J. FAC. SCI., SHINSHU UNIVERSITY, Vol. 34, No. 1, pp.1-14 1999

Some homotopy groups of the mod 4 Moore space

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(Received July 15, 1999)

Abstract

Let M^n be a Moore space of type $(\mathbf{Z}_4, n-1)$. We calculate the homotopy groups $\pi_{2n-k}(M^n)$ in the range $k=3,4$ and $n \leq 24$. The methods are based on Toda's composition methods and we use Gray's cellular structure of the homotopy fiber of the pinching map from M^n to an n-sphere S^n and also use James' exact sequence including a relative homotopy group $\pi_*(M^n, S^{n-1})$.

1 Introduction and summary

We denote by $\iota_n \in \pi_n(S^n)$ the homotopy class of the identity map of S^n and by $M_q^n = S^{n-1} \cup_{q\iota_{n-1}} e^n$ a Moore space of type $(\mathbf{Z}_q, n-1)$. In particular we set $M^n = M_4^n$. The purpose of the present note is to determine the stable group $\pi_{2n-4}(M^n) \cong \pi_{n-2}^S(M^2)$ and the metastable group $\pi_{2n-3}(M^n)$ for $n \leq 24$. For example, the notation $(\mathbf{Z}_4)^r \oplus (\mathbf{Z}_2)^s$ or $4^r + 2^s$ means the abelian group

$$\underbrace{\mathbf{Z}_4 \oplus \cdots \oplus \mathbf{Z}_4}_{r} \oplus \underbrace{\mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2}_{s}.$$

Our result is stated as follows.

Theorem 1.1 $\pi_{n-2}^S(M^2) \cong \pi_{2n-4}(M^n) \cong (\mathbf{Z}_4)^r \oplus (\mathbf{Z}_2)^s$, *where* r *and* s *are given in the following table.*

n	3	4	5	6	7	8	9	10	11	12	13	14	15
r	1	0	0	1	1	0	0	1	1	0	0	1	1
s	0	1	2	1	0	0	1	1	2	5	4	1	0

n	16	17	18	19	20	21	22	23	24	25
r	0	0	1	1	0	1	2	2	1	0
s	0	2	3	3	6	5	2	1	2	4

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Theorem 1.2 (i) *$\pi_{2n-3}(M^n)$ for $n \leq 10$ is isomorphic to the group G in the following table.*

n	2	3	4	5	6	7	8	9	10
G	4	8	2^2	$8+2^2$	$4+2$	4	2	$8+2^2$	$4+2^3$

(ii) *$\pi_{2n-3}(M^n) \cong (\mathbf{Z}_4)^r \oplus (\mathbf{Z}_2)^s$ for $n \geq 11$, where r and s are given in the following table.*

n	11	12	13	14	15	16	17	18	19	20	21	22	23	24
r	1	0	2	1	1	1	2	1	1	1	3	2	2	0
s	5	5	1	1	0	1	3	4	6	6	2	2	2	5

We determine $\pi_{2n-2}(M^n)$ for $3 \leq n \leq 7$.

Proposition 1.3 *$\pi_4(M^3) \cong (\mathbf{Z}_2)^2$, $\pi_6(M^4) \cong \mathbf{Z}_8 \oplus (\mathbf{Z}_2)^2$, $\pi_8(M^5) \cong \mathbf{Z}_4 \oplus (\mathbf{Z}_2)^2$, $\pi_{10}(M^6) \cong \mathbf{Z}_8 \oplus \mathbf{Z}_2$ and $\pi_{12}(M^7) \cong \mathbf{Z}_2$.*

Our result overlaps with that of Baues-Buth [4]. The result of Theorem 1.2 for $n = 5$ and 9 corrects the corresponding result of Shinpo [13].

Our method is to use the composition methods developed by Toda [15]. We use the second stage in the cellular decomposition of the homotopy fiber of the collapsing map $p: M^n \to S^n$ obtained by Gray [6]. We also use the James exact sequence (James [7]) including the relative homotopy group $\pi_*(M^n, S^{n-1})$. The key step determining the group extension of $\pi_*(M^n)$ is Lemma 2.5 which ensures that elements of $\pi_*(M^n)$ induced from those of $\pi_*(M_2^n)$ with lifts ([10]) are of order 2. We use the notations and the results of [15], [9] and [8] freely.

2 Fundamental facts

For a pair of spaces (X, A), let $i_{A,X}: A \to X$ be the inclusion and $p_{X,A}: X \to X/A$ be the map pinching A to one point. In particular we set $i_n = i_{A,X}$ and $p_n = p_{X,A}$ for $(X, A) = (M_q^n, S^{n-1})$. Let ι_n' be the identity class of M_q^n. Let $\eta_2 \in \pi_3(S^2)$ be the Hopf map and $\eta_n = \Sigma^{n-2}\eta_2$ for $n \geq 2$. For integers a and b, we denote by (a, b) the greatest common divisor of a and b. We set $\eta = \Sigma^\infty \eta_2$, $\iota' = \Sigma^\infty \iota_2'$, $i = \Sigma^\infty i_2$ and $p = \Sigma^\infty p_2$. Then, as is well known ([2]),

$$\{M_q^2, M_q^2\} = \mathbf{Z}_q\{\iota'\} \oplus \mathbf{Z}_{(q,2)}\{i\eta p\} \; (q \not\equiv 2 \bmod 4)$$

and

$$\{M_q^2, M_q^2\} = \mathbf{Z}_{2q}\{\iota'\} \; (q \equiv 2 \bmod 4).$$

Although we know the group structure of $[M_q^3, M_q^3]$ by Corollary III.D.15 of [3] or by Proposition 11 of [1], we show the following.

Lemma 2.1 (i) $[M_q^3, M_q^3]=\mathbf{Z}_q\{\iota_3'\}\oplus\mathbf{Z}_q\{i_3\eta_2p_3\}$ *for* $q\not\equiv 2 \bmod 4$.
(ii) $[M_q^3, M_q^3]=\mathbf{Z}_{2q}\{\iota_3'\}\oplus\mathbf{Z}_{\frac{q}{2}}\{ai_3\eta_2p_3-2\iota_3'\}$ *for* $q\equiv 2 \bmod 4$, *where* a *is an odd integer.*

Proof. First we note that $[M_q^3, M_q^3]\cong[M_q^2, \Omega M_q^3]$ is abelian by making use of Theorem X.3.10 of [16].

By Proposition 7.1 of [12], the order of ι_3' is q or $2q$ according as $q\not\equiv 2 \bmod 4$ or $q\equiv 2 \bmod 4$. We have $\pi_2(M_q^3)=\mathbf{Z}_q\{i_3\}$. We consider the following exact sequence induced from the cofibration starting with $q\iota_2$:

$$0\longleftarrow\pi_2(M_q^3)\xleftarrow{i_3^*}[M_q^3, M_q^3]\xleftarrow{p_3^*}\pi_3(M_q^3)\xleftarrow{q\iota_3^*}\pi_3(M_q^3).$$

By [14], $\pi_3(M_q^3)=\mathbf{Z}_{(2q,q^2)}\{i_3\eta_2\}$. So the assertion (i) is obtained.

Next assume that $q\equiv 2 \bmod 4$. Then, in the above exact sequence, we have $q\iota_3'=xi_3\eta_2p_3$ for an integer x. By stabilizing the relation,we have $q\iota'=xi\eta p$ and so x becomes odd. Since $0=2q\iota_3'=2xi_3\eta_2p_3$ and $i_3\eta_2p_3$ is of order q, we can set $2x=aq$ for an odd integer a. So we have the relation $q\iota_3'=a\cdot\frac{q}{2}i_3\eta_2p_3$, and hence the element $ai_3\eta_2p_3-2\iota_3'$ is of order $\frac{q}{2}$. This leads to (ii), completing the proof. □

Let F be the homotopy fiber of the map $p_n: M_q^n\to S^n$. According to [6], F has a homotopy type of a CW-complex $S^{n-1}\cup e^{2(n-1)}\cup e^{3(n-1)}\cup\cdots$ and the subcomplex $Y=S^{n-1}\cup e^{2(n-1)}$ of the second stage has the following cell strucure.

Lemma 2.2

$$Y=S^{n-1}\cup_{q[\iota_{n-1},\iota_{n-1}]}e^{2n-2},$$

where $[\ ,\ ]$ *is the Whitehead product.*

The following sequence is exact for $k\leq 2n-5\,(i=i_{Y,M_q^n})$:

$$\pi_{n+k+1}(S^n)\xrightarrow{\Delta}\pi_{n+k}(Y)\xrightarrow{i_*}\pi_{n+k}(M_q^n)\xrightarrow{p_{n*}}\pi_{n+k}(S^n). \tag{1}$$

We have a formula

$$\Delta(\alpha\circ\Sigma\beta)=\Delta(\alpha)\circ\beta\,(\alpha\in\pi_m(S^n),\ \beta\in\pi_{n+k}(S^{m-1})). \tag{2}$$

We set $i'=i_{S^{n-1},Y}$.

Lemma 2.3 $\Delta\iota_n=qi'$ *for* $n\geq 3$.

Proof. We consider the exact sequence (1) for $k=-1$:

$$\pi_n(S^n)\xrightarrow{\Delta}\pi_{n-1}(Y)\xrightarrow{i_*}\pi_{n-1}(M^n_q)\longrightarrow 0.$$

Since $\pi_{n-1}(Y)=\mathbf{Z}\{i'\}$ and $\pi_{n-1}(M^n_q)=\mathbf{Z}_q\{i_n\}$, we have the assertion. □

Next we show the following result overlapping with that of [14].

Lemma 2.4 $\pi_n(M^n_q)=\{i_n\eta_{n-1}\}\cong\begin{cases}\mathbf{Z}_{(2q,q^2)} & (n=3)\\ \mathbf{Z}_{(q,2)} & (n\geq 4).\end{cases}$

Proof. We consider the exact sequence (1) for $k=0$:

$$\pi_{n+1}(S^n)\xrightarrow{\Delta}\pi_n(Y)\xrightarrow{i_*}\pi_n(M^n_q)\xrightarrow{p_{n*}}\pi_n(S^n).$$

Since $\pi_n(M^n_q)=i_{n*}\,\pi_n(S^{n-1})$, p_{n*} is trivial and i_* is an epimorphism. Note that $q\iota_2\circ\eta_2=q^2\eta_2$. So, by (2) and Lemma 2.3, we have

$$\begin{aligned}\Delta\eta_3&=\Delta\iota_3\circ\eta_2\\&=qi'\circ\eta_2\\&=i'\circ q\iota_2\circ\eta_2\\&=q^2i'\eta_2.\end{aligned}$$

Since $\pi_3(Y)=\mathbf{Z}_{2q}\{i'\eta_2\}$, we have $\pi_3(M^3_q)\cong\pi_3(Y)/(\mathrm{Im}\Delta)\cong\mathbf{Z}_{(2q,q^2)}$.

For $n\geq 4$, we have $\pi_n(Y)=\mathbf{Z}_2\{i'\eta_{n-1}\}$ and

$$\begin{aligned}\Delta\eta_n&=\Delta\iota_n\circ\eta_{n-1}\\&=qi'\circ\eta_{n-1}\\&=qi'\eta_{n-1}.\end{aligned}$$

This leads to the assertion, completing the proof. □

Assume that q is even and set $q'=\frac{q}{2}$. Then we consider the following commutative diagram between the cofiber sequences:

$$\begin{array}{ccccccc}S^{n-1}&\xrightarrow{2\iota_{n-1}}&S^{n-1}&\xrightarrow{i}&M^n_2&\xrightarrow{p}&S^n\\ \Big\downarrow{=}&&\Big\downarrow{q'\iota_{n-1}}&&\Big\downarrow{c_n}&&\Big\downarrow{=}\\ S^{n-1}&\xrightarrow{q\iota_{n-1}}&S^{n-1}&\xrightarrow{i_n}&M^n_q&\xrightarrow{p_n}&S^n.\end{array}\qquad(3)$$

Here $p: M^n_2\to S^n$ is the collapsing map, $i: S^{n-1}\to M^n_2$ is the inclusion and $c_n: M^n_2\to M^n_q$ is the natural map.

When there exists an element $\beta\in\pi_k(M^n_q)$ satisfying $p_n\beta=\alpha$ for a given element $\alpha\in\pi_k(S^n)$, β is called a lift of α. A lift β is written as $[\alpha]$. Now we show a sharper result than that of Lemma 3.3 of [13].

Lemma 2.5 *Let* $q \equiv 0 \bmod 4$, $q' = \frac{q}{2}$ *and* $n \geq 3$. *Suppose that* $\alpha \in \pi_k(S^n)$ *has a lift* $[\alpha] \in \pi_k(M_2^n)$. *Then* $c_n[\alpha] \in \pi_k(M_q^n)$ *is a lift of* α. *Moreover, if* $2[\alpha]=0$ or $2[\alpha]=i\eta_{n-1}\alpha$, *then* $c_n[\alpha] \in \pi_k(M_q^n)$ *is of order 2.*

Proof. By the diagram (3), we have $p_n \circ c_n \circ [\alpha] = p[\alpha] = \alpha$. If $2[\alpha]=0$, then $2c_n[\alpha]=0$. Suppose that $2[\alpha]=i\eta_{n-1}\alpha$. Then we have $2c_n[\alpha] = c_n \circ i \circ \eta_{n-1} \circ \alpha = i_n \circ q'\iota_{n-1} \circ \eta_{n-1} \circ \alpha = 0$ for $n \geq 4$. Note that $2\iota_n \circ \alpha = 0$ if α is lifted to M_2^n ([10]). For $n=3$, we have

$$\begin{aligned} 2c_3[\alpha] &= c_3 \circ i \circ \eta_2 \circ \alpha \\ &= i_3 \circ q'\iota_2 \circ \eta_2 \circ \alpha \\ &= i_3 \circ \eta_2 \circ (q')^2 \iota_3 \circ \alpha \\ &= 0. \end{aligned}$$

This completes the proof. □

The following elements are known to be lifted to M_2^n ([10]): $\eta_n (n \geq 3)$, $\nu_n^2 (n \geq 5)$, $\varepsilon_n (n \geq 4)$, $\bar{\nu}_n (n \geq 7)$, $\mu_n (n \geq 4)$, $\kappa_n (n \geq 11)$, $\sigma_n^2 (n \geq 17)$, $\omega_n (n \geq 15)$, $\bar{\mu}_n (n \geq 4)$, $\bar{\sigma}_n (n \geq 8)$. The problem whether ε_3, μ_3, $\bar{\mu}_3$ can be lifted to M_2^3 and $\bar{\sigma}_7$ can be lifted to M_2^7 is open ([10]).

3 Proof of the theorems

A lift of an element $\alpha_n \in \pi_k(S^n)$ is also denoted as $\tilde{\alpha}_{n-1} \in \pi_k(M_q^n)$. We recall that $\pi_4(M_2^3)$ is isomorphic to $\mathbf{Z}_4$ and it is generated by a lift $\tilde{\eta}_2$ of η_3 and $2\tilde{\eta}_2 = i\eta_2^2$. We show the following ([4]).

Example 3.1 $\pi_4(M_q^3) = \mathbf{Z}_2\{c_3\tilde{\eta}_2\} \oplus \mathbf{Z}_2\{i_3\eta_2^2\}$ *for* $q \equiv 0 \bmod 4$.

Proof. We consider the exact sequence (1) for $n=3$ and $k=1$:

$$\pi_5(S^3) \xrightarrow{\Delta} \pi_4(Y) \xrightarrow{i_*} \pi_4(M_q^3) \xrightarrow{p_{3*}} \pi_4(S^3).$$

By (2) and the proof of Lemma 2.4, we have

$$\Delta(\eta_3^2) = \Delta\eta_3 \circ \eta_3 = q^2 i' \eta_2 \circ \eta_3 = 0.$$

Obviously $\pi_4(Y) = \mathbf{Z}_2\{i'\eta_2^2\}$. By Lemma 2.5 for η_3, $c_3\tilde{\eta}_2$ is of order 2. This completes the proof. □

Hereafter we shall work in the 2-primary components of homotopy groups, unless otherwise stated.

The stable group $\pi_k^S(M^2)$ has the exponent 4 because $4\iota' = 0 \in \{M^2, M^2\}$. By making use of the exact sequence induced from the cofibration starting with $4\iota_1$, we can determine the stable group $\pi_k^S(M^2)$ for $k \leq 23$. For example, we determine the group in the case of $k=3$, 4, 8 and 17. The rest of Theorem 1.1 is obtained by the similar argument.

Example 3.2 (i) $\pi_3^S(M^2)=\mathbf{Z}_2\{\tilde{\eta}\} \oplus \mathbf{Z}_2\{i\eta^2\}$ and $\pi_4^S(M^2)=\mathbf{Z}_2\{\tilde{\eta}\eta\} \oplus \mathbf{Z}_4\{i\nu\}$.
(ii) $\pi_8^S(M^2)=\mathbf{Z}_4\{i\sigma\} \oplus \mathbf{Z}_2\{\widetilde{\nu^2}\}$.
(iii) $\pi_{17}^S(M^2)=\mathbf{Z}_4\{\widetilde{8\rho}\} \oplus \mathbf{Z}_2\{\tilde{\eta}\kappa\} \oplus \mathbf{Z}_2\{i\eta^*\} \oplus \mathbf{Z}_2\{i\eta\rho\}$, *where* $\widetilde{8\rho}$ *is a coextension of* 8ρ.

Proof. The first half of (i) is easily obtained.

In the exact sequence

$$\cdots \xrightarrow{4\iota_*} \pi_4^S(S^1) \xrightarrow{i_*} \pi_4^S(M^2) \xrightarrow{p_*} \pi_4^S(S^2) \longrightarrow 0,$$

we have $p_*(\tilde{\eta}\eta)=\eta^2$ and the order of $i\nu$ is 4. This leads to the second half of (i).

In the exact sequence

$$\cdots \xrightarrow{4\iota_*} \pi_8^S(S^1) \xrightarrow{i_*} \pi_8^S(M^2) \xrightarrow{p_*} \pi_8^S(S^2) \longrightarrow 0,$$

we know $\pi_8^S(S^2)=\mathbf{Z}_2\{\nu^2\}$ and $\pi_8^S(S^1)=\mathbf{Z}_{16}\{\sigma\}$. The order of $i\sigma$ is 4. The order of $\widetilde{\nu^2}=\Sigma^\infty(c_5\widetilde{\nu_4^2})$ is 2 because $\widetilde{\nu_4^2} \in \pi_{11}(M_2^5)$ is of order 2 ([10]). This leads to (ii).

In the exact sequence

$$0 \longrightarrow \pi_{17}^S(S^1) \xrightarrow{i_*} \pi_{17}^S(M^2) \xrightarrow{p_*} \pi_{17}^S(S^2) \xrightarrow{4\iota_*} \cdots,$$

we know $\pi_{17}^S(S^2)=\mathbf{Z}_{32}\{\rho\} \oplus \mathbf{Z}_2\{\eta\kappa\}$ and $\pi_{17}^S(S^1)=\mathbf{Z}_2\{\eta^*\} \oplus \mathbf{Z}_2\{\eta\rho\}$. Since $p_*\widetilde{8\rho}=8\rho$, the order of $\widetilde{8\rho}$ is 4. This leads to (iii), completing the proof. □

We show

Lemma 3.3 $\pi_7(M^5)=\mathbf{Z}_8\{i_5\nu_4\} \oplus \mathbf{Z}_2\{i_5\Sigma\nu'\} \oplus \mathbf{Z}_2\{\tilde{\eta}_4\eta_6\}$.

Proof. In the exact sequence (1) for $n=5$, $k=2$:

$$\pi_8(S^5) \xrightarrow{\Delta} \pi_7(Y) \xrightarrow{i_*} \pi_7(M^5) \xrightarrow{p_{5*}} \pi_7(S^5),$$

we have $p_{5*}(\tilde{\eta}_4\eta_6)=\eta_5^2$ and $\pi_7(Y)=\mathbf{Z}_8\{i'\nu_4\} \oplus \mathbf{Z}_{12}\{i'\Sigma\nu'\}$ because $4[\iota_4, \iota_4]=8\nu_4$. Here, by abuse of notation, $\Sigma\nu'$ stands for a generator of the direct summand Z_{12} of $\pi_7(S^4)$.

By Theorem XI.8.9 of [16], we have $4\iota_4 \circ \nu_4=4\nu_4+6[\iota_4, \iota_4]=16\nu_4-6\Sigma\nu'$. So we have $\Delta\nu_5=\Delta\iota_5 \circ \nu_4=4i' \circ \nu_4=-6i'\Sigma\nu'$. This leads to the group $\pi_7(M^5)$, completing the proof. □

Let A be a connected CW-complex and $X=A \cup e^n$ be the complex formed by attaching an n-cell. Let CY be the reduced cone of a pointed space Y. For an element

$\alpha \in \pi_{k-1}(Y)$, we denote by $\hat{\alpha}' \in \pi_k(CY, Y)$ an element satisfying $\partial\hat{\alpha}' = \alpha$, where $\partial : \pi_k(CY, Y) \to \pi_{k-1}(Y)$ is the connecting isomorphism. Let $\omega \in \pi_n(X, A)$ be the characteristic map of the n-cell e^n of X. For $\alpha \in \pi_{k-1}(S^{n-1})$, we set $\hat{\alpha} = \omega\hat{\alpha}' \in \pi_k(X, A)$. Let $\omega \in \pi_n(M^n, S^{n-1})$ and $\gamma \in \pi_n(M_2^n, S^{n-1})$ be the characteristic maps of the n-cells of M^n and M_2^n respectively. Let $[\omega, \iota_{n-1}] \in \pi_{2n-2}(M^n, S^{n-1})$ be the relative Whitehead product ([5]). Then, by (3) and [5], we can take $\omega = c_n\gamma$ and we have

$$c_n[\gamma, \iota_{n-1}] = 2[\omega, \iota_{n-1}]. \tag{4}$$

Let $j : (M^n, *) \to (M^n, S^{n-1})$ be the inclusion. We show

Lemma 3.4 $\pi_{15}(M^9) = \mathbf{Z}_8\{i_9\sigma_8\} \oplus \mathbf{Z}_2\{i_9\Sigma\sigma'\} \oplus \mathbf{Z}_2\{\widetilde{\nu_8^2}\}$.

Proof. We consider the homotopy exact sequence of pair (M^9, S^8):

$$\pi_{16}(M^9, S^8) \xrightarrow{\partial} \pi_{15}(S^8) \xrightarrow{i_{9*}} \pi_{15}(M^9) \xrightarrow{j_*} \pi_{15}(M^9, S^8).$$

Since $\pi_{15}(M^9, S^8) = \mathbf{Z}_2\{\widehat{\nu_8^2}\}$ and $j_*\widetilde{\nu_8^2} = \widehat{\nu_8^2}$, j_* is a spilit epimorphism. By Theorem 2.1 of [7], $\pi_{16}(M^9, S^8)$ is generated by elements $\hat{\sigma}_8$ and $[\omega, \iota_8]$. By Theorem XI.8.9 of [16], $\partial\hat{\sigma}_8 = 4\iota_8 \circ \sigma_8 = 4\sigma_8 + 6[\iota_8, \iota_8] = 16\sigma_8 - 6\Sigma\sigma'$. By [5], $\partial[\omega, \iota_8] = -4[\iota_8, \iota_8] = -8\sigma_8 + 4\Sigma\sigma'$. So we have

$$\partial(\hat{\sigma}_8 + 2[\omega, \iota_8]) = 2\Sigma\sigma'$$

and

$$\partial(2\hat{\sigma}_8 + 3[\omega, \iota_8]) = 8\sigma_8.$$

This determines the group $\pi_{15}(M^9)$, completing the proof. □

We recall that $\pi_{13}(S^6) = \mathbf{Z}_4\{\sigma''\}$, $2\sigma'' = \Sigma\sigma'''$ and $H(\sigma'') = \eta_{11}^2$ ([15]). By the Hilton formula ([16]), $2\iota_6 \circ \sigma'' = 2\sigma'' + [\iota_6, \iota_6] \circ H(\sigma'') = \Sigma\sigma'''$, because $[\iota_6, \iota_6] \circ \eta_{11}^2 = [\eta_6, \eta_6] = \eta_6 \circ [\iota_7, \iota_7] = 0$. So we have the relation $4\iota_6 \circ \sigma'' = 2\Sigma\sigma''' = 0$ and a coextension $\widetilde{\sigma''} \in \pi_{14}(M^7)$ of σ'' is taken as a representative of a Toda bracket $\{i_7, 4\iota_6, \sigma''\}$. $\widetilde{\sigma''}$ is a lift of $\Sigma\sigma''$. We recall that $\pi_{14}(S^6) = \mathbf{Z}_8\{\bar{\nu}_6\} \oplus \mathbf{Z}_2\{\varepsilon_6\}$ ([15]). Then we show

Lemma 3.5 *By a suitable choice of a coextension $\widetilde{\sigma''}$, the order of $\widetilde{\sigma''}$ is 4.*

Proof. We have $4\widetilde{\sigma''} \in \{i_7, 4\iota_6, \sigma''\} \circ 4\iota_{14} = -i_7\{4\iota_6, \sigma'', 4\iota_{13}\}$. By Corollary 3.7 of [15], we have $\{2\iota_6, \Sigma\sigma''', 2\iota_{13}\}_1 \ni (\Sigma\sigma''')\eta_{13} = 0 \bmod 2\pi_{14}(S^6) = \{2\bar{\nu}_6\}$. So we have $\{2\iota_6, \Sigma\sigma''', 2\iota_{13}\} \ni 0 \bmod 2\iota_6 \circ \pi_{14}(S^6) + 2\pi_{14}(S^6) = \{2\bar{\nu}_6\}$. Here we have used the relations $2\iota_6 \circ \varepsilon_6 = 0$ and $2\iota_6 \circ \bar{\nu}_6 = 4\bar{\nu}_6$.

We have

$$\begin{aligned}\{4\iota_6, \sigma'', 4\iota_{13}\} &\subset \{2\iota_6, 2\iota_6 \circ \sigma'', 4\iota_{13}\}\\ &= \{2\iota_6, \Sigma\sigma''', 4\iota_{13}\}\end{aligned}$$

$$\begin{aligned}&\supset \{2\iota_6, \Sigma\sigma''', 2\iota_{13}\} \circ 2\iota_{14}\\ &\ni 0 \bmod \{4\bar{\nu}_6\} + 2\iota_6 \circ \pi_{14}(S^6) = \{4\bar{\nu}_6\}.\end{aligned}$$

So we have $4\widetilde{\sigma''} = 4ai_7\bar{\nu}_6$ for $a \in \mathbf{Z}$. We set $\widetilde{\sigma''}' = \widetilde{\sigma''} - ai_7\bar{\nu}_6$. Then $p_7\widetilde{\sigma''}' = \Sigma\sigma'' = 2\sigma'$ and $4\widetilde{\sigma''}' = 0$. By renaming $\widetilde{\sigma''}'$ as $\widetilde{\sigma''}$, we have the assertion. This completes the proof. □

We set $\widetilde{4\sigma_n} = \Sigma^{n-6}\widetilde{\sigma''} (n \geq 9)$. Since $p_{n+1}\widetilde{4\sigma_n} = 4\sigma_{n+1}$, the order of $\widetilde{4\sigma_n}$ $(n \geq 9)$ is 4. We show

Lemma 3.6 $\pi_{17}(M^{10}) = \mathbf{Z}_4\{\widetilde{4\sigma_9}\} \oplus \mathbf{Z}_2\{i_{10}[\iota_9, \iota_9]\} \oplus \mathbf{Z}_2\{i_{10}\bar{\nu}_9\} \oplus \mathbf{Z}_2\{i_{10}\varepsilon_9\}$.

Proof. We consider the exact sequence (1) for $n=10$ and $k=7$:

$$\pi_{18}(S^{10}) \xrightarrow{\Delta} \pi_{17}(Y) \xrightarrow{i_*} \pi_{17}(M^{10}) \xrightarrow{p_{10*}} \pi_{17}(S^{10}).$$

We have $\mathrm{Im}\,\Delta = 0$ and $\pi_{17}(Y) \cong \pi_{17}(S^9) = \mathbf{Z}_2\{\varepsilon_9\} \oplus \mathbf{Z}_2\{\bar{\nu}_9\} \oplus \mathbf{Z}_2\{[\iota_9, \iota_9]\}$ by[15]and Lemma 2. 2. This completes the proof. □

The group $\pi_{2n-3}(M^n)$ for $11 \leq n \leq 13$ is given as follows ([13]).

Example 3.7 (i) $\pi_{19}(M^{11}) = \mathbf{Z}_4\{i_{11}[\iota_{10}, \iota_{10}]\} \oplus \mathbf{Z}_2\{i_{11}\nu_{10}^3\} \oplus \mathbf{Z}_2\{i_{11}\mu_{10}\} \oplus \mathbf{Z}_2\{i_{11}\eta_{10}\varepsilon_{11}\} \oplus \mathbf{Z}_2\{\tilde{\bar{\nu}}_{10}\} \oplus \mathbf{Z}_2\{\tilde{\varepsilon}_{10}\}$.

(ii) $\pi_{21}(M^{12}) = \mathbf{Z}_2\{i_{12}\sigma_{11}\nu_{18}\} \oplus \mathbf{Z}_2\{i_{12}\eta_{11}\mu_{12}\} \oplus \mathbf{Z}_2\{\widetilde{\nu_{11}^2}\nu_{18}\} \oplus \mathbf{Z}_2\{\tilde{\mu}_{11}\} \oplus \mathbf{Z}_2\{\tilde{\eta}_{11}\varepsilon_{13}\}$.

(iii) $\pi_{23}(M^{13}) = \mathbf{Z}_4\{i_{13}[\iota_{12}, \iota_{12}]\} \oplus \mathbf{Z}_4\{i_{13}\zeta_{12}\} \oplus \mathbf{Z}_2\{\tilde{\eta}_{12}\mu_{14}\}$.

Although we can get the group $\pi_{31}(M^{17})$ quickly ([13]), we take a roundabout way. First we recall that $\kappa_{10} \in \pi_{24}(S^{10})$ is not lifted to M_2^{10} ([10]) and has the property ([15], [9])

$$2\iota_{10} \circ \kappa_{10} = 2\kappa_{10} = 0.$$

So we can define a coextension $\tilde{\kappa}_{10} \in \{i, 2\iota_{10}, \kappa_{10}\} \subset \pi_{25}(M_2^{11})$ of κ_{10}. We know $\pi_{25}(S^{10}) = \mathbf{Z}_{16}\{\Sigma\rho'\} \oplus \mathbf{Z}_2\{\eta_{10}\kappa_{11}\} \oplus \mathbf{Z}_2\{\sigma_{10}\bar{\nu}_{17}\}$ and $\pi_{26}(S^{11}) = \mathbf{Z}_{16}\{\Sigma^2\rho'\} \oplus \mathbf{Z}_2\{\eta_{11}\kappa_{12}\}$. Then we show

Lemma 3.8 *By a suitable choice of a coextension* $\tilde{\kappa}_{10}$,

$$2\tilde{\kappa}_{10} \equiv i\eta_{10}\kappa_{11} \bmod i\sigma_{10}\bar{\nu}_{17}$$

and the order of $\tilde{\kappa}_{10}$ *is 4.*

Proof. By Corollary 3.7 of [15], we have

$$\{2\iota_{11},\ \kappa_{11},\ 2\iota_{25}\}_1 \ni \kappa_{11}\eta_{25} = \eta_{11}\kappa_{12} \bmod 2\pi_{26}(S^{11}) = \{2\Sigma^2\rho'\}.$$

Since $\Sigma\{2\iota_{10},\ \kappa_{10},\ 2\iota_{24}\} \subset -\{2\iota_{11},\ \kappa_{11},\ 2\iota_{25}\}_1$, we have

$$\{2\iota_{10},\ \kappa_{10},\ 2\iota_{24}\} \ni \eta_{10}\kappa_{11} \bmod \{\sigma_{10}\bar{\nu}_{17}\} + 2\pi_{25}(S^{10}) = \{2\Sigma\rho',\ \sigma_{10}\bar{\nu}_{17}\}.$$

So we have

$$\begin{aligned} 2\tilde{\kappa}_{10} &\in \{i,\ 2\iota_{10},\ \kappa_{10}\} \circ 2\iota_{25} \\ &= -i\{2\iota_{10},\ \kappa_{10},\ 2\iota_{24}\} \\ &\ni i\eta_{10}\kappa_{11} \\ &\bmod \{2i\Sigma\rho',\ i\sigma_{10}\bar{\nu}_{17}\}. \end{aligned}$$

So, by a suitable choice of a coextension $\tilde{\kappa}_{10}$, we have the relation. In the stable range, we have $i\eta\kappa \neq 0$ in $\pi^S_{16}(M^2_2)$. Hence the order of $\tilde{\kappa}_{10}$ is 4. This completes the proof. □

Hereafter we set $\tilde{\kappa}_n = \Sigma^{n-10}\tilde{\kappa}_{10}$ for $n \geq 10$ for the coextension $\tilde{\kappa}_{10}$ in Lemma 3.8. Since $\sigma_{11}\bar{\nu}_{18} = 0$, we have $2\tilde{\kappa}_n = i\eta_n\kappa_{n+1}$ for $n \geq 11$.

We recall that σ^2_{16} is not lifted to M^{16}_2 ([10]). The following is a byproduct of our roundabout way.

Lemma 3.9 $[\iota_{16},\ \iota_{16}] \in \{2\iota_{16},\ \sigma^2_{16},\ 2\iota_{30}\} \bmod 2\pi_{31}(S^{16})$.

Proof. Since $2\sigma^2_{15} = [\iota_{15},\ \iota_{15}]$, a Toda bracket $\{2\iota_{16}, \sigma^2_{16}, 4\iota_{30}\}_1$ is well defined. By Proposition 2.6 of [15], we have

$$H\{2\iota_{16},\ \sigma^2_{16},\ 4\iota_{30}\}_1 = -\Delta^{-1}(2\sigma^2_{15}) \circ 4\iota_{31} = -\{4\iota_{31}\} = \pm\{2H([\iota_{16},\ \iota_{16}])\}.$$

The indeterminacy of $\{2\iota_{16},\ \sigma^2_{16},\ 4\iota_{30}\}_1$ is $2\iota_{16} \circ \Sigma\pi_{30}(S^{15}) + 4\pi_{31}(S^{16}) = \{4[\iota_{16},\ \iota_{16}],\ 2\rho_{16}\}$. So $\{2\iota_{16},\ \sigma^2_{16},\ 4\iota_{30}\}_1$ contains $2[\iota_{16},\ \iota_{16}]$ modulo elements of $\Sigma\pi_{30}(S^{15}) = \{\rho_{16},\ \eta_{16}\kappa_{17}\}$. In the stable case, $\eta\kappa \neq 0$ and $\langle 2\iota,\ \sigma^2,\ 4\iota\rangle \ni 0 \bmod 2\pi^S_{15}(S^0) = \{2\rho\}$. Hence we have

$$2[\iota_{16},\ \iota_{16}] \in \{2\iota_{16},\ \sigma^2_{16},\ 4\iota_{30}\}_1 \bmod \{4[\iota_{16},\ \iota_{16}],\ 2\rho_{16}\}.$$

We have

$$\begin{gathered} \{2\iota_{16},\ \sigma^2_{16},\ 4\iota_{30}\}_1 \subset \{2\iota_{16},\ \sigma^2_{16},\ 4\iota_{30}\} \supset 2\{2\iota_{16},\ \sigma^2_{16},\ 2\iota_{30}\} \\ \bmod 2\iota_{16} \circ \pi_{31}(S^{16}) + 4\pi_{31}(S^{16}) = \{4[\iota_{16},\ \iota_{16}],\ 2\rho_{16}\} \end{gathered}$$

So, for any element $\alpha \in \{2\iota_{16},\ \sigma^2_{16},\ 4\iota_{30}\}$, we have

$$2\alpha \equiv 2[\iota_{16},\ \iota_{16}] \bmod \{4[\iota_{16},\ \iota_{16}],\ 2\rho_{16}\}.$$

This implies the relation $\alpha \equiv [\iota_{16},\ \iota_{16}] \bmod \{2[\iota_{16},\ \iota_{16}],\ \rho_{16},\ \eta_{16}\kappa_{17}\}$. By the same argument as the above in the stable range, we have

$$\alpha \equiv [\iota_{16}, \iota_{16}] \bmod \{2[\iota_{16}, \iota_{16}], 2\rho_{16}\} = 2\pi_{31}(S^{16}).$$

The indeterminacy of $\{2\iota_{16}, \sigma_{16}^2, 2\iota_{30}\}$ is $2\iota_{16} \circ \pi_{31}(S^{16}) + 2\pi_{31}(S^{16}) = 2\pi_{31}(S^{16})$. Hence we have $[\iota_{16}, \iota_{16}] \in \{2\iota_{16}, \sigma_{16}^2, 2\iota_{30}\} \bmod 2\pi_{31}(S^{16})$. This completes the proof. □

Let $\widetilde{\sigma_{16}^2} \in \pi_{31}(M_2^{17})$ be a coextension of σ_{16}^2. Then we show

Lemma 3.10 $\pi_{31}(M_2^{17}) = \mathbf{Z}_4\{\tilde{\kappa}_{16}\} \oplus \mathbf{Z}_4\{\widetilde{\sigma_{16}^2}\} \oplus \mathbf{Z}_2\{i\rho_{16}\}$, *where* $2\tilde{\kappa}_{16} = i\eta_{16}\kappa_{17}$ *and* $2\widetilde{\sigma_{16}^2} = i[\iota_{16}, \iota_{16}]$ *for a suitable choice of a coextension* $\widetilde{\sigma_{16}^2}$.

Proof. In the exact sequence

$$\pi_{32}(M_2^{17}, S^{16}) \xrightarrow{\partial} \pi_{31}(S^{16}) \xrightarrow{i_*} \pi_{31}(M_2^{17}) \xrightarrow{j_*} \pi_{31}(M_2^{17}, S^{16}),$$

we know $\pi_{31}(M_2^{17}, S^{16}) \cong \pi_{31}(S^{17}) = \mathbf{Z}_2\{\sigma_{17}^2\} \oplus \mathbf{Z}_2\{\kappa_{17}\}$. So j_* is an epimorphism. By Theorem 2.1 of [7], we have $\pi_{32}(M_2^{17}, S^{16}) = \mathbf{Z}_{32}\{\hat{\rho}_{16}\} \oplus \mathbf{Z}_2\{\hat{\eta}_{16}\bar{\kappa}_{17}\} \oplus \mathbf{Z}\{[\omega, \iota_{16}]\}$. We have $\partial(\hat{\rho}_{16}) = 2\rho_{16}$, $\partial(\hat{\eta}_{16}\bar{\kappa}_{17}) = 0$ and $\partial([\omega, \iota_{16}]) = -2[\iota_{16}, \iota_{16}]$. We know $2\tilde{\kappa}_{16} = i\eta_{16}\kappa_{17}$.

By Lemma 3.9, we have

$$\begin{aligned} 2\widetilde{\sigma_{16}^2} &\in \{i, 2\iota_{16}, \sigma_{16}^2\} \circ 2\iota_{31} \\ &= -i\{2\iota_{16}, \sigma_{16}^2, 2\iota_{30}\} \\ &\ni i[\iota_{16}, \iota_{16}] \\ \bmod\ & 2i_*\pi_{31}(S^{16}). \end{aligned}$$

So, by a suitable choice of the coextension $\widetilde{\sigma_{16}^2}$, we have $2\widetilde{\sigma_{16}^2} = i[\iota_{16}, \iota_{16}]$. This completes the proof. □

We set $\widetilde{\sigma_n^2} = \Sigma^{n-16}\widetilde{\sigma_{16}^2}$ for $n \geq 16$. Now we show

Lemma 3.11 $\pi_{31}(M^{17}) = \mathbf{Z}_4\{i_{17}[\iota_{16}, \iota_{16}]\} \oplus \mathbf{Z}_4\{i_{17}\rho_{16}\} \oplus \mathbf{Z}_2\{i_{17}\eta_{16}\kappa_{17}\} \oplus \mathbf{Z}_2\{c_{17}\tilde{\kappa}_{16}\} \oplus \mathbf{Z}_2\{c_{17}\widetilde{\sigma_{16}^2}\}$.

Proof. We consider the exact sequence (1) for $k=14$ and $n=17$:

$$\pi_{32}(S^{17}) \xrightarrow{\Delta} \pi_{31}(Y) \xrightarrow{i_*} \pi_{31}(M^{17}) \xrightarrow{p_{17*}} \pi_{31}(S^{17}).$$

We know $\pi_{31}(S^{17}) = \mathbf{Z}_2\{\kappa_{17}\} \oplus \mathbf{Z}_2\{\sigma_{17}^2\}$ and $\pi_{32}(S^{17}) = \mathbf{Z}_{32}\{\rho_{17}\} \oplus \mathbf{Z}_2\{\eta_{17}\kappa_{18}\}$. We have $\pi_{31}(Y) = \mathbf{Z}_{32}\{i'\rho_{16}\} \oplus \mathbf{Z}_4\{i'[\iota_{16}, \iota_{16}]\} \oplus \mathbf{Z}_2\{i'\eta_{16}\kappa_{17}\}$. We have $\Delta(\rho_{17}) = 4i'\rho_{16}$ and $\Delta(\eta_{17}\kappa_{18}) = 0$. By Lemmas 2.5 and 3.10, the order of $c_{17}\tilde{\kappa}_{16}$ is 2. By Lemmas 2.5, 3.10 and by (3), we have

$$2c_{17}\widetilde{\sigma_{16}^2} = c_{17} \circ i[\iota_{16}, \iota_{16}] = i_{17} \circ 2\iota_{16} \circ [\iota_{16}, \iota_{16}] = 4i_{17}[\iota_{16}, \iota_{16}] = 0.$$

This completes the proof. □

The following result is easily obtained.

Lemma 3.12 (i) $\pi_{41}(M^{22})=\mathbf{Z}_4\{i_{22}\bar{\kappa}_{21}\}\oplus\mathbf{Z}_4\{2\widetilde{\bar{\zeta}_{21}}\}\oplus\mathbf{Z}_2\{i_{22}[\iota_{21},\iota_{21}]\}\oplus\mathbf{Z}_2\{\widetilde{\bar{\sigma}_{21}}\}$, *where* $2\widetilde{\bar{\zeta}_{21}}\in\{i_{22},4\iota_{21},2\bar{\zeta}_{21}\}$.
(ii) $\pi_{43}(M^{23})=\mathbf{Z}_4\{i_{23}[\iota_{22},\iota_{22}]\}\oplus\mathbf{Z}_4\{2\widetilde{\bar{\kappa}_{22}}\}\oplus\mathbf{Z}_2\{i_{23}\eta_{22}\bar{\kappa}_{23}\}\oplus\mathbf{Z}_2\{i_{23}\sigma^3_{22}\}$, *where* $2\widetilde{\bar{\kappa}_{22}}\in\{i_{23},4\iota_{22},2\bar{\kappa}_{22}\}$.
(iii) $\pi_{45}(M^{24})=\mathbf{Z}_2\{i_{24}[\iota_{23},\iota_{23}]\}\oplus\mathbf{Z}_2\{\widetilde{\sigma^2_{23}}\sigma_{31}\}\oplus\mathbf{Z}_2\{\tilde{\eta}_{23}\bar{\kappa}_{24}\}\oplus\mathbf{Z}_2\{i_{24}\eta^2_{23}\bar{\kappa}_{25}\}\oplus\mathbf{Z}_2\{i_{24}\nu_{23}\bar{\sigma}_{26}\}$.

The rest of Theorem 1.2 is obtained by the similar argument ([13]).

4 Some unstable homotopy groups of M^n

In this section, we shall prove Proposition 1.3. We recall that $\pi_6(M^4_2)=\mathbf{Z}_4\{\delta\}\oplus\mathbf{Z}_2\{\tilde{\eta}_3\eta_5\}$, where $j_*\delta=[\gamma,\iota_3]$ and $\Sigma\delta=2i\nu_4$ ([11]). We show

Lemma 4.1 *There exists an element* $\theta\in\pi_6(M^4)$ *satisfying* $j_*\theta=[\omega,\iota_3]$, $2\theta=\pm c_4\delta$ *and* $\pi_6(M^4)=\mathbf{Z}_8\{\theta\}\oplus\mathbf{Z}_2\{i_4\nu'+2\theta\}\oplus\mathbf{Z}_2\{c_4\tilde{\eta}_3\eta_5\}$.

Proof. By Theorem 2.1 of [7], we have $\pi_6(M^4,S^3)=\mathbf{Z}_2\{\widehat{\eta^2_3}\}\oplus\mathbf{Z}_4\{[\omega,\iota_3]\}$ and $\pi_7(M^4,S^3)=\mathbf{Z}_4\{\widehat{\nu'}\}\oplus\mathbf{Z}_2\{[\omega,\eta_3]\}$. In the homotopy exact sequence of a pair (M^4,S^3):

$$\pi_7(M^4,S^3)\xrightarrow{\partial}\pi_6(S^3)\xrightarrow{i_{4*}}\pi_6(M^4)\xrightarrow{j_*}\pi_6(M^4,S^3)\xrightarrow{\partial}\pi_5(S^3),$$

we have

$$\partial\widehat{\eta^2_3}=4\iota_3\circ\eta^2_3=0,\ \partial[\omega,\iota_3]=-[4\iota_3,\iota_3]=0,$$
$$\partial\widehat{\nu'}=4\iota_3\circ\nu'=4\nu'=0\text{ and }\partial[\omega,\eta_3]=-[4\iota_3,\eta_3]=0.$$

So there exists an element $\theta\in\pi_6(M^4)$ satisfying $j_*\theta=[\omega,\iota_3]$ and we have a short exact sequence:

$$0\longrightarrow\pi_6(S^3)\xrightarrow{i_{4*}}\pi_6(M^4)\xrightarrow{j_*}\pi_6(M^4,S^3)\longrightarrow 0.$$

By (4), we have

$$j_*(c_{4*}\delta)=c_{4*}j_*\delta=c_{4*}[\gamma,\iota_3]=2[\omega,\iota_3]=2j_*\theta.$$

So we have the relation $2\theta=c_4\delta+ai_4\nu'$ for an integer a. Note that we take $c_5=\Sigma c_4$ in the diagram (3). Then, by Lemma 3.3, we have

$$\begin{aligned} c_5\Sigma\delta &= c_5\circ 2i\nu_4 \\ &= 2(i_5\circ 2\iota_4\circ\nu_4) \\ &= 2(i_5(4\nu_4-\Sigma\nu')) \\ &= 8i_5\nu_4-2i_5\Sigma\nu' \\ &= 0. \end{aligned}$$

Therefore we have $2\Sigma\theta=ai_5\Sigma\nu'$. Since $i_5\Sigma\nu'$ is not divisible by 2 by Lemma 3.3, a becomes even. So we have the relation $2\theta\equiv c_4\delta \bmod 2i_4\nu'$. By the diagram (3), we have $4\theta=2c_4\delta=c_4\circ i\nu'=i_4\circ 2\iota_3\circ\nu'=2i_4\nu'\neq 0$. Hence the order of θ is 8 and $2i_4\nu'=4\theta=2c_4\delta$. Thus we have $2\theta=\pm c_4\delta$ and we get the group $\pi_6(M^4)$. This completes the proof. □

Let $\tilde{\nu}'\in\{i_4, 4\iota_3, \nu'\}\subset\pi_7(M^4)$ be a coextension of ν'. Then we show

Lemma 4.2 $\pi_8(M^5)=\mathbf{Z}_4\{\Sigma\tilde{\nu}'\}\oplus\mathbf{Z}_2\{i_5\nu_4\eta_7\}\oplus\mathbf{Z}_2\{i_5(\Sigma\nu')\eta_7\}$.

Proof. We consider the exact sequence

$$\pi_9(M^5, S^4)\xrightarrow{\partial}\pi_8(S^4)\xrightarrow{i_{5*}}\pi_8(M^5)\xrightarrow{j_*}\pi_8(M^5, S^4)\xrightarrow{\partial}\pi_7(S^4).$$

By Theorem 2.1 of [7], we have $\pi_8(M^5, S^4)=\mathbf{Z}\{[\omega, \iota_4]\}\oplus\mathbf{Z}_4\{\Sigma'\hat{\nu}'\}$ and $\pi_9(M^5, S^4)=\mathbf{Z}_2\{\hat{\nu}_4\hat{\eta}_7'\}\oplus\mathbf{Z}_2\{[\omega, \iota_4]\}$. Here Σ': $\pi_7(M^4, S^3)\to\pi_8(M^5, S^4)$ is the relative suspension. We have $\partial[\omega, \iota_4]=-8\nu_4$, $j_*\Sigma\tilde{\nu}'=\Sigma'\hat{\nu}'$, $\partial\hat{\nu}_4\hat{\eta}_7'=4\iota_4\circ\nu_4\circ\eta_7=(16\nu_4-6\Sigma\nu')\circ\eta_7=0$ and $\partial[\omega, \eta_4]=0$. So the following short exact sequence splits:

$$0\longrightarrow\pi_8(S^4)\xrightarrow{i_{5*}}\pi_8(M^5)\longrightarrow\mathbf{Z}_4\{\Sigma'\hat{\nu}'\}\longrightarrow 0.$$

This completes the proof. □

We set $\widetilde{2\nu_n}=\Sigma^{n-2}\tilde{\nu}'$ for $n\geq 5$. By use of the exact sequence (1) for $n=6$ and $k=3$, we have

Example 4.3 $\pi_9(M^6)=\mathbf{Z}_4\{\widetilde{2\nu_5}\}\oplus\mathbf{Z}_2\{i_6\nu_5\eta_8\}$.

By Theorem 1.2 of [7], we have $\pi_{10}(M^6, S^5)=\mathbf{Z}_8\{[\omega, \iota_5]\}$. Let $\beta\in\pi_{10}(M^6)$ be an element satisfying $j_*\beta=[\omega, \iota_5]$. Then we show

Lemma 4.4 $\pi_{10}(M^6)=\mathbf{Z}_8\{\beta\}\oplus\mathbf{Z}_2\{i_6\nu_5\eta_8^2\}$.

Proof. In the exact sequence

$$\pi_{11}(M^6, S^5) \xrightarrow{\partial} \pi_{10}(S^5) \xrightarrow{i_{6*}} \pi_{10}(M^6) \xrightarrow{j_*} \pi_{10}(M^6, S^5) \longrightarrow 0,$$

we have $\pi_{11}(M^6, S^5) = \mathbf{Z}_2\{[\omega, \eta_5]\}$ and $\partial[\omega, \eta_5]=0$. So it suffices to show $8\beta=0$. By the parallel argument to the proof of Lemma 4.1, we have a relation

$$2\beta = c_6\lambda + bi_6\nu_5\eta_8^2 \ (b=0, 1),$$

where λ is a generator of $\pi_{10}(M_2^6) \cong \mathbf{Z}_8$ satisfying $j_*\lambda = [\gamma, \iota_5]$ and $4\lambda = i\nu_5\eta_8^2$ ([11]). By the diagram (3), we have

$$8\beta = c_6 \circ i\nu_5\eta_8^2 = i_6 \circ 2\iota_5 \circ \nu_5\eta_8^2 = 0.$$

This completes the proof. □

Finally the following is easily obtained.

Example 4.5 $\pi_{12}(M^7) = \mathbf{Z}_2\{i_7\nu_6^2\}$.

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