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Nilpotency Indices of the Radicals of Finite p-Solvable Group Algebras, IV

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Abstract

In [2], we have classified the p-solvable groups G with $p^{m-2} < t(G) < p^{m-1}$ for p odd, where $t(G)$ is the nilpotency index of the (Jacobson) radical of $k[G]$, k a field of characteristic p, and p^m is the highest power of p dividing the order of G. In the paper cited above, we have given only an outline of the proof of the result for $p=3$ ([2, Theorem 11]). To complete the proof of the theorem, we need somewhat complicated calculation, and we have given in [3] and [4] two parts of such caluculations. The aim of this paper is to give one more such calculation and complete the proof of the theorem.

1 Introduction

Let k be a field of characteristic $p>0$, and G a finite p-solvable group whose order is divisible by p. We denote by $t(G)$ the nilpotency index of the (Jacobson) radical of the group algebra $k[G]$. In [2], we described a classification of G which satisfies $p^{m-2} < t(G) < p^{m-1}$ for p odd, where p^m is the highest power of p dividing the order of G. Here we restate the result :

Theorem 1. *Suppose that G is p-solvable and $p \geq 5$. Then $p^{m-2} < t(G) < p^{m-1}$ if and only if Sylow p-subgroups of G are of exponent p^{m-2} or isomorphic to*

$$\langle a,b,c,d \mid a^5=b^5=c^5=d^5=1, [c,d]=b, [b,d]=a \rangle \ (p=5).$$

Further, in this case, G has p-length 1.

Theorem 2. *Let $p=3$ and $m \geq 3$. Suppose $3^{m-2} < t(G) < 3^{m-1}$. If the 3-length of G is greater than 1 then G has 3-length 2. Suppose further that $O_{3'}(G)=1$. Then $H=O_{3,3',3}(G)$ is one of the groups of the following list :*

(1) *a nonsplit extension of*

$$\langle a,b,c \mid a^{3^{m-3}}=b^3=c^3=1, [a,b]=1, [a,c]=1, [b,c]=a^{3^{m-4}}\rangle$$

by SL(2,3) ($m \geq 5$) ;

(2) *a split extension of* $C_9 \times C_9$ *by* SL(2,3) ;

(3) *an extension of* $M(3)$ *by* SL(2,3) ;

(4) *an extension of* $C_3 \times C_3 \times C_3$ *by* SL(2,3) ;

(5) *a split extension of* $C_3 \times C_3 \times C_3$ *by* A_4 ; *and*

(6) *a nonsplit extension of* $C_{3^{m-3}} \times C_3 \times C_3$ *by* SL(2,3) ($m \geq 5$).

In the paper cited above, we have given the proof of Theorem 1 and an outline of the proof of Theorem 2. To give a complete proof of Theorem 2, we need somewhat complicated calculation, and we have given in [3] and [4] two parts of such calculations. In this paper, we shall give one more such calculation, and in consequence we complete the proof of the theorem.

In what follows, we assume that $p=3$. Let G be a 3-solvable group such that $O_3(G)$ is the direct product of a extra-special 3-group, say N_1, of order 3^3 and exponent 3 and a cyclic group, say N_2, of order 3 and $G/O_3(G) \cong SL(2,3)$. We set

$$N_1 = \langle a,b,c \mid a^3=b^3=c^3=1, [a,b]=c, [a,c]=[b,c]=1\rangle,$$
$$N_2 = \langle s \mid s^3=1\rangle.$$

Further we let

$$Q=\langle x, y \mid x^4=1,\ x^2=y^2,\ y^{-1}xy=x^{-1}\rangle$$

be a Sylow 2-subgroup of G. Then we can choose an element σ of G-NQ such that $G=\langle NQ, \sigma\rangle$ and $\sigma^3 \in Z(N)$.

We already proved in [2] that $t(G) < 3^3$ for the case when $\sigma^3 \in Z(N_1)$. We now assume that $\sigma^3 \notin Z(N_1)$. Then we may assume that $\sigma^3=s$. Further we may assume that the elements a,b,c,s,x,y,σ satisfy the following (see [2]) :

$$(*)\quad \begin{cases} a^x=b^2, & b^x=ac, & c^x=c, & s^x=s, \\ a^y=ab^2c, & b^y=a^2b^2, & c^y=c, & s^y=s, \\ a^\sigma=a, & b^\sigma=ab, & c^\sigma=c, & s^\sigma=s. \end{cases}$$

As explained in [4, Introduction], to complete the proof of Theorem 2, it suffices to prove the following :

Theorem. $t(G) \leq 3^3$ *for the group* G *given above.*

Because $G/\langle z,s\rangle \cong Qd(3)$, $J(k[G])$ is given as follows :

$$J(k[G])=A+B+C \quad \text{(see [2], Lemma 3)},$$

where

$$A=J(k[\langle\sigma\rangle])Q^{+}k[G],\ B=J(k[T])k[G],\ C=J(k[N])k[G],$$
$$T=\{f,\tau,\tau^{2}\},$$
$$f=x^{2}-1,\ \tau=\sigma(1+x+y-xy)f.$$

Because $C^{11}=0$, if we can prove that $(A+B)^{16}=0$, then the result follows. This equality will be proved in Section three. In Section two we shall give the preliminary results. We shall use the notation given in [2].

2 Preliminaries

Under the notation given in Section one, we state several preliminary results.

Lemma 1. (1) *f is a central element of $k[\langle Q,\sigma\rangle]$.*
(2) *$fuf=-f(u+u^{x^{2}})=-(u+u^{x^{2}})f=fu^{x^{2}}f$ for all $u\in N$.*
(3) *τ is a central element of $k[\langle Q,\sigma\rangle]$.*
(4) *$\tau^{3}=sf$.*
(5) *$(f-\tau)u(f-\tau)=-(f-\tau)(1+u+u^{a^{2}})(f-\tau)+(f-\tau)^{2}$ for all $u\in N$.*
(6) *$(f-\tau)^{3}=(1-s)f$.*
(7) *$J(k[\langle c\rangle]\subset J(k[N_{1}])^{2}$.*
(8) *$1+u+u^{x^{2}}\in J(k[N_{1}])^{2}k[N_{2}]+J(k[N_{2}])$ for all $u\in N$.*
(9) *$(f-\tau)k[N](f-\tau)\subset(f-\tau)J(k[N_{1}])^{2}(f-\tau)k[N_{2}]+(f-\tau)^{2}k[N_{2}]$.*

Proof. It is esay to see the validity of (1)-(6) (see [1], p. 431).
(7) follows from the equality:

$$\begin{aligned} c-1 &= x^{-1}y^{-1}xy-1\\ &= (x^{-1}y^{-1}-1)(xy-1)+(x-1)(y-1)\\ &\quad +(x^{-1}-1)(y^{-1}-1)+(x-1)^{2}+(y-1)^{2}. \end{aligned}$$

(8) If $u=c$ or c^{2} then $1+u+u^{x^{2}}=1-c$, and so the result follows from (7). If $u=a$ or $a^{2}c^{2}$ then

$$1+u+u^{x^{2}}=1+a+a^{2}c^{2}=(a-1)^{2}-a^{2}(1-c^{2}),$$

and again by (7) the result follows. We can show similarly that the result holds for the other elements of N_{1}. If $u\in N_{2}$ then evidently $1+u+u^{x^{2}}=1-u\in J(k[N_{2}])$. Finally, if $u=vs^{i}$, where $v\in N_{1}-\{1\}$ then

$$1+u+u^{x^{2}}=(1+v+v^{x^{2}})s^{i}\in J(k[N_{1}])^{2}k[N_{2}],$$

and the result follows.
(9) Let $u\in N$. Then by (5) and (8) we have

$$(f-\tau)u(f-\tau)\in(f-\tau)(J(k[N_{1}])^{2}k[N_{2}]+J(k[N_{2}]))(f-\tau)+(f-\tau)^{2}.$$

Because $k[N_{2}]$ is contained in the center of $k[G]$, the right-hand term is equal to

$$(f-\tau)J(k[N_1])^2(f-\tau)k[N_2]+(f-\tau)^2J(k[N_2])+(f-\tau)^2$$
$$=(f-\tau)J(k[N_1])^2(f-\tau)k[N_2]+(f-\tau)^2Jk[N_2],$$

and the result follows.

In what follows, we set $D=J(k[N_1])$. By a result of Jennings (see [1], p. 311), we have

$\dim D/D^2=2$, $\dim D^2/D^3=4$, $\dim D^3/N_1^4=4$, $\dim D^4/D^5=5$,

$\dim D^5/D^6=4$, $\dim D^6/D^7=4$, $\dim D^7/D^8=2$.

We now set $\alpha=a-1$, $\beta=b-1$, $\gamma=c-1$.

Lemma 2. *The power structure of D is as follows:*

$$\begin{aligned}
D &= \langle \alpha, \beta, D^2\rangle\\
D^2 &= \langle \alpha^2, \beta^2, \alpha\beta, \beta\alpha, D^3\rangle\\
D^3 &= \langle \alpha^2\beta, \alpha\beta^2, \alpha\gamma, \beta\gamma, D^4\rangle\\
D^4 &= \langle \alpha^2\beta^2, \alpha^2\gamma, \beta^2\gamma, \alpha\beta\gamma, \gamma^2, D^5\rangle\\
D^5 &= \langle \alpha^2\beta\gamma, \alpha\beta^2\gamma, \alpha\gamma^2, \beta\gamma^2, D^6\rangle\\
D^6 &= \langle \alpha^2\beta^2\gamma, \alpha^2\gamma^2, \beta^2\gamma^2, \alpha\beta\gamma, D^7\rangle\\
D^7 &= \langle \alpha\beta^2\gamma^2, \alpha^2\beta\gamma^2, D^8\rangle\\
D^8 &= \langle \alpha^2\beta^2\gamma^2\rangle
\end{aligned}$$

The following is a direct consequence of (*).

Lemma 3. *The elements σ, x, y act trivially on γ and act on α, β as follows:*

$$\begin{aligned}
&\alpha^\sigma\equiv\alpha, && \beta^\sigma\equiv\alpha+\beta && (\mathrm{mod}\ D^2),\\
&\alpha^x\equiv-\beta, && \beta^x\equiv\alpha && (\mathrm{mod}\ D^2),\\
&\alpha^y\equiv\alpha-\beta, && \beta^y\equiv-\alpha-\beta && (\mathrm{mod}\ D^2).
\end{aligned}$$

In what follows, D^0 means $k[N_1]$.

Lemma 4. *Let $0\le i\le 4$. Then:*

(1) $Q^+D^{2i}f=Q^+D^{2i+1}f\subset Q^+D^{2i+1}$,

(2) $fD^{2i}Q^+=fD^{2i+1}Q^+\subset D^{2i+1}Q^+$.

Proof. (1) The right-hand inclusion is clear. Now let $u\in N_1$. Then $Q^+uf=Q^+(u-1)f$ because $Q^+f=0$. Hence $Q^+uf\in Q^+Df$, and so the assertion holds for $i=0$. Further by Lemma 3, we have $Q^+\lambda f\in Q^+D^3$, for $\lambda=\alpha^2, \beta^2, \alpha\beta, \beta\alpha$. Hence, by Lemma 2, the assertion holds for $i=1$. By a similar method, one can show that the assertion holds for $i=2, 3, 4$.

(2) is proved similarly.

Lemma 5. *Let $0\le i\le 4$. Then*

(1) $Q^+D^{2i}(f-\tau)=Q^+D^{2i+1}(f-\tau)\subset J(k[\langle\sigma\rangle])Q^+D^{2i+1}+k[\langle\sigma\rangle]Q^+D^{2i+3}$.

(2) $(f-\tau)D^{2i}Q^+=(f-\tau)D^{2i+1}Q^+\subset D^{2i+1}Q^+J(k[\langle\sigma\rangle])+D^{2i+3}Q^+k[\langle\sigma\rangle]$.

Proof. (1) The left-hand equality is a direct consequence of Lemma 4, and so it will suffice to prove the following:

(i) $Q^+\lambda(f-\tau)\in J(k[\langle\sigma\rangle])Q^+D+k[\langle\sigma\rangle]Q^+D^3$ for $\lambda\in\{\alpha, \beta\}$,

(ii) $Q^+\lambda(f-\tau)\in J(k[\langle\sigma\rangle])Q^+D^3+k[\langle\sigma\rangle]Q^+D^5$ for $\lambda\in\{\alpha^2\beta, \alpha\beta^2, \alpha\gamma, \beta\gamma\}$,

(iii) $Q^+\lambda(f-\tau)\in J(k[\langle\sigma\rangle])Q^+D^5+k[\langle\sigma\rangle]Q^+D^7$ for $\lambda\in\{\alpha^2\beta\gamma, \alpha\beta^2\gamma, \alpha\gamma^2, \beta\gamma^2\}$,

(iv) $Q^+\lambda(f-\tau)\in J(k[\langle\sigma\rangle])Q^+D^7$ for $\lambda\in\{\alpha\beta^2\gamma^2, \alpha^2\beta\gamma^2\}$,

We first prove (i). We have

$$Q^+\alpha(f-\tau)=Q^+\alpha f(1-\sigma)-Q^+\alpha f(x+y-xy)\sigma.$$

Obviously

$$Q^+\alpha f(1-\sigma)=(1-\sigma)Q^+\alpha f\in J(k[\langle\sigma\rangle])Q^+D,$$

and

$$Q^+\alpha f(x+y-xy)=Q^+(\alpha^x+\alpha^y-\alpha^{xy})f.$$

Further by Lemma 3,

$$\alpha^x+\alpha^y-\alpha^{xy}\equiv 0 \pmod{D^2},$$

and so

$$Q^+\alpha f(x+y-xy)\in Q^+D^2f=Q^+D^3 \quad \text{(Lemma 4)}.$$

Thus we have

$$Q^+\alpha(f-\tau)\in J(k[\langle\sigma\rangle])Q^+D+k[\langle\sigma\rangle]Q^+D^3.$$

To show

$$Q^+\beta(f-\tau)\in J(k[\langle\sigma\rangle])Q^+D+k[\langle\sigma\rangle]Q^+D^3,$$

we note that $\alpha^{x^3}=bc^2$, and so $\alpha^{x^3}\equiv\beta \bmod D^2$. From this we have

$$\begin{aligned}(Q^+\alpha(f-\tau))^{x^3}&\in Q^+(\beta+D^2)(f-\tau)\\&=Q^+\beta(f-\tau)+Q^+D^2(f-\tau)\\&=Q^+\beta(f-\tau)+Q^+D^3(f-\tau).\end{aligned}$$

Because

$$Q^+\alpha(f-\tau)\in J(k[\langle\sigma\rangle])Q^+D+k[\langle\sigma\rangle]Q^+D^3,$$

we have

$$(Q^+\alpha(f-\tau))^{x^3}\in J(k[\langle\sigma\rangle])^{x^3}Q^+D+k[\langle\sigma\rangle]^{x^3}Q^+D^3.$$

It is easy to see that $J(k[\langle\sigma\rangle])^{x^3}Q^+=J(k[\langle\sigma\rangle])Q^+$ and $k[\langle\sigma\rangle]^{x^3}Q^+=k[\langle\sigma\rangle]Q^+$. Hence

$$(Q^+\alpha(f-\tau))^{x^3}\in J(k[\langle\sigma\rangle])\,Q^+D+k[\langle\sigma\rangle]\,Q^+D^3.$$

Thus we have

$$Q^+\beta(f-\tau)\in J(k[\langle\sigma\rangle])\,Q^+D+k[\langle\sigma\rangle]\,Q^+D^3+Q^+D^3(f-\tau).$$

Evidently, $Q^+D^3(f-\tau)\subset k[\langle\sigma\rangle]\,Q^+D^3$, and so

$$Q^+\beta(f-\tau)\in J(k[\langle\sigma\rangle])\,Q^+D+k[\langle\sigma\rangle]\,Q^+D^3.$$

Thus (i) is proved.

We next show (ii). Let first $\lambda=\alpha\gamma$. Then

$$\begin{aligned}Q^+\lambda f(1-\sigma)&=\gamma Q^+\alpha f(1-\sigma)\in\gamma J(k[\langle\sigma\rangle])\,Q^+D\\&=J(k[\langle\sigma\rangle])\,Q^+\gamma D\subset J(k[\langle\sigma\rangle])\,Q^+D^3,\\Q^+\lambda f(x+y-xy)\sigma&=\gamma Q^+\alpha f(x+y-xy)\sigma\in\gamma k[\langle\sigma\rangle]\,Q^+D^3\\&=k[\langle\sigma\rangle]\,Q^+\gamma D^3\subset k[\langle\sigma\rangle]\,Q^+D^5.\end{aligned}$$

Hence (ii) holds for $\lambda=\alpha\gamma$. Similarly it holds for $\lambda=\beta\gamma$. We next let $\lambda=\alpha^2\beta$. Then $\lambda^\sigma\equiv\lambda \bmod D^4$ and so

$$\begin{aligned}Q^+\lambda f(1-\sigma)&\in(1-\sigma)(Q^+\alpha^2\beta f+\sigma Q^+D^4f)=J(k[\langle\sigma\rangle])\,Q^+D^3+k[\langle\sigma\rangle]\,Q^+D^4f\\&=J(k[\langle\sigma\rangle])\,Q^+D^3+k[\langle\sigma\rangle]\,Q^+D^5f,\\Q^+\lambda f(x+y-xy)\sigma&\in\sigma Q^+(\alpha^2\beta+D^4)(x+y-xy).\end{aligned}$$

Further by (*) we have

$$(\alpha^2\beta)^x+(\alpha^2\beta)^y-(\alpha^2\beta)^{xy}\equiv 0 \pmod{D^4}.$$

This together with Lemma 4 implies that

$$Q^+\lambda f(x+y-xy)\sigma\in\sigma Q^+D^4f\subset k[\langle\sigma\rangle]\,Q^+D^5.$$

Thus (ii) holds for $\lambda=\alpha^2\beta$. Using the congruence $(\alpha^2\beta)^x\equiv\alpha\beta^2+\beta\gamma \bmod D^4$, we see that (ii) holds for $\lambda=\alpha\beta^2$. Thus (ii) is proved. By a similar method, we can prove (iii) and (iv).

(2) is proved similarly.

The following is a direct consequence of Lemma 5.

Lemma 6. *Let* $0\le i\le 4$. *Then*

(1) $Q^+D^{2i}(f-\tau)^2=Q^+D^{2i+1}(f-\tau)^2$
$\subset J(k[\langle\sigma\rangle])^2\,Q^+D^{2i+1}+J(k[\langle\sigma\rangle])\,Q^+D^{2i+3}+k[\langle\sigma\rangle]Q^+D^{2i+5}$.

(2) $(f-\tau)^2D^{2i}Q^+=(f-\tau)^2D^{2i+1}Q^+$
$\subset D^{2i+1}\,Q^+J(k[\langle\sigma\rangle])^2+D^{2i+3}Q^+J(k[\langle\sigma\rangle])+D^{2i+5}\,Q^+k[\langle\sigma\rangle]$.

Lemma 7. *Let* $0 \le i \le 4$. *Then* $Q^+D^{2i+1}Q^+ = Q^+D^{2i+2}Q^+$.

Proof. We have $Q^+\alpha Q^+ = Q^+\sum_{g\in Q}\alpha^g$, and $\sum_{g\in Q}\alpha^g \equiv \mathrm{mod}\ D^2$ by (*). Thus $Q^+\alpha Q^+ \in Q^+D^2Q^+$. Similarly, $Q^+\beta Q^+ \in Q^+D^2Q^+$. We therefore see that the lemma holds for $i=0$. By a similar method, one can prove the lemma for $i=1, 2, 3, 4$.

The following is a direct consequence of Lemmas 4-6.

Lemma 8. (1) $fk[N_1]Af \subset DQ^+J(k[\langle\sigma\rangle])Q^+Dk[G]$.

(2) $fk[N_1]A(f-\tau) \subset DQ^+J(k[\langle\sigma\rangle])^2Q^+Dk[G] + DQ^+J(k[\langle\sigma\rangle])Q^+D^3k[G]$.

(3) $fk[N_1]A(f-\tau)^2 \subset DQ^+J(k[\langle\sigma\rangle])^3Q^+Dk[G] + DQ^+J(k[\langle\sigma\rangle])^2Q^+D^3k[G] + DQ^+J(k[\langle\sigma\rangle])Q^+D^5k[G]$.

(4) $(f-\tau)k[N_1]Af \subset DQ^+J(k[\langle\sigma\rangle])^2Q^+Dk[G] + D^3Q^+J(k[\langle\sigma\rangle])Q^+Dk[G]$.

(5) $(f-\tau)k[N_1]A(f-\tau) \subset DQ^+J(k[\langle\sigma\rangle])^3Q^+Dk[G] + DQ^+J(k[\langle\sigma\rangle])^2Q^+D^3k[G] + D^3Q^+J(k[\langle\sigma\rangle])^2Q^+Dk[G] + D^3Q^+J(k[\langle\sigma\rangle])Q^+D^3k[G]$.

(6) $(f-\tau)k[N_1]A(f-\tau)^2 \subset DQ^+J(k[\langle\sigma\rangle])^4Q^+Dk[G] + DQ^+J(k[\langle\sigma\rangle])^3Q^+D^3k[G] + DQ^+J(k[\langle\sigma\rangle])^2Q^+D^5k[G] + D^3Q^+J(k[\langle\sigma\rangle])^3Q^+Dk[G] + D^3Q^+J(k[\langle\sigma\rangle])^2Q^+D^3k[G] + D^3Q^+J(k[\langle\sigma\rangle])Q^+D^5k[G]$.

(7) $(f-\tau)^2k[N_1]Af \subset DQ^+J(k[\langle\sigma\rangle])^3Q^+Dk[G] + D^3Q^+J(k[\langle\sigma\rangle])^2Q^+Dk[G] + D^5Q^+J(k[\langle\sigma\rangle])Q^+Dk[G]$.

(8) $(f-\tau)^2k[N_1]A(f-\tau) \subset DQ^+J(k[\langle\sigma\rangle])^4Q^+Dk[G] + DQ^+J(k[\langle\sigma\rangle])^3Q^+D^3k[G] + D^3Q^+J(k[\langle\sigma\rangle])^3Q^+Dk[G] + D^3Q^+J(k[\langle\sigma\rangle])^2Q^+D^3k[G] + D^5Q^+J(k[\langle\sigma\rangle])^2Q^+Dk[G] + D^5Q^+J(k[\langle\sigma\rangle])Q^+D^3k[G]$.

(9) $(f-\tau)^2k[N_1]A(f-\tau)^2 \subset DQ^+J(k[\langle\sigma\rangle])^5Q^+Dk[G] + DQ^+J(k[\langle\sigma\rangle])^4Q^+D^3k[G] + DQ^+J(k[\langle\sigma\rangle])^3Q^+D^5k[G] + D^3Q^+J(k[\langle\sigma\rangle])^4Q^+Dk[G] + D^3Q^+J(k[\langle\sigma\rangle])^3Q^+D^3k[G] + D^3Q^+J(k[\langle\sigma\rangle])^2Q^+D^5k[G] + D^5Q^+J(k[\langle\sigma\rangle])^3Q^+Dk[G] + D^5Q^+J(k[\langle\sigma\rangle])^2Q^+D^3k[G]$.

3 Proof of Theorem

We here give the proof of Theorem. We already know that

$$J(k[G]) = A + B + C,$$

and so it suffices to prove that $(A+B)^{16}=0$ because $C^{11}=0$. We first show that $A^9=0$ and $B^{15}=0$. We start with the following:

Lemma 9. σ acts trivially on the k-space $Q^+k[N]Q^+$.

Proof. $Q^+k[N]Q^+$ is a k-space generated by

$$\{Q^+\Delta^+ \mid \Delta \text{ is a } Q\text{-orbit of } N\}.$$

By (*), we see that each Q-orbit is invariant under the action of σ, and the result follows.

Lemma 10. $A^9=0$.

Proof. Since Q^+ and $J(k[\langle\sigma\rangle])$ commute, we have

$$\begin{aligned} A^2 &= J(k[\langle\sigma\rangle])\, Q^+k[G]J(k[\langle\sigma\rangle])\, Q^+k[G] \\ &= J(k[\langle\sigma\rangle])\, Q^+k[N]\, Q^+J(k[\langle\sigma\rangle])k[G]. \end{aligned}$$

Hence by Lemma 9,

$$A^2 = J(k[\langle\sigma\rangle])^2\, Q^+k[N]\, Q^+k[G].$$

By repeating the same argument, we have

$$A^2 = J(k[\langle\sigma\rangle])^i\, Q^+k[N]\, Q^+k[G]$$

for any i. Thus we get $A^9=0$ because $J(k[\langle\sigma\rangle])^9=0$.

Lemma 11. $B^{15}=0$.

Proof. Since $B=J(k[T])k[G]=(f-\tau)k[G]$, we have

$$\begin{aligned} B^2 &= (f-\tau)k[N](f-\tau)k[G] \\ &\subset (f-\tau)^2k[G]+(f-\tau)D^2(f-\tau)k[G] \quad \text{(Lemma 1 (9))}, \end{aligned}$$

and

$$\begin{aligned} B^3 \subset\ & (f-\tau)^3k[G]+(f-\tau)^2D^2(f-\tau)k[G]+(f-\tau)D^2(f-\tau)^2k[G] \\ & +(f-\tau)D^2(f-\tau)D^2(f-\tau)k[G]. \end{aligned}$$

Repeating this procedure, we get

$$B^i \subset (f-\tau)^ik[G]+\sum_{i_0+\cdots+i_t=i}(f-\tau)^{i_0}D^2(f-\tau)^{i_1}\cdots(f-\tau)^{i_{t-1}}D^2(f-\tau)^{i_t}k[G]$$

for any i. Now set

$$X_i=\sum_{i_0+\cdots+i_t=i}(f-\tau)^{i_0}D^2(f-\tau)^{i_1}\cdots(f-\tau)^{i_{t-1}}D^2(f-\tau)^{i_t}.$$

Because $(f-\tau)^9=0$, it suffices to prove that $X_i=0$ for some i with $i\leq 15$. If $t\geq 5$ then $X_i\subset D^{10}k[G]=0$ because $D^9=0$. Therefore we may assume $t\leq 4$. Then we show that

(i) if $t=1$ then $X_{11}=0$,

(ii) if $t=2$ then $X_{13}=0$,

(iii) if $t=3$ then $X_{15}=0$, and

(iv) if $t=4$ then $X_{15}=0$.

Because s commutes with D, by Lemma 1 (6), (i) is clear. If $t=2$, then we have

$$X_{13}\subset (f-\tau)^iD^4(f-\tau)^jk[G], \quad i+j=11,$$

and if $t=3$, we have

$$X_{15}\subset(f-\tau)^i D^6(f-\tau)^j k[G],\quad i+j=11.$$

Hence (ii) and (iii) follow from (i). If $t=4$, then

$$X_{15}\subset(f-\tau)^i D^8(f-\tau)^j k[G],\quad i+j=9.$$

Because $D^8=N_1^+$ is a central element of $k[G]$, obviously we have

$$X_{15}\subset(f-\tau)^9 D^8 k[G]=0,$$

and (iv) is proved.

We are now in a position to prove the following lemma, from which our theorem is deduced.

Lemma 12. $(A+B)^{16}=0$.

We have

$$(A+B)^k=\sum A^{m_1}B^{n_1}\cdots A^{m_t}B^{n_t},$$

where $\sum_{i=1}^t m_i+n_i=k$, $m_i,\ n_i\geq 0$. Now set

$$Z_k=A^{m_1}B^{n_1}\cdots A^{m_t}B^{n_t}\quad\left(k=\sum_{i=1}^t m_i+n_i\right).$$

To prove the lemma, it suffices to prove that $Z_{16}=0$. In what follows, we set $I=\sum_{i=1}^t m_i$, $J=\sum_{i=1}^t n_i$ and assume $I+J=16$.

If $I\geq 9$, then $Z_{16}\subset k[G]A^9=0$ by Lemma 10. Accordingly, we may assume $I\leq 8$. We distinguish three cases.

Case 1. $m_1\neq 0$. If $m_1=I=1$ then $Z_{16}=AB^{15}=0$ by Lemma 11. If $m_1=1$ and $I\geq 2$ then $Z_{16}\subset AB^iAB^j$ with $i+j=9$ because $J\geq 9$. If $m_1\geq 2$, then $Z_{16}\subset A^2B^9$. Therefore it suffices to prove the following:

Lemma 13. (1) $A^2B^9=0$.

(2) $AB^iAB^j=0$ *if* $i+j=9$.

Case 2. $n_t=0$. Similarly, it suffices to show the following:

Lemma 14. (1) $B^9A^2=0$.

(2) $B^iAB^jA=0$ *if* $i+j=9$.

Case 3. $m_1=0$ and $n_t\neq 0$. If $m_2=I=1$, then $Z_{16}\subset B^{15}=0$. If $m_2=1$ and $I\geq 2$, then $Z_{16}\subset B^iAB^jAB^k$ with $i+j+k=9$. If $m_2=I=2$, then $Z_{16}=B^iA^2B^j$ with $i+j=14$. If $m_2=2$, and $I\geq 3$, then $Z_{16}\subset B^iA^2B^jAB^k$ with $i+j+k=9$. Finally, if $m_2\geq 3$, then $Z_{16}\subset B^iA^3B^j$ with $i+j=9$. Thus it suffices to prove the following:

Lemma 15. (1) $B^iA^2B^j=0$ if $i+j=11$.
(2) $B^iA^3B^j=0$ if $i+j=9$.
(3) $B^iAB^jAB^k=0$ if $i+j+k=9$

Because Lemmas 13 and 14 can be proved by a similar method, in the rest of the paper, we shall prove Lemmas 13 and 15.

Proof of Lemma 13. (1) Since $(f-\tau)^9=0$, we have

$$A^2B^9 \subset A^2 \sum_{i_0+\cdots+i_t=9} (f-\tau)^{i_0}D^2(f-\tau)^{i_1}\cdots D^2(f-\tau)^{i_t}k[G].$$

This shows that if $t \geq 5$ then $A^2B^9 \subset D^{10}k[G]=0$. Assume $t \geq 4$ and set

$$X_t=A^2(f-\tau)^{i_0}D^2(f-\tau)^{i_1}\cdots D^2(f-\tau)^{i_t}.$$

Consider first the case $t=1$. Because $(f-\tau)^3=f(1-s)$, if $(i_0, i_1)=(6, 3)$ or $(3, 6)$ then

$$(f-\tau)^{i_0}D^2(f-\tau)^{i_1} \subset J(k[\langle\sigma\rangle])^9k[G]=0.$$

We next assume that $(i_0, i_1)\neq(6, 3), (3, 6)$. Then

$$(f-\tau)^{i_0}D^2(f-\tau)^{i_1}=s^+(f-\tau)^{i_0'}D^2(f-\tau)^{i_1'},$$

where $(i_0', i_1')=(2, 1)$ or $(1, 2)$. Assume $(i_0', i_1')=(2, 1)$. We already know that

$$A^2=J(k[\langle\sigma\rangle])^2Q^+k[N_1]\,Q^+k[G].$$

Hence we have

$$\begin{aligned} X_1 &\subset J(k[\langle\sigma\rangle])^2\,Q^+k[N_1]\,Q^+k[N_1]s^+(f-\tau)^2D^2(f-\tau)k[G] \\ &= J(k[\langle\sigma\rangle])^2s^+Q^+k[N_1]\,Q^+k[N_1]\,(f-\tau)^2D^2(f-\tau)k[G] \\ &\subset J(k[\langle\sigma\rangle])^8\,Q^+k[N_1]\,Q^+k[N_1]\,(f-\tau)^2D^2(f-\tau)k[G]. \end{aligned}$$

Because

$$Q^+k[N_1]\,(f-\tau)^2 \subset J(k[\langle\sigma\rangle])^2Q^+D+J(k[\langle\sigma\rangle])\,Q^+D^3+k[\langle\sigma\rangle]Q^+D^5 \quad \text{(Lemma 6)},$$

and $J(k[\langle\sigma\rangle])^9=0$, the above implies that

$$X_1 \subset J(k[\langle\sigma\rangle])^8\,Q^+D^7(f-\tau)k[G].$$

Further by Lemma 5,

$$Q^+D^7(f-\tau)\subset J(k[\langle\sigma\rangle])\,Q^+D^7.$$

Thus we have $X_1=0$ for the case $(i_0', i_1')=(2, 1)$. By a similar argument, one can also show the same for the case $(i_0', i_1')=(1, 2)$.

Suppose next $t=2$. Because

$$(f-\tau)^3D^2(f-\tau)^3D^2(f-\tau)^3 \subset J(k[\langle\sigma\rangle])^9k[G]=0,$$

we have $X_2=0$ for $(i_0, i_1, i_2)=(3,3,3)$. Assume $(i_0, i_1, i_2)\neq(3,3,3)$. Then

$$(f-\tau)^{i_0}D^2(f-\tau)^{i_1}D^2(f-\tau)^{i_2}=\begin{cases}(1-s)(f-\tau)^2D^2(f-\tau)^2D^2(f-\tau)^2, \text{ or}\\ s^+(f-\tau)^{i'_0}D^2(f-\tau)^{i'_1}D^2(f-\tau)^{i'_2},\end{cases}$$

where $\sum_{h=0}^{2} i'_h=3$, $0\leq i'_h\leq 2$. For the former case, by making use of Lemma 6, we have

$$\begin{aligned}X_2 &\subset J(k[\langle\sigma\rangle])^5 Q^+k[N_1]\,Q^+k[N_1]\,(f-\tau)^2D^2(f-\tau)^2D^2(f-\tau)^2k[G]\\ &\subset (J(k[\langle\sigma\rangle])^7 Q^+D^3k[G]+J(k[\langle\sigma\rangle])^6Q^+D^5k[G]+J(k[\langle\sigma\rangle])^5Q^+D^7k[G])\cdot\\ &\quad (f-\tau)^2D^2(f-\tau)^2k[G]\\ &= J(k[\langle\sigma\rangle])^7 Q^+D^3(f-\tau)^2D^2(f-\tau)^2k[G]\\ &\quad +J(k[\langle\sigma\rangle])^6Q^+D^5(f-\tau)^2D^2(f-\tau)^2k[G]\\ &\subset J(k[\langle\sigma\rangle])^8Q^+D^7(f-\tau)^2k[G]=0.\end{aligned}$$

For the latter case, we have

$$X_2 \subset J(k[\langle\sigma\rangle])^8Q^+k[N_1]\,Q^+k[N_1]\,(f-\tau)^{i'_0}D^2(f-\tau)^{i'_1}D^2(f-\tau)^{i'_2}k[G].$$

Because $\sum_{h=0}^{2} i'_h=3$, $0\leq i'_h\leq 2$, one of i'_0, i'_1, i'_2 is zero. If $i'_0=0$, becauce $Q^+k[N_1]f=Q^+D$ (Lemma 4), we have

$$X_2 \subset J(k[\langle\sigma\rangle])^8Q^+D^3(f-\tau)^{i'_1}D^2(f-\tau)^{i'_2}k[G],$$

where $(i'_1, i'_2)=(2,1)$ or $(1,2)$. Hence $X_2=0$ by the argument given above. If $i'_1=0$ or $i'_2=0$, then we also get $X_2=0$ by Lemmas 5 and 6.

If $t=3$ then we have either

$$X_3 \subset J(k[\langle\sigma\rangle])^8Q^+k[N_1]\,Q^+k[N_1]\,(f-\tau)^{i'_0}D^2(f-\tau)^{i'_1}D^2(f-\tau)^{i'_2}D^2(f-\tau)^{i'_3}k[G],$$

where $\sum_{h=0}^{3} i'_h=3$, $0\leq i'_h\leq 2$, or

$$X_3 \subset (J(k[\langle\sigma\rangle])^5Q^+k[N_1]\,Q^+k[N_1]\,(f-\tau)^{i^*_0}D^2(f-\tau)^{i^*_1}D^2(f-\tau)^{i^*_2}D^2(f-\tau)^{i^*_3}k[G],$$

where $\sum_{h=0}^{3} i^*_h=6$, $0\leq i^*_h\leq 2$. For each case, we get $X_3=0$ by a similar argument.

If $t=4$ then we have

$$\begin{aligned}X_4 &\subset J(k[\langle\sigma\rangle])^2Q^+k[N_1]\,Q^+k[N_1]\,(f-\tau)^{i_0}D^8(f-\tau)^{i_4}k[G]\\ &\subset D^9k[G]=0.\end{aligned}$$

Thus we complete the proof of (1).

(2) We have $AB^iA\subset Y_1+Y_2$, where

$$\begin{aligned}Y_1&=A(f-\tau)^ik[G]A,\\ Y_2&=A(\textstyle\sum_{i_0+\cdots+i_t=i}(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[G])A.\end{aligned}$$

Hence we have to prove $Y_1B^j=0$ and $Y_2B^j=0$. We first show $Y_1B^j=0$. This will suffice to prove that

$$Y_1(f-\tau)^j=0 \text{ and } Y_1(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}=0,$$

where $\sum_{h=0}^{u} j_h = j$. Because

$$Y_1 = J(k[\langle\sigma\rangle])\,Q^+k[N_1]\,(f-\tau)^i k[N_1]\,Q^+J(k[\langle\sigma\rangle])k[G],$$

noting that

$$Q^+k[N_1]\,(f-\tau)^i k[N_1]\,Q^+ \subset k[\langle\sigma\rangle]\,Q^+DQ^+ \quad \text{(Lemma 5)}$$

and that $k[\langle\sigma\rangle]\,Q^+DQ^+$ and $k[\langle\sigma\rangle]$ commute, we have

$$Y_1 \subset J(k[\langle\sigma\rangle])^2\,Q^+k[N_1]\,(f-\tau)^i k[N_1]\,Q^+k[G].$$

Hence by Lemmas 4–7, we have

$$(\dagger)\begin{cases} i=1: & Y_1 \subset J(k[\langle\sigma\rangle])^3\,Q^+D^2Q^+k[G]+J(k[\langle\sigma\rangle])^2\,Q^+D^4Q^+k[G],\\ i=2: & Y_1 \subset J(k[\langle\sigma\rangle])^4\,Q^+D^2Q^+k[G]+J(k[\langle\sigma\rangle])^3\,Q^+D^4Q^+k[G]\\ & \qquad +J(k[\langle\sigma\rangle])^2\,Q^+D^6Q^+k[G],\\ i=3: & Y_1 \subset J(k[\langle\sigma\rangle])^5\,Q^+D^2Q^+k[G],\\ i=4: & Y_1 \subset J(k[\langle\sigma\rangle])^6\,Q^+D^2Q^+k[G]+J(k[\langle\sigma\rangle])^5\,Q^+D^4Q^+k[G],\\ i=5: & Y_1 \subset J(k[\langle\sigma\rangle])^7\,Q^+D^2Q^+k[G]+J(k[\langle\sigma\rangle])^6\,Q^+D^4Q^+k[G]\\ & \qquad +J(k[\langle\sigma\rangle])^5\,Q^+D^6Q^+k[G],\\ i=6: & Y_1 \subset J(k[\langle\sigma\rangle])^8\,Q^+D^2Q^+k[G],\\ i=7: & Y_1 \subset J(k[\langle\sigma\rangle])^8\,Q^+D^4Q^+k[G],\\ i=8: & Y_1 \subset J(k[\langle\sigma\rangle])^8\,Q^+D^6Q^+k[G]. \end{cases}$$

Because $j=9-i$, by the above we see that
if $i\equiv 0 \bmod 3$, $Y_1(f-\tau)^j=0$,
if $i\equiv 1 \bmod 3$, $Y_1(f-\tau)^j \subset J(k[\langle\sigma\rangle])^8\,Q^+D^4Q^+k[N_1]\,(f-\tau)^2k[G]$, and
if $i\equiv 2 \bmod 3$, $Y_1(f-\tau)^j \subset J(k[\langle\sigma\rangle])^8\,Q^+D^6Q^+k[N_1]\,(f-\tau)k[G]$.
Further, by Lemmas 5 and 6, the right-hand term in the second and third inclusion is each equal to 0, and so we get $Y_1(f-\tau)^j=0$ for each case. We next show that

$$Y_1(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}=0.$$

where $\sum_{h=0}^{u} j_h = j$. We already know that $Y_1 \subset Q^+D^2Q^+k[G]$ (see $(\dagger)$), and so setting

$$Y_1^* = Y_1(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u},$$

we have

$$Y_1^* \subset Q^+D^2Q^+k[N_1]\,(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}k[G].$$

Because $Q^+k[N_1]f \subset Q^+D$, the above shows that if $u\geq 3$ then $Y_1^* \subset Q^+D^9k[G]=0$.

Assume $u=2$. Then

$$Y_1^* = Y_1(f-\tau)^{j_0}D^2(f-\tau)^{j_1}D^2(f-\tau)^{j_2}.$$

If $i=1$, then by (†), we have

$$\begin{aligned} Y_1^* &\subset (J(k[\langle\sigma\rangle])^3Q^+D^2Q^+ + J(k[\langle\sigma\rangle])^2Q^+D^4Q^+)k[N_1]\cdot \\ &\quad (f-\tau)^{j_0}D^2(f-\tau)^{j_1}D^2(f-\tau)^{j_2}k[G] \\ &\subset J(k[\langle\sigma\rangle])^3Q^+D^2Q^+k[N_1](f-\tau)^{j_0}D^2(f-\tau)^{j_1}D^2(f-\tau)^{j_2} + D^9k[G] \\ &= J(k[\langle\sigma\rangle])^3Q^+D^2Q^+k[N_1](f-\tau)^{j_0}D^2(f-\tau)^{j_1}D^2(f-\tau)^{j_2}. \end{aligned}$$

Because $\sum_{h=0}^2 j_h = 8$, we have either

$$\begin{aligned} Y_1^* &\subset J(k[\langle\sigma\rangle])^9Q^+D^2Q^+k[N_1](f-\tau)^{j_0'}D^2(f-\tau)^{j_1'}D^2(f-\tau)^{j_2'}, \text{ or} \\ Y_1^* &\subset J(k[\langle\sigma\rangle])^6Q^+D^2Q^+k[N_1](f-\tau)^{j_0^*}D^2(f-\tau)^{j_1^*}D^2(f-\tau)^{j_2^*}, \end{aligned}$$

where $\sum_{h=0}^2 j_h' = 2$, $0 \le j_h' \le 2$, and $\sum_{h=0}^2 j_h^* = 5$, $0 \le j_h^* \le 2$. For the former case, we have $Y_1^* = 0$ because $J(k[\langle\sigma\rangle])^9 = 0$. For the latter case, we have

$$Y_1^* \subset J(k[\langle\sigma\rangle])^6Q^+D^2Q^+k[N_1](f-\tau)D^2(f-\tau)D^2(f-\tau)k[G],$$

and by Lemma 5 the right-hand term is equal to 0. Let $i=2$. Then by (†), we have

$$Y_1^* \subset J(k[\langle\sigma\rangle])^4Q^+D^2Q^+k[N_1](f-\tau)^{j_0}D^2(f-\tau)^{j_1}D^2(f-\tau)^{j_2}k[G],$$

where $\sum_{h=0}^2 j_h = 7$. From this we have either

$$Y_1^* \subset J(k[\langle\sigma\rangle])^{10}Q^+D^2Q^+k[N_1](f-\tau)^{j_0'}D^2(f-\tau)^{j_1'}D^2(f-\tau)^{j_2'}k[G],$$

where one of j_0', j_1', j_2' is 1 and the other two are 0, or

$$Y_1^* \subset J(k[\langle\sigma\rangle])^7Q^+D^2Q^+k[N_1](f-\tau)^{j_0^*}D^2(f-\tau)^{j_1^*}D^2(f-\tau)^{j_2^*}k[G],$$

where $\sum_{h=0}^2 j_h^* = 4$, $0 \le j_h^* \le 2$. For the former case evidently we have $Y_1^* = 0$, and for the latter case, by Lemmas 4-6, we also have $Y_1^* = 0$. If $i=3$ then by (†) we have

$$Y_1^* \subset J(k[\langle\sigma\rangle])^5Q^+D^2Q^+k[N_1](f-\tau)^{j_0}D^2(f-\tau)^{j_1}D^2(f-\tau)^{j_2}k[G],$$

where $\sum_{h=0}^2 j_h = 6$, and so we have either

$$Y_1^* \subset J(k[\langle\sigma\rangle])^8Q^+D^2Q^+k[N_1](f-\tau)^{j_0'}D^2(f-\tau)^{j_1'}D^2(f-\tau)^{j_2'}k[G],$$

where $\sum_{h=0}^2 j_h' = 3$, $0 \le j_h' \le 2$, or

$$Y_1^* \subset J(k[\langle\sigma\rangle])^5Q^+D^2Q^+k[N_1](f-\tau)^2D^2(f-\tau)^2D^2(f-\tau)^2k[G].$$

For either case, by Lemmas 4-6, we have $Y_1^* = 0$. Further if $i=4, 5, 6$ then by a similar method one can prove $Y_1^* = 0$. Finally if $i \ge 7$ then by (†), clearly $Y_1^* \subset D^9k[G] = 0$.

Assume now $u=1$. Then

$$Y_1^* = Y_1(f-\tau)^{j_0}D^2(f-\tau)^{j_1}.$$

By making use of (†) and Lemmas 4-6, one can show $Y_1^*=0$ similarly. Thus we have proved that $Y_1B^j=0$.

We next show that $Y_2B^j=0$. It will suffice to prove that

$$Y_2^* = A(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A(f-\tau)^j=0 \text{ and}$$
$$Y_2^{**} = A(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}=0,$$

where $\sum_{h=0}^{t} i_h = i$ and $\sum_{h=0}^{u} j_h = j$.

We first show $Y_2^*=0$. Since

$$A(f-\tau)^{i_0}=J(k[\langle\sigma\rangle])\,Q^+k[N_1]\,(f-\tau)^{i_0}\subset Dk[G]$$

and

$$(f-\tau)^{i_t}k[N_1]A(f-\tau)^j=(f-\tau)^{i_t}k[N_1]Q^+J(k[\langle\sigma\rangle])k[N_1]\,(f-\tau)^j$$
$$=(f-\tau)^{i_t}k[N_1]Q^+J(k[\langle\sigma\rangle])Q^+k[N_1]\,(f-\tau)^j\subset D^2k[G],$$

we see that if $t\geq 3$ then $Y_2^*\subset D^9k[G]=0$. Assume now $t=2$ then

$$Y_2^* = J(k[\langle\sigma\rangle])Q^+k[N_1]\,(f-\tau)^{i_0}D^2(f-\tau)^{i_1}D^2(f-\tau)^{i_2}k[N_1]\cdot$$
$$Q^+J(k[\langle\sigma\rangle])Q^+k[N_1]\,(f-\tau)^j.$$

Since

$$Q^+k[N_1]\,(f-\tau)^{i_0}D^2(f-\tau)^{i_1}D^2(f-\tau)^{i_2}k[N_1]Q^+\subset Q^+k[N_1]Q^+k[\langle\sigma\rangle],$$

and σ commutes with $Q^+k[N_1]Q^+k[\langle\sigma\rangle]$, we have

$$Y_2^* = J(k[\langle\sigma\rangle])^2Q^+k[N_1]\,(f-\tau)^{i_0}D^2(f-\tau)^{i_1}D^2(f-\tau)^{i_2}k[N_1]\,Q^+k[N_1]\,(f-\tau)^j.$$

Then, because $(\sum_{h=0}^{2} i_h)+j=9$, one of the following holds:

$$Y_2^* = J(k[\langle\sigma\rangle])^8Q^+k[N_1]\,(f-\tau)^{i_0'}D^2(f-\tau)^{i_1'}D^2(f-\tau)^{i_2'}k[N_1]\,Q^+k[N_1]\,(f-\tau)^{j'},$$
$$Y_2^* = J(k[\langle\sigma\rangle])^5Q^+k[N_1]\,(f-\tau)^{i_0^*}D^2(f-\tau)^{i_1^*}D^2(f-\tau)^{i_2^*}k[N_1]\,Q^+k[N_1]\,(f-\tau)^{j^*},$$

where $(\sum_{h=0}^{2} i_h')+j'=3$, $0\leq i_h', j'\leq 2$, and $(\sum_{h=0}^{2} i_h^*)+j^*=6$, $0\leq i_h^*, j^*\leq 2$. We first show $Y_2^*=0$ for the former case. If $i_0'\neq 0$ then

$$Q^+k[N_1]\,(f-\tau)^{i_0'}\subset (J(k[\langle\sigma\rangle])\,Q^+D+k[\langle\sigma\rangle]\,Q^+D^3)(f-\tau)^{i_0'-1}\quad \text{(Lemma 5).}$$

Further Lemma 4 implies

$$(f-\tau)^{i_2'}k[N_1]\,Q^+k[N_1]\,(f-\tau)^{j'}\subset D^2k[G].$$

Therefore

$$Y_2^* \subset J(k[\langle\sigma\rangle])^9 k[G] + D^9 k[G] = 0.$$

If $i_0' = 0$ and $i_1' \neq 0$ then

$$Y_2^* \subset J(k[\langle\sigma\rangle])^8 Q^+ D^3 (f-\tau)^{i_1'} D^2 (f-\tau)^{i_2'} k[N_1]\, Q^+ k[N_1]\, (f-\tau)^{j'} k[G].$$

Because

$$Q^+ D^3 (f-\tau)^{i_1'} \subset J(k[\langle\sigma\rangle])\, Q^+ D^3 + k[\langle\sigma\rangle] Q^+ D^5 \quad \text{(Lemma 5)},$$

we obtain $Y_2^* \subset J(k[\langle\sigma\rangle])^9 k[G] + D^9 k[G] = 0$ again. For the other case, we also have $Y_2^* \subset J(k[\langle\sigma\rangle])^9 k[G] + D^9 k[G] = 0$. Further for the latter case, one can show $Y_2^* = 0$ by the same method.

If $t=1$ then

$$Y_2^* = J(k[\langle\sigma\rangle])^2 Q^+ k[N_1]\, (f-\tau)^{i_0} D^2 (f-\tau)^{i_1} k[N_1]\, Q^+ k[N_1]\, (f-\tau)^j.$$

Because $i_0 + i_1 + j = 9$, one of the following holds:

$$\begin{aligned} Y_2^* &\subset J(k[\langle\sigma\rangle])^{11} k[G] = 0, \\ Y_2^* &= J(k[\langle\sigma\rangle])^8 Q^+ k[N_1]\, (f-\tau)^{i_0'} D^2 (f-\tau)^{i_1'} k[N_1]\, Q^+ k[N_1]\, (f-\tau)^{j'}, \\ Y_2^* &= J(k[\langle\sigma\rangle])^5 Q^+ k[N_1]\, (f-\tau)^2 D^2 (f-\tau)^2 k[N_1]\, Q^+ k[N_1]\, (f-\tau)^2, \end{aligned}$$

where $i_0' + i_1' + j' = 3$, $0 \leq i_0', i_1', j' \leq 2$. For the second case, we obtain

$$Y_2^* \subset J(k[\langle\sigma\rangle])^9 k[G] + D^9 k[G] = 0$$

by the argument similar to the above. For the third case, by Lemma 6, we have $Y_2^* = 0$.

We next show that $Y_2^{**} = 0$. Recall that

$$\begin{aligned} Y_2^{**} = {} & J(k[\langle\sigma\rangle]) Q^+ k[N_1]\, (f-\tau)^{i_0} D^2 \cdots D^2 (f-\tau)^{i_t} k[N_1] Q^+ J(k[\langle\sigma\rangle]) k[N_1] \cdot \\ & (f-\tau)^{j_0} D^2 \cdots D^2 (f-\tau)^{j_u} k[G], \end{aligned}$$

where $\sum_{h=0}^{t} i_h + \sum_{k=1}^{u} j_k = 9$. Since

$$Q^+ k[N_1]\, (f-\tau)^{i_0} \subset D k[G]$$

and

$$\begin{aligned} & (f-\tau)^{i_t} k[N_1]\, Q^+ J(k[\langle\sigma\rangle]) k[N_1]\, (f-\tau)^{j_0} \\ & = (f-\tau)^{i_t} k[N_1] Q^+ J(k[\langle\sigma\rangle]) Q^+ k[N_1]\, (f-\tau)^2 \\ & \subset D^2 k[G] \end{aligned}$$

we see that if $t+u \geq 3$ then $Y_2^{**} \subset D^9 k[G] = 0$. Thus we may assume that $t=u=1$. Then

$$Y_2^{**} = J(k[\langle\sigma\rangle])Q^+k[N_1](f-\tau)^{i_0}D^2(f-\tau)^{i_1}k[N_1]J(k[\langle\sigma\rangle])Q^+k[N_1]\cdot (f-\tau)^{j_0}D^2(f-\tau)^{j_1}k[G].$$

Because

$$Q^+k[N_1](f-\tau)^{i_0}D^2(f-\tau)^{i_1}k[N_1]\,Q^+ \subset Q^+k[N_1]\,Q^+k[\langle\sigma\rangle],$$

and σ commutes with $Q^+k[N_1]\,Q^+k[\langle\sigma\rangle]$,

$$Y_2^{**} \subset J(k[\langle\sigma\rangle])Q^+k[N_1](f-\tau)^{i_0}D^2(f-\tau)^{i_1}k[N_1]Q^+k[N_1](f-\tau)^{j_0}D^2(f-\tau)^{j_1}k[G].$$

Because $i_0+i_1+j_0+j_1=9$, one of the following holds:

$$Y_2^{**} \subset J(k[\langle\sigma\rangle])^8Q^+k[N_1](f-\tau)^{i_0'}D^2(f-\tau)^{i_1'}k[N_1]Q^+k[N_1](f-\tau)^{j_0'}D^2(f-\tau)^{j_1'}k[G],$$
$$Y_2^{**} \subset J(k[\langle\sigma\rangle])^5Q^+k[N_1](f-\tau)^{i_0^*}D^2(f-\tau)^{i_1^*}k[N_1]Q^+k[N_1](f-\tau)^{j_0^*}D^2(f-\tau)^{j_1^*}k[G],$$

where $i_0'+i_1'+j_0'+j_1'=3, 0\leq i_0', i_1', j_0', j_1'\leq 2$ and $i_0^*+i_1^*+j_0^*+j_1^*=6, 0\leq i_0^*, i_1^*, j_0^*, j_1^*\leq 2$. For the former case, if $i_0'\neq 0$ then because

$$Q^+k[N_1](f-\tau)^{i_0'}\subset J(k[\langle\sigma\rangle])Q^+D+k[\langle\sigma\rangle]\,Q^+D^3 \quad \text{(Lemma 5)},$$

and both $(f-\tau)^{i_1'}k[N_1]\,Q^+$ and $Q^+k[N_1](f-\tau)^{j_0'}$ are contained in $Dk[G]$ (Lemma 4), we have

$$Y_2^{**} \subset J(k[\langle\sigma\rangle])^9k[G]+D^9k[G]=0.$$

By a similar method, we see that if one of i_1', j_0', j_1' is nonzero, then

$$Y_2^{**} \subset J(k[\langle\sigma\rangle])^9k[G]+D^9k[G]=0.$$

This shows that $Y_2^{**}=0$ for the former case. For the latter case, if none of $i_0^*, i_1^*, j_0^*, j_1^*$ are 0 then

$$Y_2^{**} \subset J(k[\langle\sigma\rangle])^5Q^+k[N_1](f-\tau)D^2(f-\tau)k[N_1]\,Q^+k[N_1](f-\tau)D^2(f-\tau)k[G],$$

and by Lemmas 4 and 6, we have $Y_2^{**}=0$. If one of $i_0^*, i_1^*, j_0^*, j_1^*$ is 0 then the other three are 2. Hence we have either

$$Y_2^{**} \subset J(k[\langle\sigma\rangle])^5Q^+k[N_1](f-\tau)^2D^2(f-\tau)^2k[N_1]\,Q^+k[N_1]fD^2fk[G], \text{ or}$$
$$Y_2^{**} \subset J(k[\langle\sigma\rangle])^5Q^+k[N_1]fD^2fk[N_1]\,Q^+k[N_1](f-\tau)^2D^2(f-\tau)^2k[G].$$

For each case, we have $Y_2^{**}=0$ by Lemmas 5 and 6. Thus we have proved that $Y_2B^j=0$ and we complete the proof of (2).

Proof of Lemma 15. (1) It suffices to prove the following:

(i) $(f-\tau)^ik[N_1]A^2(f-\tau)^j=0$,

(ii) $(f-\tau)^ik[N_1]A^2(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}=0$,

(iii) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A^2(f-\tau)^j=0$,

(iv) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A^2(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}=0$,

where $\sum_{h=0}^{t} i_h=i$, $\sum_{h=0}^{u} j_h=j$, and $i+j=11$.

(i) is trivial because

$$(f-\tau)^i k[N_1]A^2(f-\tau)^j\subset(f-\tau)^9 k[G]=0.$$

(ii) Set $L=(f-\tau)^i k[N_1]A^2(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}$. Then we have

$$L\subset(f-\tau)^i k[N_1]J(k[\langle\sigma\rangle])^2Q^+k[N_1]\,Q^+k[N_1](f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}k[G].$$

Since $Q^+k[N_1](f-\tau)^{j_0}\subset Dk[G]$, we see that if $u\geq 4$ then $L\subset D^9k[G]=0$. Let $u=3$. Because $i+\sum_{h=0}^{3} j_h=11$, at lest one and at most three of i, $j_0,\ldots,j_3$ are greater than or equal to 3. Assume first three of them are greater than or equal to 3. Then they are of course equal to 3, and $L\subset J(k[\langle\sigma\rangle])^9k[G]=0$. Assume that exactly two of i, $j_0,\ldots,j_3$ are greater than or equal to 3. Then noting that Q^+ commutes with $J(k[\langle\sigma\rangle])$, we have

(a) $L\subset J(k[\langle\sigma\rangle])^6(f-\tau)^{i'}k[N_1]\,Q^+J(k[\langle\sigma\rangle])^2k[N_1]\,Q^+k[N_1]\cdot$
$(f-\tau)^{j'_0}D^2\cdots D^2(f-\tau)^{j'_3}k[G]$.

where $i'+\sum_{h=0}^{3} j'_h=5,\ 0\leq i',j'_h\leq 2$. Assume next that exactly one of i, $j_0,\ldots,j_3$ is greater than or equal to 3. If it is greater than or equal to 6 then (a) holds again. If it is at most 5, then we have

(b) $L\subset J(k[\langle\sigma\rangle])^3(f-\tau)^{i^*}k[N_1]\,Q^+J(k[\langle\sigma\rangle])^2k[N_1]\,Q^+k[N_1]\cdot$
$(f-\tau)^{j^*_0}D^2\cdots D^2(f-\tau)^{j^*_3}k[G]$,

where $i^*+\sum_{h=0}^{3} j^*_h=8,\ 0\leq i^*,j^*_h\leq 2$.

We now show $L=0$ for case (a). If $i'\neq 0$, then

$$(f-\tau)^{i'}k[N_1]\,Q^+\subset DQ^+J(k[\langle\sigma\rangle])+D^3Q^+k[\langle\sigma\rangle]\quad\text{(Lemma 5)}.$$

Hence noting that

$$Q^+\,k[N_1](f-\tau)^{j'_0}=Q^+D(f-\tau)^{j'_0}\quad\text{(Lemma 4)},$$

we have

$$\begin{aligned}L&\subset J(k[\langle\sigma\rangle])^6DQ^+J(k[\langle\sigma\rangle])^3k[N_1]Q^+D\cdot\\&\qquad(f-\tau)^{j'_0}D^2\cdots D^2(f-\tau)^{j'_3}k[G]+D^{10}k[G]\\&\subset J(k[\langle\sigma\rangle])^9k[G]=0.\end{aligned}$$

If $i'=0$, then $j'_v\neq 0$ for $v=2$ or 3, and

$$\begin{aligned}&Q^+k[N_1](f-\tau)^{j'_0}D^2(f-\tau)^{j'_1}D^2\cdots D^2(f-\tau)^{j'_v}\\&\subset Q^+D^{2v+1}(f-\tau)k[G]\\&\subset(J(k[\langle\sigma\rangle])Q^+D^{2v+1}+k[\langle\sigma\rangle]Q^+D^{2v+3})k[G].\end{aligned}$$

Hence we have $L\subset J(k[\langle\sigma\rangle])^9k[G]+D^{10}k[G]=0$ again.

We next show that $L=0$ for case (b). If $i^*=2$, then by Lemma 6,

$$(f-\tau)^{i^*}k[N_1]Q^+ \subset DQ^+J(k[\langle\sigma\rangle])^2+D^3Q^+J(k[\langle\sigma\rangle])+D^5Q^+k[\langle\sigma\rangle],$$

and so we have

$$\begin{aligned} L &\subset J(k[\langle\sigma\rangle])^3DQ^+J(k[\langle\sigma\rangle])^4k[N_1]Q^+k[N_1]\cdot \\ &\quad (f-\tau)^{j_0^*}D^2\cdots D^2(f-\tau)^{j_3^*}k[G]+D^{10}k[G] \\ &= DQ^+J(k[\langle\sigma\rangle])^7k[N_1]Q^+k[N_1]\cdot \\ &\quad (f-\tau)^{j_0^*}D^2\cdots D^2(f-\tau)^{j_3^*}k[G]. \end{aligned}$$

Because $j_v^*=2$ for some v and

$$\begin{aligned} &Q^+k[N_1](f-\tau)^{j_0^*}D^2\cdots D^2(f-\tau)^{j_v^*} \\ &\quad\subset Q^+D^{2v+1}(f-\tau)^2k[G] \\ &\quad\subset J(k[\langle\sigma\rangle])^2Q^+D^{2v+1}+J(k[\langle\sigma\rangle])\,Q^+D^{2v+3}+k[\langle\sigma\rangle]\,Q^+D^{2v+5}, \end{aligned}$$

we have $L\subset J(k[\langle\sigma\rangle])^9k[G]+D^9k[G]=0$. If $i^*<2$ then at least three of $j_0^*,\cdots,j_3^*$ are 2, and we have $L=0$ similarly. Now let $u=2$. Then

$$L \subset (f-\tau)^ik[N_1]J(k[\langle\sigma\rangle])^2Q^+k[N_1]\,Q^+k[N_1](f-\tau)^{j_0}D^2(f-\tau)^{j_1}D^2(f-\tau)^{j_2}k[G],$$

and by an argument simliar to the above, one can see that either $L\subset J(k[\langle\sigma\rangle])^9k[G]=0$ or one of the following holds:

$$\begin{aligned} L &\subset J(k[\langle\sigma\rangle])^6(f-\tau)^{i'}k[N_1]J(k[\langle\sigma\rangle])^2Q^+k[N_1]Q^+k[N_1]\cdot \\ &\quad (f-\tau)^{j_0'}D^2(f-\tau)^{j_1'}D^2(f-\tau)^{j_2'}k[G], \\ L &\subset J(k[\langle\sigma\rangle])^3(f-\tau)^2k[N_1]\,Q^+J(k[\langle\sigma\rangle])^2Q^+k[N_1]Q^+k[N_1]\cdot \\ &\quad (f-\tau)^2D^2(f-\tau)^2D^2(f-\tau)^2k[G], \end{aligned}$$

where $i'+\sum_{h=0}^{2}j_h'=5$, $0\le i', j_h'\le 2$. For each case, one can show that $L=0$ as the above.

Finally let $u=1$. Then

$$L \subset (f-\tau)^ik[N_1]\,Q^+J(k[\langle\sigma\rangle])^2Q^+k[N_1]\,Q^+k[N_1](f-\tau)^{j_0}D^2(f-\tau)^{j_1}k[G].$$

If one of i, j_0, j_1 is 9 then clearly $L\subset J(k[\langle\sigma\rangle])^9k[G]=0$. Assume i, j_0, $j_1\le 8$. If one of them is at least 6 and the other one is at least 3, or all of them are at least 3, then the same holds. For the other case, we have

$$L = J(k[\langle\sigma\rangle])^6(f-\tau)^{i'}Q^+k[N_1]\,Q^+k[N_1](f-\tau)^{j_0'}D^2(f-\tau)^{j_1'},$$

where $i'+j_0'+j_1'=5$, $0\le i', j_0', j_1'\le 2$. Clearly $i'\neq 0$, and one can see that $L=0$ by an argument given above.

(iii) and (iv) can be proved similarly.

(2) Since

$$\begin{aligned} B^iA^3B^j &= B^iJ(k[\langle\sigma\rangle])^3Q^+k[N_1]\,Q^+k[N_1]B^j \\ &= J(k[\langle\sigma\rangle])^3B^iQ^+k[N_1]\,Q^+k[N_1]B^j, \end{aligned}$$

it suffices to prove that

$$B^iQ^+k[N_1]\,Q^+k[N_1]B^j \subset J(k[\langle\sigma\rangle])^6k[G],$$

that is, it suffices to prove the following:

(i) $(f-\tau)^ik[N_1]\,Q^+k[N_1]\,Q^+k[N_1](f-\tau)^j \subset J(k[\langle\sigma\rangle])^6k[G]$,
(ii) $(f-\tau)^ik[N_1]\,Q^+k[N_1]\,Q^+k[N_1](f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u} \subset J(k[\langle\sigma\rangle])^6k[G]$,
(iii) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]\,Q^+k[N_1]Q^+k[N_1](f-\tau)^j \subset J(k[\langle\sigma\rangle])^6k[G]$,
(iv) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]\,Q^+k[N_1]Q^+k[N_1](f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}$
$\subset J(k[\langle\sigma\rangle])^6k[G]$,

where $\sum_{h=0}^{t} i_h = i$, $\sum_{h=0}^{u} j_h = j$. We can prove these by using the argument given in the proof of (1), and we omit the details.

(3) It suffices to prove the following:

(i) $(f-\tau)^ik[N_1]A(f-\tau)^jk[N_1]A(f-\tau)^k=0$,
(ii) $(f-\tau)^ik[N_1]A(f-\tau)^jk[N_1]A(f-\tau)^{k_0}D^2\cdots D^2(f-\tau)^{k_v}=0$,
(iii) $(f-\tau)^ik[N_1]A(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}k[N_1]A(f-\tau)^k=0$,
(iv) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A(f-\tau)^jk[N_1]A(f-\tau)^k=0$,
(v) $(f-\tau)^ik[N_1]A(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}D^2\cdot$
$k[N_1]A(f-\tau)^{k_0}D^2\cdots D^2(f-\tau)^{k_v}=0$,
(vi) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A(f-\tau)^j\cdot$
$k[N_1]A(f-\tau)^{k_0}D^2\cdots D^2(f-\tau)^{k_v}=0$,
(vii) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}\cdot$
$k[N_1]A(f-\tau)^k=0$,
(viii) $(f-\tau)^{i_0}D^2\cdots D^2(f-\tau)^{i_t}k[N_1]A(f-\tau)^{j_0}D^2\cdots D^2(f-\tau)^{j_u}\cdot$
$k[N_1]A(f-\tau)^{k_0}D^2\cdots D^2(f-\tau)^{k_v}=0$,

where $\sum_{h=1}^{t} i_h = i$, $\sum_{h=1}^{u} j_h = j$, $\sum_{h=1}^{v} k_h = k$ and $i+j+k=9$.

We now prove (i). If $(i, j, k)=(3, 3, 3)$, then

$$(f-\tau)^ik[N_1]A(f-\tau)^jk[N_1]A(f-\tau)^k \subset J(k[\langle\sigma\rangle])^9k[G]=0.$$

Assume $(i, j, k)\neq(3, 3, 3)$. Then it is contained in

$$J(k[\langle\sigma\rangle])^6(f-\tau)^{i'}k[N_1]A(f-\tau)^{j'}k[N_1]A(f-\tau)^{k'}k[G]$$

where $i'+j'+k'=3$, $0 \leq i', j', k' \leq 2$. If $(i', j', k')=(2, 1, 0)$, then noting that $J(k[\langle\sigma\rangle])^9=0$ and $D^9=0$, by Lemma 8 (1) and (8) we have

$$
\begin{aligned}
&(f-\tau)^{i'}k[N_1]A(f-\tau)^{j'}k[N_1]A(f-\tau)^{k'}k[G] \\
&=(f-\tau)^2k[N_1]A(f-\tau)k[N_1]Afk[G] \\
&\subset (D^3Q^+J(k[\langle\sigma\rangle])^2Q^+D^3+D^5Q^+J(k[\langle\sigma\rangle])^2Q^+D)(DQ^+J(k[\langle\sigma\rangle])Q^+D)k[G] \\
&=D^3Q^+J(k[\langle\sigma\rangle])^2Q^+D^4Q^+J(k[\langle\sigma\rangle])Q^+Dk[G] \\
&\quad +D^5Q^+J(k[\langle\sigma\rangle])^2Q^+D^2Q^+J(k[\langle\sigma\rangle])Q^+Dk[G] \\
&\subset J(k[\langle\sigma\rangle])^3k[G].
\end{aligned}
$$

Thus (i) holds in this case. For the other cases of (i', j', k'), (i) can be proved similarly.

(ii) Set

$$S=(f-\tau)^ik[N_1]A(f-\tau)^jk[N_1]A(f-\tau)^{k_0}D^2\cdots D^2(f-\tau)^{k_v}.$$

Lemma 8 (2) and (5) imply that

$$(f-\tau)^ik[N_1]A(f-\tau)^jk[N_1]A(f-\tau)^{k_0}\subset D^4k[G].$$

Hence if $v\geq 3$, $S\subset D^{10}k[G]=0$. Assume $v=2$. Then S is contained in either

$$J(k[\langle\sigma\rangle])^6(f-\tau)^{i'}k[N_1]A(f-\tau)^{j'}k[N_1]A(f-\tau)^{k'_0}D^2(f-\tau)^{k'_1}D^2(f-\tau)^{k'_2}, \text{ or}$$
$$J(k[\langle\sigma\rangle])^3(f-\tau)^{i^*}k[N_1]A(f-\tau)^{j^*}k[N_1]A(f-\tau)^{k^*_0}D^2(f-\tau)^{k^*_1}D^2(f-\tau)^{k^*_2},$$

where $i'+j'+\sum_{h=0}^2 k'_h=3, 0\leq i', j', k'_h\leq 2$, and $i^*+j^*+\sum_{h=0}^2 k^*_h=3, 0\leq i^*, j^*, k^*_h\leq 2$. For each case, by Lemma 8, we have

$$S \subset J(k[\langle\sigma\rangle])^9k[G]+D^9k[G]=0.$$

If $v=1$, then similarly one can prove the same.

(iii)-(viii) also can be proved by using Lemma 8.

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