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Nilpotency Indices of the Radicals of Finite p-Solvable Group Algebras, III

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Abstract

In [2], we have classified the p-solvable groups G with $p^{m-2} < t(G) < p^{m-1}$ for p odd, where $t(G)$ is the nilpotency index of the (Jacobson) radical of $k[G]$, k a field of characteristic p, and p^m is the highest power of p dividing the order of G. In the paper cited above, we have given only an outline of the proof of the result for $p=3$ ([2, Theorem 11]). The aim of this paper is to give the complete proof of part (1) in the theorem.

1 Introduction

Let k be a field of characteristic $p>0$, and G a finite group whose order is divisible by p. The (Jacobson) radical of the group algebra $k[G]$ will be denoted by $J(k[G])$. As is well known, $J(k[G])$ is a nilpotent ideal. We denote by $t(G)$ the nilpotency index of $J(k[G])$, that is, $t(G)$ is the least positive integer t such that $J(k[G])^t=0$. Let p^m be the highest power of p dividing the order of G. In the case of p-groups, the groups G with $t(G) \geq p^{m-2}$ are already classified. On the other hand, for p-solvable groups, only the groups G with $t(G) \geq p^{m-1}$ are classified for an odd prime p. In [2], we classified the p-solvable groups G with $p^{m-2} < t(G) < p^{m-1}$ for an odd prime p under the assumption that $O_{p'}(G)=1$. Here we restate the result:

Theorem 1. *Suppose that G is p-solvable and $p \geq 5$. Then $p^{m-2} < t(G) < p^{m-1}$ if and only if Sylow p-subgroups of G are of exponent p^{m-2} or isomorphic to*

$$\langle a,b,c,d \mid a^5=b^5=c^5=d^5=1, [c,d]=b, [b,d]=a \rangle \ (p=5).$$

Further, in this case, G has p-length 1.

Theorem 2. *Let $p=3$ and $m \geq 3$. Suppose $3^{m-2} < t(G) < 3^{m-1}$. If the 3-length of G is greater than 1 then G has 3-length 2. Suppose further that $O_{3'}(G)=1$. Then $H=O_{3,3',3}(G)$*

is one of the groups of the following list :

(1) *a nonsplit extension of*

$$\langle a,b,c \mid a^{3^{m-3}}=b^3=c^3=1,[a,b]=1,[a,c]=1,[b,c]=a^{3^{m-4}}\rangle$$

by SL(2,3) ($m\geq 5$);

(2) *a split extension of* $C_9\times C_9$ *by* SL(2,3) ;

(3) *an extension of* $M(3)$ *by* SL(2,3) ;

(4) *an extension of* $C_3\times C_3\times C_3$ *by* SL(2,3) ;

(5) *a split extension of* $C_3\times C_3\times C_3$ *by* A_4 ; *and*

(6) *a nonsplit extension of* $C_{3^{m-3}}\times C_3\times C_3$ *by* SL(2,3) ($m\geq 5$).

In the presentation of the groups given in Theorem 1, all relations of the form $[x, y]=1$ (with x,y generators) are omitted. In the paper cited above, we have given the proof of Theorem 1 and an outline of the proof of Theorem 2.

Now let $p=3$ and G a finite 3-solvable group. Assume that G satisfies the inequality $3^{m-2}<t(G)<3^{m-1}$ and $O_{3'}(G)=1$. We have proved in [2] that the 3-part of $|G/O_3(G)|$ is 3. Hence the 3-length of G is 2, and so, because $t(G)=t(O_{3,3',3}(G))$, we may assume that $G=O_{3,3',3}(G)$. We already know that $O_3(G)$ is probably isomorphic to one of the following groups ([2, Lemma 4]):

(i) $C_{3^{m-3}}\times C_3\times C_3$.

(ii) $\langle a,b,c \mid a^{3^{m-3}}=b^3=c^3=1,[a,b]=1,[a,c]=1,[b,c]=a^{3^{m-4}}\rangle$.

(iii) $C_9\times C_9$

(iv) $M(3)\times C_3$.

(v) $M(3)$.

In [2], we proved that : if $O_3(G)\cong C_{3^{m-3}}\times C_3\times C_3$ then Theorem 2 (4), (5) or (6) holds; and if $O_3(G)\cong M(3)$ then Theorem 2 (3) holds. Further we proved in [2,3] that if $O_3(G)\cong C_9\times C_9$ then Theorem 2 (2) holds. In the case when $O_3(G)$ is a group given in (ii), we proved in [2] that if $O_3(G)$ does not have a complement then Theorem 2 (1) holds. Suppose $O_3(G)\cong M(3)\times C_3$, and set $O_3(G)=N_1\times N_2$, where $N_1\cong M(3)$ and $N_2\cong C_3$. Then a Sylow 3-subgroup of G is given by $\langle O_3(G),\sigma\rangle$, where $\sigma^3\in Z(O_3(G))=Z(N_1)\times N_2$. We proved in [2] that if $\sigma^3\in Z(N_1)$ then $t(G)<3^3$. Thus to complete the proof of Theorem 2, it suffices to prove the following :

(a) If $O_3(G)$ is a goup given in (ii) and has a complement then $t(G)<3^{m-2}$.

(b) If $O_3(G)=N_1\times N_2$ and $\sigma^3\notin Z(N_1)$ then $t(G)<3^3$.

To show these, we need somewhat complicated calculation. As for (a), this assertion has been proved in [2] for the case $|O_3(G)|>3^4$. In this paper, we shall give a complete proof of this assertion. The proof of (b) will be given in [4].

Our group G discussed here is given as follows : Set

$$N=\langle a,b,c \mid a^{3^{r-2}}=b^3=c^3=1,[a,b]=[a,c]=1,[b,c]=a^{3^{r-3}}\rangle,$$

where $r \geq 4$, and let $Q=\langle x,y\rangle$ be a quaternion group of order 8, and $W=\langle\sigma\rangle$ a cyclic group of order 3; and let $H=QW$ be a semidirect product of Q by W with respect to the action:

$$x^{\sigma}=x^{3}y,\ y^{\sigma}=x.$$

Now let $G=NH$ be a semidirect product of N by H with respect to the action:

$$\begin{array}{lll} a^{x}=a, & a^{y}=a, & a^{\sigma}=a, \\ b^{x}=c^{2}, & b^{y}=a^{2\cdot 3^{r-3}}bc^{2}, & b^{\sigma}=b, \\ c^{x}=b, & c^{y}=a^{3^{r-3}}b^{2}c^{2}, & c^{\sigma}=a^{3^{m-4}}bc. \end{array}$$

The aim of this paper is to prove the following:

Theorem. *Under the above notation, we have* $t(G)<3^{r-1}$.

We now set

$$f=x^{2}-1,\ \tau=f(1+x+y-xy)\sigma, T=\{f,\tau,\tau^{2}\}.$$

Then T is a group of order 3 with identity f. We now set

$$A=J(k[W])Q^{+}k[G],\ B=J(k[T])k[G],\ C=J(k[N])k[G],$$

where $Q^{+}=\sum_{g\in Q}g$. Then, because $G/\langle a\rangle\cong Qd(3)$, by Motose's result (see [2] Lemma 3), we have

$$J(k[G])=A+B+C.$$

By Jennings' formula (see [1], p. 311), we see that

$$t(C)=3^{r-2}+2(3-1)=3^{r-2}+4,$$

and so, if $(A+B)^{11}=0$, then

$$(A+B+C)^{3^{r-2}+14}=0,$$

and hence we have $t(G)\leq 3^{r-1}$ as desired. Hence to prove the theorem it suffices to prove that $A^{3}=0$ and $B^{9}=0$. We shall prove this assertion in Section three after giving preliminary results.

2 Preliminaries

Given a finite subset X of $k[G]$ and $g\in G$, we set $X^{+}=\sum_{x\in X}x$ and $g^{+}=\sum_{x\in\langle g\rangle}x$ respectively. We here give three lemmas which will be used in the proof of our theorem.

Lemma 1. σ acts trivially on the k-space $Q^{+}k[N]Q^{+}$.

Proof. $Q^{+}k[N]Q^{+}$ is a k-space generated by

$$\{Q^{+}\Delta^{+} \mid \Delta \text{ is a } Q\text{-orbit of } N\}.$$

We easily see that each Q-orbit is invariant under the action of σ, and the result follows.

We now set $S=\langle a^{3^{r-3}},b,c\rangle$ and $D=J(k[S])$.

Lemma 2.

(1) $(f-\tau)k[N](f-\tau)\subset k[\langle a\rangle](f-\tau)D^2(f-\tau)+k[\langle a\rangle]T^{+}$.

(2) $(f-\tau)k[N]T^{+}\subset k[\langle a\rangle](f-\tau)D^2T^{+}$.

(3) $T^{+}k[N](f-\tau)\subset k[\langle a\rangle]T^{+}D^2(f-\tau)$.

(4) $T^{+}k[N]T^{+}\subset k[\langle a\rangle]T^{+}D^2T^{+}$.

Proof. (1) Let $u\in N$. Then it suffices to prove that

$$(f-\tau)u(f-\tau)\in k[\langle a\rangle](f-\tau)D^2(f-\tau)+k[\langle a\rangle]T^{+}.$$

If $u\in\langle a\rangle$, then u is a central element, and so we have

$$(f-\tau)u(f-\tau)=uT^{+}\in k[\langle a\rangle]T^{+}.$$

Assume next that $u\notin\langle a\rangle$. Then $u=a^k v$, where $v=b^ic^j$, $(i,j)\neq(0,0)$ and

$$(f-\tau)u(f-\tau)=a^k(f-\tau)v(f-\tau).$$

We note that

$$\begin{aligned} fvf &= (x^2-1)b^ic^j(x^2-1) \\ &= -(b^ic^j+b^{2i}c^{2j})f=-f(b^ic^j+b^{2i}c^{2j})f. \end{aligned}$$

Suppose first $v\in\langle b\rangle$ or $\langle c\rangle$. Then we have $fvf=-f(v+v^2)f$ by the above, and so

$$\begin{aligned} (f-\tau)v(f-\tau) &= -(f-\tau)(v+v^2)(f-\tau) \\ &= -(f-\tau)(1+v+v^2)(f-\tau)+(f-\tau)^2 \\ &= -(f-\tau)v^{+}(f-\tau)+T^{+}. \end{aligned}$$

Hence

$$(f-\tau)u(f-\tau)\in k[\langle a\rangle](f-\tau)D^2(f-\tau)+k[\langle a\rangle]T^{+}.$$

We next suppose $v\notin\langle b\rangle$ and $v\notin\langle c\rangle$. Then $v=b^ic^j$ with $(i,j)=(1,1)$, $(2,1)$, $(1,2)$ or $(2,2)$, and we have

$$\begin{aligned} (f-\tau)v(f-\tau) &= -(f-\tau)(b^ic^j+b^{2i}c^{2j})(f-\tau) \\ &= -(f-\tau)(1+b^ic^j+b^{2i}c^{2j})(f-\tau)+(f-\tau)^2 \\ &= -(f-\tau)(1+b^ic^j+b^{2i}c^{2j})(f-\tau)+T^{+}. \end{aligned}$$

Further we have $(b^ic^j)^2=a^{l\cdot 3^{r-3}}b^{2i}c^{2j}$, where $l=1$ if $i\neq j$ and 2 if $i=j$. Hence

$$1+b^ic^j+b^{2i}c^{2j}=(1+b^ic^j+(b^ic^j)^2)+(1-a^{l\cdot 3^{r-3}})b^{2i}c^{2j}.$$

Thus we get

$$\begin{aligned}(f-\tau)v(f-\tau)&=-(f-\tau)(v^++(1-a^{l\cdot 3^{r-3}})b^{2i}c^{2j})(f-\tau)+T^+\\&=-(f-\tau)v^+(f-\tau)-(1-a^{l\cdot 3^{r-3}})(f-\tau)b^{2i}c^{2j}(f-\tau)+T^+.\end{aligned}$$

But because $(f-\tau)b^{2i}c^{2j}(f-\tau)=(f-\tau)b^ic^j(f-\tau)$, we have

$$(f-\tau)v(f-\tau)=-(f-\tau)\,v^+(f-\tau)-(1-a^{l\cdot 3^{r-3}})(f-\tau)v(f-\tau)+T^+.$$

From this we get

$$(1+a^{l\cdot 3^{r-3}})(f-\tau)v(f-\tau)=(f-\tau)v^+(f-\tau)-T^+.$$

Because $1+a^{l\cdot 3^{r-3}}$ is a unit in $k[\langle a^{3^{r-3}}\rangle]$, there exists $\alpha\in k[\langle a^{3^{r-3}}\rangle]$ such that

$$(f-\tau)v(f-\tau)=\alpha((f-\tau)v^+(f-\tau)+T^+).$$

This shows that

$$(f-\tau)u(f-\tau)\in k[\langle a\rangle](f-\tau)D^2(f-\tau)+k[\langle a\rangle]T^+.$$

Thus (1) is proved. (2)—(4) follow directly from (1).

Since S is a extra special 3-group of order 3^3 and exponent 3, $t(S)=9$ and $S^+=D^8$ is a central element of $k[G]$.

Lemma 3. (1) $(f-\tau)D^8k[G](f-\tau)\subset S^+T^+k[G]$.

(2) $(f-\tau)D^8k[G]T^+=0$.

(3) $T^+D^8k[G](f-\tau)=0$.

Proof. (1) By using the fact noted above, we have

$$\begin{aligned}(f-\tau)D^8k[G](f-\tau)&\subset S^+(f-\tau)k[N](f-\tau)k[G]\\&\subset S^+(f-\tau)D^2(f-\tau)k[G]+S^+T^+k[G]\quad \text{(Lemma 2 (1))}\\&=(f-\tau)S^+D^2(f-\tau)k[G]+S^+T^+k[G]\\&=S^+T^+k[G].\end{aligned}$$

(2) By (1), we have

$$\begin{aligned}(f-\tau)D^8k[G]T^+&\subset S^+T^+k[G](f-\tau)\\&\subset S^+T^+k[N](f-\tau)k[G]\\&\subset S^+T^+D^2(f-\tau)k[G]\quad \text{(Lemma 2 (3))}\\&=T^+S^+D^2(f-\tau)k[G]\\&=0.\end{aligned}$$

(3) follows from (1) directly.

3 Proof of Theorem

As stated in Section one, our theorem is deduced from the following:

Lemma 4. $A^3=0$ and $B^9=0$.

Proof. We first prove that $A^3=0$. A^2 is given by

$$\begin{aligned} A^2 &= J(k[W])Q^+k[G]\cdot J(k[W])Q^+k[G] \\ &= J(k[W])Q^+k[N]J(k[W])Q^+k[G] \\ &= J(k[W])\cdot Q^+k[N]Q^+\cdot J(k[W])k[G]. \end{aligned}$$

Because σ acts trivially on $Q^+k[N]Q^+$ (Lemma 1), we have

$$\begin{aligned} A^2 &= J(k[W])^2\cdot Q^+k[N]Q^+\cdot k[G] \\ &\subset J(k[W])^2Q^+k[G], \end{aligned}$$

and consequently

$$\begin{aligned} A^3 &\subset J(k[W])^2Q^+k[G]\cdot J(k[W])^2Q^+k[G] \\ &= J(k[W])^2\cdot Q^+k[N]Q^+\cdot J(k[W])k[G] \\ &= J(k[W])^3\cdot Q^+k[N]Q^+\cdot k[G] \\ &= 0. \end{aligned}$$

We next prove that $B^9=0$. We begin with the following:

(a) $\quad B^2\subset(f-\tau)D^2(f-\tau)k[G]+T^+k[G].$

Because f and τ are central elements of $k[QW]$, we have

$$\begin{aligned} B^2 &= J(k[T])k[G]\cdot J(k[T])k[G] \\ &= J(k[T])k[N]\cdot J(k[T])k[G] \\ &= (f-\tau)k[N](f-\tau)k[G]. \end{aligned}$$

Thus (a) follows by Lemma 2 (1). We next show that

(b) $\quad B^4\subset T^+D^2T^+k[G]+T^+D^4(f-\tau)k[G]+(f-\tau)D^4T^+k[G]$
$\quad +(f-\tau)D^2T^+D^2(f-\tau)k[G]+(f-\tau)D^6(f-\tau)k[G].$

By (a), we have

$$\begin{aligned} B^4 \subset\ & (f-\tau)D^2(f-\tau)k[G]\cdot(f-\tau)D^2(f-\tau)k[G] \\ & +(f-\tau)D^2(f-\tau)k[G]\cdot T^+k[G]+T^+k[G]\cdot(f-\tau)D^2(f-\tau)k[G] \\ & +T^+k[G]\cdot T^+k[G] \\ =\ & (f-\tau)D^2(f-\tau)k[N](f-\tau)D^2(f-\tau)k[G] \\ & +(f-\tau)D^2(f-\tau)k[N]T^+k[G] \\ & +T^+k[N](f-\tau)D^2(f-\tau)k[G] \\ & +T^+k[N]T^+k[G] \end{aligned}$$

Now we calculate the first term:

$$\begin{aligned}&(f-\tau)D^2(f-\tau)k[N](f-\tau)D^2(f-\tau)k[G]\\ &\quad\subset(f-\tau)D^2((f-\tau)D^2(f-\tau)k[\langle a\rangle]+T^+k[\langle a\rangle])D^2(f-\tau)k[G] \quad \text{(Lemma 2 (1))}\\ &\quad=(f-\tau)D^2(f-\tau)D^2(f-\tau)D^2(f-\tau)k[G]+(f-\tau)D^2T^+D^2(f-\tau)k[G]\\ &\quad\subset(f-\tau)D^6(f-\tau)k[G]+(f-\tau)D^2T^+D^2(f-\tau)k[G].\end{aligned}$$

We next calculate the second term:

$$\begin{aligned}&(f-\tau)D^2(f-\tau)k[N]T^+k[G]\\ &\quad\subset(f-\tau)D^2(f-\tau)D^2T^+k[G] \quad \text{(Lemma 2 (2))}\\ &\quad\subset(f-\tau)D^4T^+k[G].\end{aligned}$$

We show that the third term is contained in $T^+D^4(f-\tau)k[G]$:

$$\begin{aligned}&T^+k[N](f-\tau)D^2(f-\tau)k[G]\\ &\quad\subset T^+D^2(f-\tau)D^2(f-\tau)k[G] \quad \text{(Lemma 2 (3))}\\ &\quad\subset T^+D^4(f-\tau)k[G].\end{aligned}$$

Finally, by Lemma 2 (4), we easily see that the forth term in contained in $T^+D^2T^+k[G]$. Thus (b) is proved. We now show, by using (b), the following:

(c) $B^8\subset T^+D^6T^+k[G]+S^+T^+k[G]$.

By (b), $B^4\subset\sum_{i=1}^5 X_i$, where

$X_1 = T^+D^2T^+k[G]$,

$X_2 = T^+D^4(f-\tau)k[G]$,

$X_3 = (f-\tau)D^4T^+k[G]$,

$X_4 = (f-\tau)D^2T^+D^2(f-\tau)k[G]$,

$X_5 = (f-\tau)D^6(f-\tau)k[G]$.

We easily see that X_iX_5 and X_5X_i $(2\leq i\leq 5)$ are contained in $D^{10}k[G]$, and so each of them is equal to 0. Further, X_1X_5, X_5X_1 and X_iX_j $(2\leq i,j\leq 4)$ are contained in $(f-\tau)D^8(f-\tau)k[G]$, and so, by Lemma 3 (1), each of them is contained in $S^+T^+k[G]$. Hence, it suffices to prove the following:

(i) $X_1^2\subset T^+D^6T^+k[G]$.

(ii) $X_1X_i=0$ for $i=2, 3, 4$.

(iii) $X_iX_1=0$ for $i=2, 3, 4$.

We now prove (i):

$$\begin{aligned}X_1^2 &= T^+D^2T^+k[G]\cdot T^+D^2T^+k[G]\\ &= T^+D^2T^+k[N]T^+D^2T^+k[G]\\ &\subset T^+D^2T^+D^2T^+D^2T^+k[G] \quad \text{(Lemma 2 (4))}\\ &\subset T^+D^6T^+k[G].\end{aligned}$$

We next prove (ii):

$$\begin{aligned} X_1X_2 &= T^+D^2T^+k[G]\cdot T^+D^4(f-\tau)k[G] \\ &= T^+D^2T^+k[N]T^+D^4(f-\tau)k[G] \\ &\subset T^+D^2T^+D^2T^+D^4(f-\tau)k[G] \quad \text{(Lemma 2 (4))} \\ &\subset T^+D^8k[G](f-\tau)k[G]=0. \quad \text{(Lemma 3 (3))} \end{aligned}$$

By a similar argument, we have $X_1X_3=0$ and $X_1X_4=0$. Further (iii) is also proved similarly.

Now we have $B^9=0$ as a direct consequence of (c). In fact, we have

$$\begin{aligned} B^9 &\subset (T^+D^6T^+k[G]+S^+T^+k[G])J(k[T])k[G] \\ &= T^+D^6T^+k[N](f-\tau)k[G]+S^+T^+k[N](f-\tau)k[G] \\ &= T^+D^6T^+D^2(f-\tau)k[G]+S^+T^+D^2(f-\tau)k[G] \\ &\subset T^+D^8(f-\tau)k[G]=0, \end{aligned}$$

as desired.

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