J. FAC. SCI., SHINSHU UNIVERSITY, Vol. 34, No. 2, pp.69-79 1999

On a Theorem of G.Benke and D.-C. Chang

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(Received January 7, 2000)

Abstract

Let B denote the unit ball in $\mathbf{C}^n$, and ν the normalized Lebesgue measure on B. For $\alpha > -1$, define $d\nu_\alpha(z) = (1-|z|^2)^\alpha d\nu(z)$, $z \in B$. Let $H(B)$ denote the space of all holomorphic functions in B. G.Benke and D.-C. Chang [1] have recently characterized the weighted Bergman spaces $A^p(\nu_\alpha) \equiv L^p(\nu_\alpha) \cap H(B)$ as those functions in $H(B)$ whose images under the action of a certain set of differential operators lie in $L^p(\nu_\alpha)$. In the present paper we introduce some new operators and give another proof of their theorem.

1 Introduction

Let $n \geq 1$ be a fixed integer. Let $H(B)$ denote the space of all holomorphic functions in the open unit ball B of the complex n-dimensional Euclidean space $\mathbf{C}^n$. For each $\alpha \in (-1, \infty)$ and $p \in (0, \infty)$, we define the *weighted Bergman space* $A^p(\nu_\alpha)$ by $A^p(\nu_\alpha) \equiv L^p(\nu_\alpha) \cap H(B)$, where ν is the normalized Lebesgue measure on B, $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$, $z \in B$, and $L^p(\nu_\alpha)$ is the Lebesgue space with respect to the measure ν_α. Here $c_\alpha = \Gamma(n+\alpha+1)/\Gamma(n+1)\Gamma(\alpha+1)$, and so $\nu_\alpha(B) = 1$. (cf. [5], pp.120-121.)

As usual $\mathbf{Z}_+$ stands for the set of all non-negative integers. A multi-index $I = (i_1, \ldots, i_n)$ is an element in the cartesian product $\mathbf{Z}_+^n$. For each $I = (i_1, \ldots, i_n) \in \mathbf{Z}_+^n$ and $f \in H(B)$, we define

$$(Q_I f)(z) = (1-|z|^2)^{|I|}(D^I f)(z), \quad z = (z_1, \ldots, z_n) \in B,$$

where $|z|^2 = \sum_{j=1}^n |z_j|^2$, $|I| = \sum_{j=1}^n i_j$, $D^I = D_1^{i_1} \cdots D_1^{i_n}$, $D_1 = \partial/\partial z_1, \cdots, D_n = \partial/\partial z_n$. As a characterization of the weighted Bergman spaces, G.Benke and D.-C. Chang[1] have recently proved the following theorem:

Theorem (G.Benke and D.-C. Chang)

2000 *Mathematics Subject Classification.* 32A36.

Partly supported by the Grants-in-Aid for Scientific Reserch, the Ministry of Education, Science and Culture, Japan.

Let $1 \leq p < \infty, \alpha > -1$, *N be a fixed positive integer, and* $f \in H(B)$. *Then* $f \in A^p(\nu_\alpha)$ *if and only if* $Q_I f \in L^p(\nu_\alpha)$ *for all* $I \in \mathbf{Z}_+^n$ *with* $|I| = N$. *Moreover,*

$$\|f\|_{L^p(\nu_\alpha)} \approx \left(\sum_{|I|=0}^{N-1} |(D^I f)(0)| + \sum_{|I|=N} \|Q_I f\|_{L^p(\nu_\alpha)} \right).$$

It seems to me that their proof of the above theorem in [1] has some passages not easy to understand. The main purpose of this paper is to introduce some new operators and give another proof of the theorem of G.Benke and D.-C. Chang, in the case $1 < p < \infty$.

2 Preliminaries

For each $\alpha \in (-1, \infty)$, we define two operators P_α and P'_α as follows:

$$(P_\alpha f)(z) = \int_B \frac{f(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+\alpha+1}},$$

$$(P'_\alpha f)(z) = \int_B \frac{f(w) d\nu_\alpha(w)}{|(1 - \langle z, w \rangle)^{n+\alpha+1}|},$$

for any measurable function f in B and $z \in B$, provided that each integral exists. Here $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ for $z = (z_1, \cdots, z_n) \in B, w = (w_1, \cdots, w_n) \in B$.

Lemma 1 *Let* $1 \leq p < \infty$, $-1 < \alpha < \infty$ *and* $-1 < \beta < \infty$.
(a) P_α *is a bounded operator on* $L^p(\nu_\beta)$ *if and only if* $p(\alpha+1) > \beta+1$. *Moreover, in this case* P_α *is a bounded projection of* $L^p(\nu_\beta)$ *onto* $A^p(\nu_\beta)$.
(b) *If* $p(\alpha+1) > \beta+1$, *then* P'_α *is a bounded operator on* $L^p(\nu_\beta)$.

For the proof see [2], pp.33-36 and pp.128-136. See also [4], Theorem 1.4 and [6], Theorem 4.2.3.

Lemma 2 *Let* $-1 < \alpha < \infty, 1 < p < \infty$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Then* $(A^p(\nu_\alpha))^* \cong A^q(\nu_\alpha)$ *under the usual integral pairing* $\langle f, g \rangle = \int_B f\bar{g} d\nu_\alpha$.

For the proof see [2], p.44, Theorem 2.4. See also [4], Theorem 2.1 and [3], Theorem 2.

Proposition 1 *Let* $-1 < \alpha < \infty, 1 < p < \infty$ *and* $I \in \mathbf{Z}_+^n$. *Then* Q_I *is a bounded operator of* $A^p(\nu_\alpha)$ *into* $L^p(\nu_\alpha)$.

Proof. (cf. [1], Lemma 1.4.) Since $p(\alpha+1) > \alpha+1$, it follows from Lemma 1 that P_α is a bounded projection of $L^p(\nu_\alpha)$ onto $A^p(\nu_\alpha)$. Let $f \in A^p(\nu_\alpha)$ be fixed. For $z \in B$,

$$f(z) = (P_\alpha f)(z) = \int_B \frac{f(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+\alpha+1}}.$$

By the differentiation under the sign of integral,

$$(Q_I f)(z)=(n+1+\alpha)\cdots(n+|I|+\alpha)(1-|z|^2)^{|I|}\int_B \frac{\bar{w}^I f(w)d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+|I|+\alpha}}, \tag{1}$$

where $I=(i_1,\cdots,i_n)$ and $\bar{w}^I=\bar{w}_1^{i_1}\cdots\bar{w}_n^{i_n}$ for $w=(w_1,\cdots,w_n)\in B$.

Let q be the exponent conjugate to p. Then $q(\alpha+1+|I|)\geq q(\alpha+1)>\alpha+1$. By Lemma 1, $P_{\alpha+|I|}$ is a bounded operator on $L^q(\nu_\alpha)$. It follows from the duality that the adjoint $P^*_{\alpha+|I|}$ is a bounded operator on $L^p(\nu_\alpha)$. For $g\in L^q(\nu_\alpha)$, $z\in B$,

$$(P_{\alpha+|I|}g)(z)=\int_B \frac{g(w)d\nu_{\alpha+|I|}(w)}{(1-\langle z, w\rangle)^{n+\alpha+|I|+1}}=\frac{c_{\alpha+|I|}}{c_\alpha}\int_B \frac{(1-|w|^2)^{|I|}g(w)d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+|I|+\alpha}}.$$

Hence we have, for $g\in L^q(\nu_\alpha)$ and $z\in B$,

$$(P^*_{\alpha+|I|}g)(z)=\frac{c_{\alpha+|I|}}{c_\alpha}\int_B \frac{(1-|z|^2)^{|I|}g(w)d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+|I|+\alpha}}. \tag{2}$$

(see e.g. [6], pp.4-5.) By (1) and (2),

$$Q_I f=\frac{\Gamma(|I|+\alpha+1)}{\Gamma(\alpha+1)}P^*_{\alpha+|I|}F$$

in B, where $F(w)=\bar{w}^I f(w)$, $w\in B$. Hence

$$\begin{aligned}\|Q_I f\|_{L^p(\nu_\alpha)}&\leq\frac{\Gamma(|I|+\alpha+1)}{\Gamma(\alpha+1)}\|P^*_{\alpha+|I|}\|\cdot\|F\|_{L^p(\nu_\alpha)}\\&\leq\frac{\Gamma(|I|+\alpha+1)}{\Gamma(\alpha+1)}\|P^*_{\alpha+|I|}\|\cdot\|f\|_{L^p(\nu_\alpha)}.\end{aligned}$$

This means that Q_I is a bounded operator of $A^p(\nu_\alpha)$ into $L^p(\nu_\alpha)$.

Proposition 2 *Let* $-1<\alpha<\infty$ and $I\in\mathbf{Z}^n_+$. *Then* Q_I *is a bounded operator of* $A^1(\nu_\alpha)$ *into* $L^1(\nu_\alpha)$.

Proof. (cf. [1], Lemma 1.5.) Choose $\beta\in\mathbf{R}$ so that $-1<\alpha<\beta$. Since $1(\beta+1)=\beta+1>\alpha+1$, by Lemma 1, P_β is a bounded projection of $L^1(\nu_\alpha)$ onto $A^1(\nu_\alpha)$. Let $f\in A^1(\nu_\alpha)$ be fixed. For $z\in B$,

$$f(z)=(P_\beta f)(z)=\int_B \frac{f(w)d\nu_\beta(w)}{(1-\langle z, w\rangle)^{n+\beta+1}},$$

and so,

$$(Q_I f)(z)=(n+1+\beta)\cdots(n+|I|+\beta)(1-|z|^2)^{|I|}\int_B \frac{\bar{w}^I f(w)d\nu_\beta(w)}{(1-\langle z, w\rangle)^{n+1+|I|+\beta}}.$$

Hence we have

$$\int_B|Q_I f|d\nu_\alpha\leq c_\alpha C_1\int_B(1-|w|^2)^\beta|f(w)|d\nu(w)\int_B\frac{(1-|z|^2)^{|I|+\alpha}d\nu(z)}{|1-\langle z, w\rangle|^{n+1+|I|+\beta}}, \tag{3}$$

where $C_1=c_\beta(n+1+\beta)\cdots(n+|I|+\beta)$. By [5], Proposition 1.4.10, for $w\in B$

$$\int_B\frac{(1-|z|^2)^{|I|+\alpha}d\nu(z)}{|1-\langle z, w\rangle|^{n+1+|I|+\beta}}\leq C_2(1-|w|^2)^{\alpha-\beta}, \tag{4}$$

where $C_2<\infty$ is a constant. By (3) and (4), we have

$$\|Q_I f\|_{L^1(\nu_\alpha)} \leq C_1 C_2 \|f\|_{L^1(\nu_\alpha)}.$$

This completes the proof.

Proposition 3 *Let* $-1<\alpha<\infty$, $1\leq p<\infty$ *and* $I\in \mathbf{Z}_+^n$. *Then there exists a constant* $C=C(n,\alpha,I)<\infty$ *such that* $|(D^I f)(0)|\leq C\|f\|_{L^p(\nu_\alpha)}$ *for* $f\in A^p(\nu_\alpha)$.

Proof. (cf. [1], Corollary 1.6.) Choose $\beta\in\mathbf{R}$ so that $-1<\alpha<\beta$. Let $f\in A^p(\nu_\alpha)$ be fixed. Then $f\in A^1(\nu_\alpha)$. As in the proof of Proposition 2, $f=P_\beta f$ in B, and so, for $z\in B$,

$$(D^I f)(z)=(n+1+\beta)\cdots(n+|I|+\beta)\int_B \frac{\bar{w}^I f(w)\,d\nu_\beta(w)}{(1-\langle z,w\rangle)^{n+1+|I|+\beta}}.$$

It follows that

$$|(D^I f)(0)|=C_1\left|\int_B \bar{w}^I f(w)\,d\nu_\beta(w)\right|\leq \frac{C_1 c_\beta}{c_\alpha}\int_B (1-|w|^2)^{\beta-\alpha}|f(w)|\,d\nu_\alpha(w)$$
$$\leq \frac{C_1 c_\beta}{c_\alpha}\|f\|_{L^1(\nu_\alpha)}\leq \frac{C_1 c_\beta}{c_\alpha}\|f\|_{L^p(\nu_\alpha)},$$

where $C_1=(n+1+\beta)\cdots(n+|I|+\beta)$.

Proposition 4 *Let j be an integer with* $1\leq j\leq n$. *For* $f\in H(B)$, *we define the function* $R_j f$ *by*

$$(R_j f)(z)=\int_0^1 (D_j f)(tz)\,dt,\ z\in B.$$

(a) R_j *is a linear mapping on* $H(B)$ *and for* $f\in H(B)$ *and* $z\in B$,

$$f(z)-f(0)=\sum_{j=1}^n z_j (R_j f)(z).$$

(b) *If* $-1<\alpha<\infty$, $1\leq p\leq\infty$, *then* R_j *is a bounded linear operator on* $A^p(\nu_\alpha)$.

Proof. (a) is trivial. Let $-1<\alpha<\infty$, $1\leq p<\infty$. Choose $\beta\in\mathbf{R}$ so that $-1+\frac{\alpha+1}{p}<\beta$. Then $-1<\beta<\infty$ and $\alpha+1<p(\beta+1)$. By Lemma 1, P_β is a bounded projection of $L^p(\nu_\alpha)$ onto $A^p(\nu_\alpha)$. Let $f\in A^p(\nu_\alpha)$ be fixed. For $z\in B$,

$$f(z)=(P_\beta f)(z)=\int_B \frac{f(w)\,d\nu_\beta(w)}{(1-\langle z,w\rangle)^{n+\beta+1}},$$

and so,

$$(D_j f)(z)=(n+1+\beta)\int_B \frac{\bar{w}_j f(w)\,d\nu_\beta(w)}{(1-\langle z,w\rangle)^{n+2+\beta}}.$$

Then

$$(R_j f)(z)=(n+1+\beta)\int_0^1 dt\int_B \frac{\bar{w}_j f(w)\,d\nu_\beta(w)}{(1-t\langle z,w\rangle)^{n+2+\beta}}$$

$$=(n+1+\beta)\int_B \bar{w}_j f(w)d\nu_\beta(w)\int_0^1 \frac{dt}{(1-t\langle z, w\rangle)^{n+2+\beta}}$$
$$=\int_B \frac{\bar{w}_j f(w)}{(1-\langle z, w\rangle)^{n+1+\beta}}\frac{1-(1-\langle z, w\rangle)^{n+1+\beta}}{\langle z, w\rangle}d\nu_\beta(w).$$

Define for $\lambda \in \mathbf{D}\setminus\{0\}$,

$$\psi(\lambda)=\frac{1-(1-\lambda)^{n+1+\beta}}{\lambda},$$

where $\mathbf{D}=\{\lambda \in \mathbf{C}: |\lambda|<1\}$. Then $\psi \in H(\mathbf{D})$. A simple computation shows that $|\psi(\lambda)|\leq 2(1+2^{n+1+\beta})$ for $\lambda \in \mathbf{D}$. Hence we have

$$|(R_j f)(z)|\leq C\int_B \frac{|f(w)|d\nu_\beta(w)}{|1-\langle z, w\rangle|^{n+1+\beta}}=CP'_\beta(|f|)(z),$$

where $C=2(1+2^{n+1+\beta})$. By Lemma 1, P'_β is a bounded operator on $L^p(\nu_\alpha)$. We therefore have

$$\|R_j f\|_{L^p(\nu_\alpha)}\leq C\|P'_\beta(|f|)\|_{L^p(\nu_\alpha)}\leq C\|P'_\beta\|\cdot\|f\|_{L^p(\nu_\alpha)}.$$

This shows that R_j is a bounded linear operator on $A^p(\nu_\alpha)$.

Proposition 5 (a) *To each $I \in \mathbf{Z}^n_+\setminus\{0\}$ corresponds an linear operator R_I on $H(B)$ such that*

$$f(z)-\sum_{J\in \mathbf{Z}^n_+, |J|\leq m-1}\frac{1}{J!}(D^J f)(0)z^J=\sum_{K\in \mathbf{Z}^n_+, |K|=m} z^K(R_K f)(z)$$

for $m \in \mathbf{N}$, $f \in H(B)$ and $z \in B$.
(b) *If $-1<\alpha<\infty, 1\leq p<\infty$ and $I \in \mathbf{Z}^n_+\setminus\{0\}$, then R_I is a bounded linear operator on $A^p(\nu_\alpha)$.*

Proof. We prove (a) and (b) by induction on the length $|I|=m \in \mathbf{N}$ of $I \in \mathbf{Z}^n_+\setminus\{0\}$. By Proposition 4, the claim is true in the case $m=1$. Now we take $m \in \mathbf{N}$ with $m>1$ and make the following induction hypothesis: If $J \in \mathbf{Z}^n_+\setminus\{0\}$, $|J|\leq m-1$, then (a) and (b) are true for J.
Let $I \in \mathbf{Z}^n_+\setminus\{0\}$ with $|I|=m$. For $f \in H(B)$, we define

$$f_m(z)=f(z)-\sum_{J\in \mathbf{Z}^n_+, |J|\leq m-1}\frac{1}{J!}(D^J f)(0)z^J,\ z \in B. \tag{5}$$

Then $f_m \in H(B)$, $(D^J f_m)(0)=0$ for $J \in \mathbf{Z}^n_+$ with $|J|\leq m-1$ and $(R_K f_m)(0)=\frac{1}{K!}(D^K f_m)(0)=0$ for $K \in \mathbf{Z}^n_+$ with $|K|=m-1$. Define $\Lambda_I=\{k \in \mathbf{N}: 1\leq k\leq n,\ i_k\geq 1\}$, and

$$R_I f=\sum_{k\in\Lambda_I} R_{e_k}(R_{I-e_k} f_m),$$

where $e_1=(1,0,\cdots,0),\cdots,e_n=(0,\cdots,0,1)$, $R_{e_1}=R_1,\cdots,R_{e_n}=R_n$. Clearly, R_I is a linear operator on $H(B)$. Using the induction hypothesis and Proposition 4, we have for $z \in B$

$$
\begin{aligned}
f(z) &- \sum_{J\in\mathbf{Z}_+^n,|J|\le m-1} \frac{1}{J!}(D^J f)(0)z^J \\
&= f_m(z) = f_m(z) - \sum_{J\in\mathbf{Z}_+^n,|J|\le m-2} \frac{1}{J!}(D^J f_m)(0)z^J \\
&= \sum_{J\in\mathbf{Z}_+^n,|J|=m-1} z^J(R_J f_m)(z) \\
&= \sum_{J\in\mathbf{Z}_+^n,|J|=m-1} z^J\{(R_J f_m)(z)-(R_J f_m)(0)\} \\
&= \sum_{J\in\mathbf{Z}_+^n,|J|=m-1} z^J\left\{\sum_{k=1}^n z_k(R_{e_k}(R_J f_m))(z)\right\} \\
&= \sum_{I\in\mathbf{Z}_+^n,|I|=m} z^I\left\{\sum_{k\in\Lambda_I}(R_{e_k}(R_{I-e_k} f_m))(z)\right\} \\
&= \sum_{I\in\mathbf{Z}_+^n,|I|=m} z^I(R_I f)(z).
\end{aligned}
$$

Let $-1<\alpha<\infty$ and $1\le p<\infty$. If $f\in A^p(\nu_\alpha)$, then, by (5), we have $f_m\in A^p(\nu_\alpha)$ and for $I\in\mathbf{Z}_+^n$ with $|I|=m$,

$$\|R_I f\|_{L^p(\nu_\alpha)} = \left\|\sum_{k\in\Lambda_I} R_{e_k}(R_{I-e_k} f_m)\right\|_{L^p(\nu_\alpha)} \le \sum_{k\in\Lambda_I}\|R_{e_k}\|\cdot\|R_{I-e_k}\|\cdot\|f_m\|_{L^p(\nu_\alpha)}.$$

By (5) and Proposition 3,

$$\|f_m\|_{L^p(\nu_\alpha)} \le \|f\|_{L^p(\nu_\alpha)} + \sum_{J\in\mathbf{Z}_+^n,|J|\le m-1}\frac{1}{J!}|(D^J f)(0)| \le C\|f\|_{L^p(\nu_\alpha)},$$

where $C=C(n,\alpha,m)<\infty$ is a constant. Put $C_I=C\sum_{k\in\Lambda_I}\|R_{e_k}\|\cdot\|R_{I-e_k}\|$. Then by the induction hypothesis $C_I<\infty$ and $\|R_I f\|_{L^p(\nu_\alpha)}\le C_I\|f\|_{L^p(\nu_\alpha)}$ for $f\in A^p(\nu_\alpha)$. Hence R_I is a bounded operator on $A^p(\nu_\alpha)$. This completes the proof.

For $\alpha\in(-1,\infty)$, $p\in[1,\infty)$, $m\in\mathbf{N}$ and $f\in H(B)$, we define

$$\|f\|_{m,p,\alpha} = \sum_{J\in\mathbf{Z}_+^n,|J|\le m-1}|(D^J f)(0)| + \sum_{J\in\mathbf{Z}_+^n,|I|=m}\|Q_I f\|_{L^p(\nu_\alpha)}.$$

Proposition 6 *Let $\alpha\in(-1,\infty)$, $p\in(1,\infty)$ and $m\in\mathbf{N}$. Then there exists a constant $C=C(n,\alpha,p,m)<\infty$ such that $\|f\|_{L^p(\nu_\alpha)}\le C\|f\|_{m,p,\alpha}$ for $f\in A^p(\nu_\alpha)$.*

Proof. Let q be the exponent conjugate to p. Then $1<q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. By Lemma 2, $A^p(\nu_\alpha)\cong(A^q(\nu_\alpha))^*$ and

$$\|f\|_{L^p(\nu_\alpha)} \le C_1 \sup\left\{\left|\int_B g\bar{f}d\nu_\alpha\right| : g\in A^q(\nu_\alpha), \|g\|_{L^q(\nu_\alpha)}=1\right\} \tag{6}$$

for $f\in A^p(\nu_\alpha)$, where $C_1=C_1(n,\alpha,p)<\infty$ is a constant.

Let $f\in A^p(\nu_\alpha)$ and $g\in A^q(\nu_\alpha)$ with $\|g\|_{L^q(\nu_\alpha)}=1$. By Proposition 5,

$$f(z) = \sum_{J\in\mathbf{Z}_+^n,|J|\le m-1}\frac{1}{J!}(D^J f)(0)z^J + \sum_{I\in\mathbf{Z}_+^n,|I|=m} z^I(R_I f)(z),$$

$$g(z) = \sum_{J\in\mathbf{Z}_+^n,|J|\le m-1}\frac{1}{J!}(D^J g)(0)z^J + \sum_{I\in\mathbf{Z}_+^n,|I|=m} z^I(R_I g)(z),$$

for $z\in B$. It follows that

$$\begin{aligned}\overline{\int_B g\bar{f}d\nu_\alpha} &= \int_B f\bar{g}d\nu_\alpha \\ &= \sum_{|J|\le m-1}\sum_{|K|\le m-1}\frac{1}{J!}\frac{1}{K!}(D^Kf)(0)\overline{(D^Jg)(0)}\int_B z^K\overline{z^J}\,d\nu_\alpha(z) \\ &+ \sum_{|J|\le m-1}\sum_{|I|=m}\frac{1}{J!}\overline{(D^Jg)(0)}\int_B z^I(R_If)(z)\overline{z^J}\,d\nu_\alpha(z) \\ &+ \sum_{|I|=m}\int_B f(z)\overline{z^I(R_Ig)(z)}\,d\nu_\alpha(z). \end{aligned} \tag{7}$$

By [5], Propositions 1.4.8 and 1.4.9,

$$\begin{aligned}&\sum_{|J|\le m-1}\sum_{|K|\le m-1}\frac{1}{J!}\frac{1}{K!}(D^Kf)(0)\overline{(D^Jg)(0)}\int_B z^K\overline{z^J}\,d\nu_\alpha(z) \\ &= \sum_{|J|\le m-1}\sum_{|K|\le m-1}\left\{\frac{1}{J!}\frac{1}{K!}(D^Kf)(0)\overline{(D^Jg)(0)}\right. \\ &\quad\left.\times 2nc_\alpha\int_0^1 r^{2n-1+|J|+|K|}(1-r^2)^\alpha dr\int_S \zeta^K\overline{\zeta^J}\,d\sigma(\zeta)\right\} \\ &= \sum_{|J|\le m-1}\left(\frac{1}{J!}\right)^2(D^Jf)(0)\overline{(D^Jg)(0)} \\ &\quad\times 2nc_\alpha\frac{(n-1)!J!}{(n-1+|J|)!}\int_0^1 r^{2n-1+2|J|}(1-r^2)^\alpha dr \\ &= \sum_{|J|\le m-1}\frac{1}{J!}(D^Jf)(0)\overline{(D^Jg)(0)}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+|J|+\alpha+1)}. \end{aligned} \tag{8}$$

Let $\{I, J\}\subset \mathbf{Z}_+^n$ with $|I|=m$ and $|J|\le m-1$. By Proposition 5, $R_If\in A^p(\nu_\alpha)$. Define $f_I(z)=z^I(R_If)(z)$, $z\in B$. Then clearly $f_I\in A^p(\nu_\alpha)$. Since $|I|=m>m-1\ge|J|$, we have $(D^Jf_I)(0)=0$. Since $p(1+\alpha)>1+\alpha$, Lemma 1 yields that

$$f_I(z)=(P_\alpha f_I)(z)=\int_B\frac{f_I(w)d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}},\quad z\in B.$$

By the differentiation under the sign of integral,

$$(D^Jf_I)(z)=(n+\alpha+1)\cdots(n+\alpha+|J|)\int_B\frac{\bar{w}^Jf_I(w)d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha+|J|}},\quad z\in B.$$

Hence

$$\int_B z^I(R_If)(z)\bar{z}^Jd\nu_\alpha(z)=\int_B f_I(w)\bar{w}^Jd\nu_\alpha(w)=\frac{(D^Jf_I)(0)}{(n+\alpha+1)\cdots(n+\alpha+|J|)}=0. \tag{9}$$

Since $g\in A^q(\nu_\alpha)$, by Proposition 5 $R_Ig\in A^q(\nu_\alpha)$. Since $q(m+\alpha+1)>m+\alpha+1>\alpha+1$, by Lemma 1 $P_{m+\alpha}$ is a bounded projection of $L^q(\nu_\alpha)$ onto $A^p(\nu_\alpha)$. We therefore have

$$(R_Ig)(z)=(P_{m+\alpha}(R_Ig))(z)=\int_B\frac{(R_Ig)(w)d\nu_{m+\alpha}(w)}{(1-\langle z, w\rangle)^{n+1+m+\alpha}},\quad z\in B.$$

It follows that

$$\int_B f(z)\overline{z^I(R_Ig)(z)}\,d\nu_\alpha(z)=\int_B \bar{z}^If(z)d\nu_\alpha(z)\int_B\frac{\overline{(R_Ig)(w)d\nu_{m+\alpha}(w)}}{(1-\langle w, z\rangle)^{n+1+m+\alpha}}. \tag{10}$$

Since $q(m+\alpha+1)>\alpha+1$, by Lemma 1 $P'_{m+\alpha}$ is a bounded operator on $L^q(\nu_\alpha)$. By this and Proposition 5,

$$\int_B |\bar{z}^I f(z)| d\nu_\alpha(z) \int_B \frac{|\overline{(R_I g)(w)}| d\nu_{m+\alpha}(w)}{|(1-\langle w, z\rangle)|^{n+1+m+\alpha}} \le \int_B |f(z)| P'_{m+\alpha}(|R_I g|)(z) d\nu_\alpha(z)$$
$$\le \|f\|_{L^p(\nu_\alpha)} \cdot \|P'_{m+\alpha}(|R_I g|)\|_{L^q(\nu_\alpha)} \le \|f\|_{L^p(\nu_\alpha)} \cdot \|P'_{m+\alpha}\| \cdot \|R_I g\|_{L^q(\nu_\alpha)}$$
$$\le \|f\|_{L^p(\nu_\alpha)} \cdot \|P'_{m+\alpha}\| \cdot \|R_I\| \cdot \|g\|_{L^q(\nu_\alpha)} < \infty.$$

Hence we can apply the Fubini theorem to (10):

$$\int_B f(z) \overline{z^I (R_I g)(z)}\, d\nu_\alpha(z) = \int_B \overline{(R_I g)(w)}\, d\nu_{m+\alpha}(w) \int_B \frac{\bar{z}^I f(z) d\nu_\alpha(z)}{(1-\langle w, z\rangle)^{n+1+m+\alpha}}. \tag{11}$$

Since $P_{m+\alpha}$ is a bounded operator on $L^q(\nu_\alpha)$, the adjoint $P^*_{m+\alpha}$ is a bounded operator on $L^p(\nu_\alpha)$. For $h \in L^q(\nu_\alpha)$ and $z \in B$,

$$(P_{m+\alpha} h)(z) = \int_B \frac{h(w) d\nu_{m+\alpha}(w)}{(1-\langle z, w\rangle)^{n+1+m+\alpha}} = \frac{c_{m+\alpha}}{c_\alpha} \int_B \frac{(1-|w|^2)^m h(w) d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+m+\alpha}}.$$

Hence, for $h \in L^p(\nu_\alpha)$ and $z \in B$,

$$(P^*_{m+\alpha} h)(z) = \frac{c_{m+\alpha}}{c_\alpha} \int_B \frac{(1-|z|^2)^m h(w) d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+m+\alpha}}. \tag{12}$$

Put $F(z) = \bar{z}^I f(z)$, $z \in B$. Then $F \in L^p(\nu_\alpha)$. By (11) and (12),

$$\int_B f(z) \overline{z^I (R_I g)(z)}\, d\nu_\alpha(z)$$
$$= c_{m+\alpha} \int_B (1-|w|^2)^\alpha \overline{(R_I g)(w)} d\nu(w) \int_B \frac{(1-|w|^2)^m F(z) \nu_\alpha(z)}{(1-\langle w, z\rangle)^{n+1+m+\alpha}}$$
$$= c_{m+\alpha} \frac{c_\alpha}{c_{m+\alpha}} \int_B (1-|w|^2)^\alpha \overline{(R_I g)(w)} (P^*_{m+\alpha} F)(w) d\nu(w)$$
$$= \int_B \overline{(R_I g)(w)} (P^*_{m+\alpha} F)(w) d\nu_\alpha(w). \tag{13}$$

On the other hand, since $p(\alpha+1)>\alpha+1$, by Lemma 1, P_α is a bounded projection of $L^p(\nu_\alpha)$ onto $A^p(\nu_\alpha)$. Since $f \in A^p(\nu_\alpha)$, we have for $z \in B$,

$$f(z) = (P_\alpha f)(z) = \int_B \frac{f(w) d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}},$$

and so,

$$(Q_I f)(z) = (1-|z|^2)^m (D^I f)(z)$$
$$= (1-|z|^2)^m (n+\alpha+1)\cdots(n+\alpha+m) \int_B \frac{\bar{w}^I f(w) d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+m+\alpha}}$$
$$= \frac{c_\alpha}{c_{m+\alpha}} (n+\alpha+1)\cdots(n+\alpha+m) \frac{c_{m+\alpha}}{c_\alpha} \int_B \frac{(1-|z|^2)^m F(w) d\nu_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+m+\alpha}}$$
$$= \frac{c_\alpha}{c_{m+\alpha}} (n+\alpha+1)\cdots(n+\alpha+m) (P^*_{m+\alpha} F)(z)$$
$$= \frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} (P^*_{m+\alpha} F)(z). \tag{14}$$

By (13) and (14), we have

$$\int_B f(z)\overline{z^I(R_I g)(z)}\,d\nu_\alpha(z)=\frac{\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)}\int_B (Q_I f)\overline{(R_I g)}\,d\nu_\alpha. \tag{15}$$

By (7), (8), (9) and (15),

$$\begin{aligned}\left|\int_B g\bar{f}\,d\nu_\alpha\right|&=\left|\sum_{|J|\le m-1}\frac{1}{J!}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+|J|+\alpha+1)}(D^J f)(0)\overline{(D^J g)(0)}\right.\\&\quad\left.+\frac{\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)}\sum_{|I|=m}\int_B (Q_I f)\overline{(R_I g)}\,d\nu_\alpha\right|\\&\le\sum_{|J|\le m-1}\frac{1}{J!}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+|J|+\alpha+1)}|(D^J f)(0)||(D^J g)(0)|\\&\quad+\frac{\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)}\sum_{|I|=m}\|Q_I f\|_{L^p(\nu_\alpha)}\cdot\|R_I g\|_{L^q(\nu_\alpha)}.\end{aligned} \tag{16}$$

By Proposition 3, for $J\in\mathbf{Z}_+^n$ with $|J|\le m-1$,

$$|(D^J g)(0)|\le C_2\|g\|_{L^q(\nu_\alpha)}=C_2 \tag{17}$$

where $C_2=C_2(n,\alpha,p,m)<\infty$ is a constant. By Proposition 5, for $I\in\mathbf{Z}_+^n$ with $|I|=m$,

$$\|R_I g\|_{L^q(\nu_\alpha)}\le C_3\|g\|_{L^q(\nu_\alpha)}=C_3 \tag{18}$$

where $C_3=C_3(n,\alpha,p,m)<\infty$ is a constant. By (16), (17) and (18),

$$\left|\int_B g\bar{f}\,d\nu_\alpha\right|\le C_4\left\{\sum_{|J|\le m-1}|(D^J f)(0)|+\sum_{|I|=m}\|Q_I f\|_{L^p(\nu_\alpha)}\right\}=C_4\|f\|_{m,p,\alpha} \tag{19}$$

where $C_4=C_4(n,\alpha,p,m)<\infty$ is a constant. By (6) and (19), we have

$$\|f\|_{L^p(\nu_\alpha)}\le C\,\|f\|_{m,p,\alpha}$$

where $C=C(n,\alpha,p,m)<\infty$ is a constant. This completes the proof.

Proposition 7 *Let* $\alpha\in(-1,\infty)$, $p\in(1,\infty)$ *and* $m\in\mathbf{N}$. *If* $f\in H(B)$ *and* $\|f\|_{m,p,\alpha}<\infty$, *then* $f\in A^p(\nu_\alpha)$.

Proof. Suppose $f\in H(B)$ and $\|f\|_{m,p,\alpha}<\infty$. Put $tB=\{z\in\mathbf{C}^n:|z|<t\}$ for $t\in(0,\infty)$. For $r\in(0,1)$ and $z\in r^{-1}B$, we dedine $f_r(z)=f(rz)$. Then $f_r\in H(r^{-1}B)$ and $r^{-1}B$ is an open neighborhood of the closure $\bar{B}$ of B. Hence $\{f_r\}_{0<r<1}\subset A^p(\nu_\alpha)$. By Proposition 6,

$$\|f_r\|_{L^p(\nu_\alpha)}\le C\|f_r\|_{m,p,\alpha}\qquad(0<r<1) \tag{20}$$

where $C=C(n,\alpha,p,m)<\infty$ is a constant.
Noting that $mp+\alpha>mp-1\ge p-1>0$, we have, for $I\in\mathbf{Z}_+^n$ with $|I|=m$ and $r\in(0,1)$,

$$\begin{aligned}\|Q_I f_r\|_{L^p(\nu_\alpha)}^p&=\int_B|Q_I f_r|^p d\nu_\alpha\\&=c_\alpha\int_B(1-|z|^2)^{mp+\alpha}\,r^{mp}|(D^I f)(rz)|^p d\nu(z)\end{aligned}$$

$$= c_\alpha r^{-mp-2n-2\alpha}\int_{rB}(r^2-|w|^2)^{mp+\alpha}|(D^I f)(w)|^p d\nu(w)$$
$$\leq c_\alpha r^{-mp-2n-2\alpha}\int_{rB}(1-|w|^2)^{mp+\alpha}|(D^I f)(w)|^p d\nu(w)$$
$$\leq c_\alpha r^{-mp-2n-2\alpha}\int_{B}(1-|w|^2)^{mp+\alpha}|(D^I f)(w)|^p d\nu(w)$$
$$= r^{-mp-2n-2\alpha}\|Q_I f_r\|^p_{L^p(\nu_\alpha)}.$$

Hence, for $r\in(0,1)$,

$$\sum_{I\in \mathbf{Z}^n_+,|I|=m}\|Q_I f_r\|_{L^p(\nu_\alpha)} \leq r^{-m-2p^{-1}(n+\alpha)}\sum_{I\in \mathbf{Z}^n_+,|I|=m}\|Q_I f_r\|_{L^p(\nu_\alpha)}. \tag{21}$$

Moreover, using (21), we have

$$\|f_r\|_{m,p,\alpha} = \sum_{J\in \mathbf{Z}^n_+,|J|\leq m-1}|(D^J f_r)(0)| + \sum_{I\in \mathbf{Z}^n_+,|I|=m}\|Q_I f_r\|_{L^p(\nu_\alpha)}$$
$$\leq \sum_{|J|\leq m-1} r^{|J|}|(D^J f)(0)| + r^{-m-2p^{-1}(n+\alpha)}\sum_{|I|=m}\|Q_I f\|_{L^p(\nu_\alpha)}$$
$$\leq r^{-m-2p^{-1}(n+\alpha)}\|f\|_{m,p,\alpha}. \tag{22}$$

By (20) and (22),

$$\|f_r\|_{L^p(\nu_\alpha)} \leq C r^{-m-2p^{-1}(n+\alpha)}\|f\|_{m,p,\alpha} \qquad (0<r<1). \tag{23}$$

It follows from Fatou's lemma and (23) that

$$\|f\|^p_{L^p(\nu_\alpha)} = \int_B |f|^p d\nu_\alpha = \int_B \lim_{r\uparrow 1}|f_r|^p d\nu_\alpha$$
$$\leq \liminf_{r\uparrow 1}\int_B |f_r|^p d\nu_\alpha = \liminf_{r\uparrow 1}\|f_r\|^p_{L^p(\nu_\alpha)}$$
$$\leq \liminf_{r\uparrow 1}\{C r^{-m-2p^{-1}(n+\alpha)}\|f\|_{m,p,\alpha}\}^p = (C\|f\|_{m,p,\alpha})^p < \infty.$$

This means that $f\in A^p(\nu_\alpha)$.

3 Theorem of G.Benke and D.-C. Chang

Theorem *Let* $\alpha\in(-1,\infty)$, $p\in(1,\infty)$ *and* $m\in\mathbf{N}$.
(a) *For* $f\in H(B)$, *it holds that*
$f\in A^p(\nu_\alpha)$ *if and only if* $Q_I f\in L^p(\nu_\alpha)$ *for all* $I\in\mathbf{Z}^n_+$ *with* $|I|=m$.
(b) $\|\cdot\|_{m,p,\alpha}$ *is a norm on* $A^p(\nu_\alpha)$ *which is equivalent to the norm* $\|\cdot\|_{L^p(\nu_\alpha)}$. *Precicely, there are two constants* $C_j=C_j(n,\alpha,p,m)<\infty (j=1,2)$ *such that*

$$C_1\|f\|_{L^p(\nu_\alpha)} \leq \|f\|_{m,p,\alpha} \leq C_2\|f\|_{L^p(\nu_\alpha)}$$

for all $f\in A^p(\nu_\alpha)$.

Proof. (a) is an immediate consequence of Propositions 1,3 and 7.
(b) follows from (a), Propositions 1,3 and 6.

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