \tag{7}$$

Lastly the condition of the entire falling is written as

$$|n\xi_1+(n-1)\xi_2+(n-2)\xi_3+\cdots+2\xi_{n-1}+\xi_n|>nL/2. \tag{8}$$

Computing $Q(n;L)$ is equivalent to count the number $A(n;L)$ of sequences $\{\xi_1, \xi_2, \xi_3, \cdots, \xi_n\}$ which satisfy eq. (4)-eq. (8), that is, $Q(n;L)=2A(n;L)/2^n$. The prefactor 2 arises from the result that we have fixed the first step to $\xi_1=+1$. For example, $A(7;3)=1$ and $A(5;5)=2$ because only $\{1, 1, -1, 1, -1, 1, 1\}$ for the former and $\{1, 1, 1, 1, 1\}$ and $\{1, 1, 1, 1, -1\}$ for the latter satisfy the conditions, respectively.

With aid of a computer, we have counted $A(n;L)$ for $2\le n\le 27$ and $2\le L\le 25$ of which results are summarized in Table 1. Algorithm for computing is as follows. Note that $A(n;L)=0$ trivially for $n<L$ because even $\xi_1=\xi_2=\cdots=\xi_n=+1$ cannot satisfy eq. (8). First we memorize the sequences $\{\xi_1, \xi_2, \xi_3, \cdots, \xi_L\}$ which satisfy eq. (8). Next the sequences $\{\xi_1, \xi_2, \cdots, \xi_L, \xi_{L+1}\}$ including the subsequences $\{\xi_1, \xi_2, \cdots, \xi_L\}$ and $\{\xi_2, \xi_3, \cdots, \xi_{L+1}\}$ which identify with the sequences counted among $A(L;L)$ above are eliminated and examine that the remaining sequences satisfy eq. (8) for $n=L+1$, being then counted among $A(L+1;L)$ after the check. We repeat the similar procedures.

Interesting sequences are embedded in Table 1. For example, for odd L, $A(n;L)$ increases exponentially with oscillation of period two. On the other hand, for even L (except for $L=2$) it oscillates with period four and increases up to huge numbers. If $A(n;L)$ is larger than 2^k with a positive integer k, whether the pile falls entirely or not has been already decided at the $(n-k)$th step. The general term $A(n;L)$ is expected to be represented as a function of n and L. Let us investigate some simpler cases.

Case of $L=2$: For $n=2$, the condition which should be considered is only $|2\xi_1+\xi_2|>2$, which is satisfied by $\xi_1=\xi_2$, leading to $A(2,2)=1$. For $n\ge 3$, we have from eq. (5) such a relation as $\xi_k=-\xi_{k+1}$ for $k=1, 2, \cdots, n-1$, which leads for even n to

$$|n\xi_1-(n-1)\xi_1+(n-2)\xi_1-\cdots+2\xi_1-\xi_1|=n/2 \tag{9}$$

and for odd n to

$$|n\xi_1-(n-1)\xi_1+(n-2)\xi_1-\cdots-2\xi_1+\xi_1|=(n+1)/2, \tag{10}$$

respectively. Both of them do not satisfy eq. (8), then $A(n;2)=0$ for $n\ge 3$.

Case of $L=3$: Note that eqs. (4) and (5) are satisfied automatically. Let us study a matter by induction. For $n=3$ the inequality which should be considered is only $|3\xi_1+2\xi_2+\xi_3|>4.5$, which has a unique solution $\xi_1=\xi_2=\xi_3=+1$. Then $A(3;3)=1$. For $n=4$, there are three inequalities which should be considered such as

$$|3\xi_k+2\xi_{k+1}+\xi_{k+2}|\le 4.5, \qquad (k=1, 2) \tag{11}$$

$$|4\xi_1+3\xi_2+2\xi_3+\xi_4|>6. \tag{12}$$

Table 1 : Table of $A(n;L)$ for $2 \leq n \leq 27$ and $2 \leq L \leq 25$.

n/L	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	1	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	1	0	3	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	2	0	1	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	1	3	6	2	4	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	3	0	9	3	5	1	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	1	0	12	8	12	3	11	4	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	5	0	5	12	13	4	5	4	3	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	1	10	27	23	37	41	38	17	13	8	5	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	9	1	64	34	50	15	45	19	15	4	5	0	0	0	0	0	0	0	0	0	0	0
15	0	1	0	61	62	104	48	134	69	55	22	27	9	5	0	0	0	0	0	0	0	0	0	0
16	0	0	17	3	42	100	151	61	76	74	65	25	17	9	5	0	0	0	0	0	0	0	0	0
17	0	1	36	141	168	318	421	450	239	226	183	136	56	32	15	7	0	0	0	0	0	0	0	0
18	0	0	31	9	455	315	465	203	659	311	290	169	162	64	38	12	7	0	0	0	0	0	0	0
19	0	1	1	332	472	952	446	1479	876	914	389	612	279	190	74	60	19	7	0	0	0	0	0	0
20	0	0	63	25	324	923	1558	646	1002	1198	1133	440	408	328	220	84	44	19	7	0	0	0	0	0
21	0	1	135	793	1321	2870	4255	4977	3010	3355	3140	2654	1170	927	648	399	148	71	29	10	0	0	0	0
22	0	0	115	67	3567	2824	4954	2215	8078	4312	4653	1973	3382	1486	1168	469	463	170	83	23	10	0	0	0
23	0	1	7	1914	3684	8810	4878	16682	10365	11956	5973	10876	5676	4329	1860	2166	950	533	195	121	35	10	0	0
24	0	0	239	176	2612	8720	15650	7756	11873	15226	17108	8001	8230	6918	5211	2309	1704	1092	611	223	97	35	10	0
25	0	1	518	4661	10420	26782	42647	55093	37241	42885	44320	41750	22508	19555	14747	10742	4814	3188	1896	1011	345	143	50	14
26	0	0	441	459	28216	26516	51614	25892	98772	55093	62311	30863	59327	30486	26143	10596	13241	5986	3970	1542	1157	395	167	42
27	0	1	35	11435	29751	82184	51195	182962	129121	156802	79040	157316	91327	82406	40609	50869	24775	16604	7387	6595	2716	1321	452	231

It is clear from eqs. (11) that the sequences with such a relation as $\xi_k=\xi_{k+1}=\xi_{k+2}$ $(k=1, 2)$ should be eliminated. Since the sequence which satisfies eqs. (11) and makes the left hand side of eq. (12) maximum is $\xi_1=\xi_2=-\xi_3=\xi_4$, eq. (12) cannot be always satisfied. Thus $A(4;3)=0$. By similar considerations we have to exclude the sequences with following relations for even n,

$$\xi_k=\xi_{k+1}=\xi_{k+2}, \qquad (k=1, 2, \cdots, n-2) \tag{13}$$

$$\xi_k=\xi_{k+1}=-\xi_{k+2}=\xi_{k+3}=\xi_{k+4}, \qquad (k=1, 2, \cdots, n-4) \tag{14}$$

$$\xi_k=\xi_{k+1}=-\xi_{k+2}=\xi_{k+3}=-\xi_{k+4}=\xi_{k+5}=\xi_{k+6}, \qquad (k=1, 2, \cdots, n-6) \tag{15}$$

....

$$\xi_1=\xi_2=-\xi_3=\xi_4=-\xi_5=\cdots=-\xi_{n-2}=\xi_{n-1}=\xi_n. \tag{16}$$

Therefore we find no sequences that satisfy eq. (8), that is, $A(n;3)=0$ for even n. On the other hand, for odd n, only the sequence with the relation of eq. (16) satisfies eq. (8), leading to $A(n;3)=1$ for odd n.

Case of $n=L$: In this case, the inequality which we should take into account is only eq. (8). Let $W(L;\xi_k)$ denote the left hand of eq. (8). Since $W(L;\xi_k)$ takes its maximum value, $L(L+1)/2$, when $\xi_1=\xi_2=\cdots=\xi_L=+1$, changing signs of some random variables in $\{\xi_1, \xi_2, \cdots, \xi_L\}$ are allowed within bounds of $L(L+1)/2-L^2/2=L/2$. We define a positive integer N through such a condition as $4N<L\leq 4(N+1)$ for $L\geq 5$. Suppose that the sign of ξ_J with $J=L+1-M$ is replaced from $+1$ to -1. Equation (8) remains to be satisfied if M is a positive integer less than or equal to N $(M\leq N)$, because this replacement reduces $W(L;\xi_k)$ only by $2M$ $(<L/2)$. Moreover suppose that M is represented by a sum of $M_1, M_2, \cdots, M_s$ which are positive integers different from each other, $M=M_1+M_2+\cdots+M_s$. Further note that changing the sign of ξ_J is equivalent for $\xi_{J1}, \xi_{J2}, \cdots, \xi_{Js}$ to change all signs of them, where $J_k=L+1-M_k$ $(k=1, 2, \cdots, s)$. When L is increased by every 4, that is, N is increased by every 1, therefore, $A(L;L)$ is added by $g(N)$, where $g(N)$ is a number of such cases that N is represented by the sum of positive integers under the conditions that no integer can occur more than once as a part and the order of summands is neglected. We allow the sum to have only one term. For small positive integers, $g(1)=g(2)=1$, $g(3)=g(4)=2$, $g(5)=3$ and $g(6)=4$. In conclusion, $A(L;L)=1$ for $L\leq 4$ and $A(L;L)=1+\sum_M Mg(M)$ for $L\geq 5$, where the sum $\sum_M$ is taken from $M=1$ to $M=N$.

A partition of a positive integer is a way of writing it as a sum of positive integers, ignoring the order of the summands [7]. The subject of partitions has a long history beginning with G. W. von Leibniz (1646-1716) and L. Euler (1707-1783) and has not come only from within mathematics itself but also from the outside. It is known that $g(N)=h(N)$, the latter being the number of the partitions of N into odd parts. The above derivation of $A(L;L)$ suggests that the general term $A(n;L)$ can be evaluated through considering the partition under more complicated conditions.

Let us turn to the problem of the partial fallings. Computing $P(n;L)$ is much more difficult and needs much longer time than computing $Q(n;L)$. Using a computer, we have also evaluated $B(n;L)$ which is a number of the sequences satisfying eqs. (3) up to $n-1$ but breaking one of corresponding equations for n, thus, $P(n;L)=2B(n;L)/2^n$. Remember that ξ_1 has been fixed as $\xi_1=+1$. We note again that $B(n;L)=0$ if $n<L$. Therefore we first search the sequences $\{\xi_1, \xi_2, \cdots, \xi_L\}$ for fixed L satisfying all of eqs. (3) and another sequences breaking one of eqs. (3). The number of the latter sequences is just $B(L;L)$, which is equal to $A(L;L)$. Next we add $\xi_{L+1}=\pm 1$ to the former sequences and examine that a new sequence $\{\xi_1, \xi_2, \cdots, \xi_{L+1}\}$ does not satisfy one of eqs. (3). If so, the sequence is counted among $B(L+1;L)$. If not so, it is memorized to compute $B(n;L)$ for $n>L+2$ further. We repeat this procedure up to a desired integer, n.

The resulting $B(n;L)$ for $2\leq n\leq 27$ and $2\leq L\leq 16$ that we have computed are summarized in Table 2 and $P(n;L)$ for $L=3$, 4, 5 and 6 are displayed in Fig.1. Similar to the entire falling, $B(n;L)$ oscillates with period 2 for odd L and with period 4 for

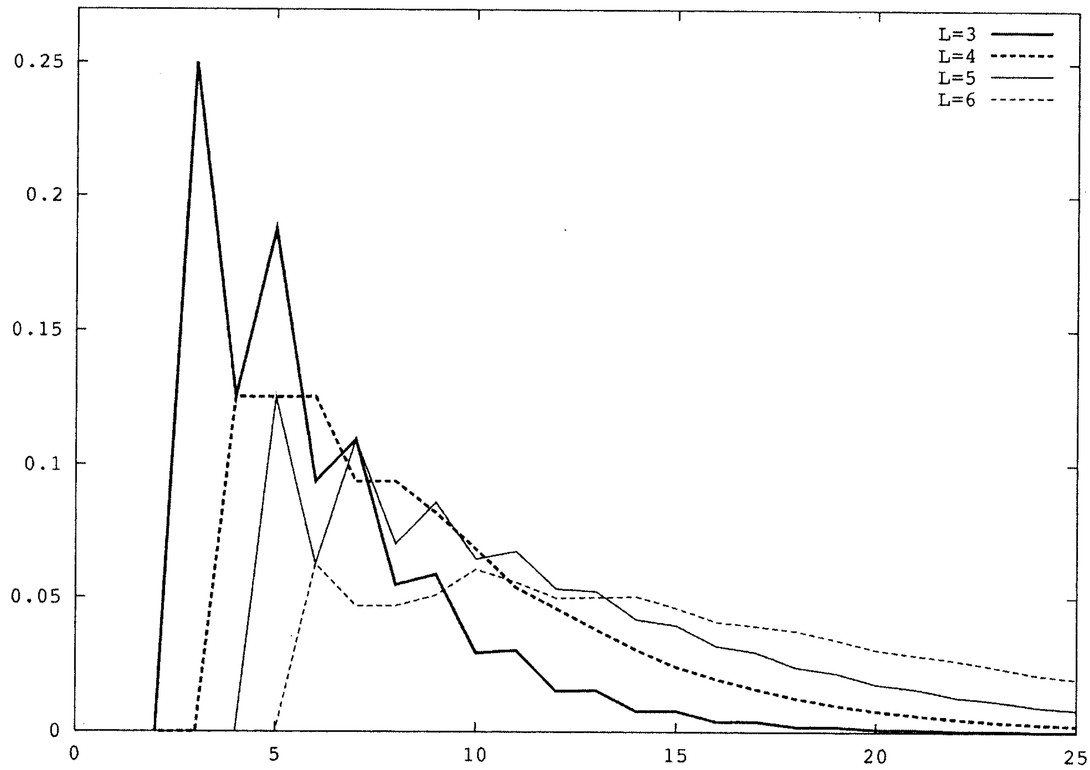

Figure 1 : Graphs of $P(n;L)$ for $L=3$ (bold solid curve), $L=4$ (bold broken curve), $L=5$ (thin solid curve) and $L=6$ (thin broken curve).

Table 2 : Table of $B(n;L)$ for $2 \leq n \leq 27$ and $2 \leq L \leq 16$.

n/L	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
5	1	3	2	2	0	0	0	0	0	0	0	0	0	0	0
6	1	3	4	2	2	0	0	0	0	0	0	0	0	0	0
7	1	7	6	7	3	2	0	0	0	0	0	0	0	0	0
8	1	7	12	9	6	3	2	0	0	0	0	0	0	0	0
9	1	15	21	22	13	9	5	3	0	0	0	0	0	0	0
10	1	15	35	33	31	17	12	4	3	0	0	0	0	0	0
11	1	31	55	69	57	41	22	18	7	3	0	0	0	0	0
12	1	31	94	109	102	78	52	29	15	7	3	0	0	0	0
13	1	63	156	214	205	169	129	89	42	23	11	5	0	0	0
14	1	63	250	343	413	319	253	148	109	52	29	9	5	0	0
15	1	127	395	649	755	665	480	405	238	137	65	41	14	5	0
16	1	127	644	1048	1345	1246	995	691	460	289	170	80	36	14	5
17	1	255	1040	1938	2580	2542	2163	1720	1041	715	450	266	111	51	20
18	1	255	1644	3145	4934	4723	4169	3008	2416	1449	990	501	323	134	63
19	1	511	2587	5722	8981	9437	7878	7055	4873	3389	1999	1478	754	389	162
20	1	511	4130	9315	15992	17477	15877	12493	9458	6732	4589	2685	1620	907	465
21	1	1023	6570	16743	29884	34480	32573	28321	19858	15111	10808	7465	3931	2381	1343
22	1	1023	10309	27312	55845	63693	62298	50514	43165	29821	22375	13498	9875	5242	3168
23	1	2047	16149	48632	100888	124341	117248	111466	85326	64994	44215	35063	21294	13098	6939
24	1	2047	25490	79442	179654	229263	230653	200657	164904	127778	95114	63846	43369	27839	16964
25	1	4095	40176	140400	330786	443463	460625	431971	338846	273492	210022	157454	97164	66368	42650
26	1	4095	62747	229577	609334	816861	877659	783868	708052	535508	425088	289714	223581	138081	93669
27	1	8191	97938	403265	1095936	1569380	1647932	1657859	1387309	1131536	834665	687090	465196	317844	194386

even L and increases exponentially as a whole. The similar oscillation are seen in $P(n;L)$, meaning that adding a block to the pile can occasionally make it steadier contrary to our experience. By taking a glance, we can see that $B(n;2)=1$ for $n \geq 2$ and $B(n;3)=2B(n-2;3)+1$ with $B(3;3)=B(4;3)=1$ for $n \geq 3$, the latter giving $B(n;3)=2^{(n-2)/2}-1$ for even n and $B(n;3)=2^{(n-1)/2}-1$ for odd n. However we have not succeeded in representing $B(n;L)$ in general as a function of n and L.

Provided that a positive integer k satisfies $B(n;L)>2^k$, such destiny of the pile that a partial falling occurs has been already decided before the $(n-k)$th step. If $B(n;L)>2^k>A(n;L)$, the destiny is replaced with the one that the entire falling does not occur but the partial falling does. Any way the destiny cannot be changed how to pile blocks afterwards.

In conclusion, we have proposed the novel idea that the simple play with piling blocks can be a subject of physics, mathematics and other related fields such as computer science. Two tables for the numbers of cases bringing about entire or partial fallings have been presented, which reveals some facts unpredicted from our experiences, However, deriving the equation for $A(n;L)$ is required to obtain an asymptotic form of $P(n;L)$. The authors expect to give rise further studies on this line.

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References

[1] S. Iwasaki and K. Honda ; J. Phys. Soc. Jpn. **69**, 1579 (2000).

[2] P. Bak, C. Tang, and K. Wiesenfeld ; Phys. Rev. Lett. **59**, 381 (1987).

[3] e.g. H. E. Stanley ; *Introduction to Phase Transition and Critical Phenomena* (Oxford, New York, 1971).

[4] P. Blanchard and M. Hongler ; J. Phys. Soc. Jpn. **71**, 9 (2002).

[5] V. Seshadri ; *The Inverse Gaussian Distribution-A Case Study in Exponential Families* (Oxford Science Publication, Oxford, 1994).

[6] T. Matsunaga ; master thesis of Shinshu University (unpublished).

[7] e.g. R. Stanley ; *Enumerative Combinatorics* Vol.1 (Wadsworth Inc., Belmont, California, 1986).