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# *A Remark on Infinite Dimensional Gaussian Integral In a Sobolev Space*

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**Abstract** : In [2] $(\infty-p)$-form on a $k$-th Sobolev space $W^k(X)$, $X$ a compact (spin) manifold, was defined by using Sobolev duality. Integrals of $(\infty-p)$-form on an $(\infty-p)$-form on a cube in $W^k(X)$ were defined without using measure. We show when the lenghth of sides of the cube tends to $\infty$, infinite dimensional Gaussian integral that is principal on application converges if and only if the cube is imbedded in $W^k(X)$, $k<-d+\frac{1}{2}$.

## 0. Introduction

Analysis on infinite dimensional spaces together with its geometric applications, has been treated mostly by using probablitic methods(e.g.[4],[8]). But more classical analysis related to the geometry of infinite demensional spaces seems not so well developed. We define an $(\infty-p)$-form on $U$, an open set of $k$-th Sobolev space $W^k(X)$ over a $d$-dimensional compact (spin) manifold $X$ to be a smooth map $f$ from $U$ to $\Lambda^p W^k(X)$, the $k$-th Sobolev space of alternating functions (spinors) on $p$-th direct product $X_{\times\cdots\times}X$ of $X$([2]). Then we treat differential and integral calculuses of $(\infty-p)$-forms. The outline of the paper is as follows ; In sec. 1, we fix the Sobolev metric of $W^k(X)$ by apointing a non degenerate 1-st order selfadjoint elliptic (pseudo) differential operator $D$ on $X$. By using spectral eta and zeta functions of $D$ and $|D|$, we define virtual dimension $\nu_-$ of $W^k(X)$ and volumes of cubes (powers of det $|D|$) in $W^{-l-\alpha}(X)$. Some caluculations related to these quantities are also done. In sec. 2, integrals of a function $f$ on a cube in $W^{-l-\alpha}(X)$ is defined in the spirit of Riemannian integral. Some complete continuity of $f$ is necessary (and sufficient) to the existence of the existence of the integral. Then $\infty$-forms are introduced. In this paper, we do not discuss these developed details. We show how infinite dimensional Gaussian integral $e^{-\pi(x,Dx)}$ on $Q(l, t)=\{\sum c_n e_n || c_n|<|t\lambda_n|^l\}$ converges to $1/\sqrt{\det|D|}$ when $t\to\infty$ if and only if $l>(d-1)/2$. As a consequence, we make clear that the convergence of infinite

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dimensional Gaussian integral that appears in various field depends on the dimension of compact (spin) manifold, especially, and that dimension 1 is important.

## 1 Virtual dimension of a sobolev space

We review virtual dimension of a Sobolev space and the definition. Let $X$ be a compact (spin) manifold with a fixed Riemanian metric, $E$ a Hermitian vector bundle over $X$ and $L^2(X)$ is the Hilbert space of sections of $E$. We denote $L^2$-metric of $f \in L^2(X)$ by $||f||$. It is fixed by the Rieman metric of $X$. We take a non degenerate 1-st order selfadjoint elliptic (pseudo) differential operator $D$ acting on the section of $E$ and fix the $k$-th Sobolev metric $||f||_k$ of $f$ by

$$||f||_k = ||D^k f|| \tag{1}$$

The $k$-th Sobolev space of sections of $E$ is denoted by $W^k(X)$. By Sobolev' imbedding Theorem, $W^k(X)$ is contained in the space of continuous section of $E$ if $k > d/2$, $d$ is the dimension of $X$.

Since $X$ is compact, $D$ can be written as

$$Df = \sum \lambda(f, e_\lambda)e_\lambda, \tag{2}$$

$\{e_\lambda\}$ is an O.N.-basis of $L^2(X)$. Then, to set

$$e_{\lambda,k} = \text{sgn}\,\lambda\,|\lambda|^{-k}\,e_\lambda, \tag{3}$$

$\{e_{\lambda,k}\}$ is an O.N.-basis of $W^k(X)$.

The spectral eta function $\eta_D(s)$ of $D$ and $\zeta_{|D|}(s)$ of $|D|$ are defined by

$$\eta_D(s) = \sum \text{sgn}\,\lambda\,|\lambda|^{-s},\ \zeta_{|D|}(s) = \eta_{D^2}(s/2) = \sum |\lambda|^{-s}. \tag{4}$$

It is known ([3],[7],[9],[10])

1. These function are continued meromorphically on the whole complex plane with possible poles at $s = d, d-1, \cdots$ with the order at most 1.
2. They are holomophic at $s = 0$.

**Definition 1.1** *We say $\zeta_{|D|}(0) = \nu$ to be the virtual dimension of $W^k(x)$ (with respect to D).* We also define the determinant det $|D|$ of $|D|$ and det $D$ of $D$ by

$$\begin{aligned} &\det|D| = \exp(-\zeta'_{|D|}(0)), \\ &\det D = \exp(\pi\sqrt{-1}\,\zeta_{D_-}(0))\det|D|,\ \zeta_{D_-}(0) = (\nu_- - \eta_D(0))/2. \end{aligned} \tag{5}$$

Then we have

$$\begin{aligned} &\det(tD) = t^{\nu_-}\det D,\ t > 0, \\ &\det|D^k| = (\det|D|)^k. \end{aligned} \tag{6}$$

## 2. Integrals on a cube in a Sobolev space

In $W^{-l-\alpha}(X)$, $\alpha>d/2$, we set

$$Q(l,t)=\{\sum c_n e_n \| c_n| \le |t\lambda_n|^l\},$$
$$Q(l,t,+)=\{\sum c_n e_n | 0<c_n\le |t\lambda_n|^l\},\ t>0. \tag{7}$$

For simple, we assume $l\neq 0$, and set

$$\mathrm{vol}(Q(l,t))=(2t)^{l\nu_-}(\det|D|)^l,\ \mathrm{vol}(Q(l,t+))=t^{l\nu_-}(\det|D|)^l. \tag{8}$$

Let $s$ be in $I=[0,1]$ with the binary expansion $0.s_1\cdots s_n\cdots$. Then we difine a subset $D(s)$ of $Q(l,t)$ by

$$D(s)=\{\sum c_n e_n | -|t\lambda_n|^l\le c_n\le 0,\ \text{if } s_n=0,\ 0\le c_n\le |t\lambda_n|^l,\ \text{if } s_n=1\}. \tag{9}$$

By definition $Q(l,t)=\cup_{s\in I}D(s)$. For a function $f(x)$ on $Q(l,t)$, we define functions $\bar f$ and $\underline{f}$ on $I$ by

$$\bar f(s)=\sup_{x\in D(s)} f(x),\ \underline{f}(s)=\inf_{x\in D(s)} f(x) \tag{10}$$

Then the integrals $\int_I \bar f ds$ vol$Q(l,t)$ and $\int_I \underline{f} ds$ vol$Q(l,t)$ are upper and lower Riemannian sums of $f(x)$ with respect to the partition $\{D(s)\}$ of $Q(l,t)$.

We assume for $(s^1,\cdots,s^{m-1})\in I^{m-1}$, the partition $D(s^1,\cdots,s^{m-1})$ of $Q(l,t)$ has been defined to be $\{\sum c_n e_n | a_n\le c_n\le b_n\}$. Then for $s^m=0.s_1^m s_2^m\cdots\in I$, we set

$$D(s^1,\cdots,s^m)=\Big\{\sum c_n e_n | a_n\le c_n\le a_n+\frac{b_n-a_n}{2},\ \text{if } s_n^m=0,$$
$$a_n+\frac{b_n-a_n}{2}\le c_n\le b_n,\ \text{if } s_n^m=1\Big\}. \tag{11}$$

The functions $\bar f(s^1,\cdots,s^m)$ and $\underline{f}(s^1,\cdots,s^m)$ are defined to be

$$\bar f(s^1,\cdots,s^m)=\sup_{x\in D(s^1,\cdots,s^m)} f(x),$$
$$\underline{f}(s^1,\cdots,s^m)=\inf_{x\in D(s^1,\cdots,s^m)} f(x). \tag{12}$$

**Lemma 2.1** *$\bar f$ and $\underline{f}$ are continuous if $f$ is continuous by the topology of $W^{-l-\alpha}(X)$, $\alpha>d/2$.* Therefore we obtain

**Theorem 2.1** *if $f(x)$ is continuous by the topology of $W^{-l-\alpha}(X)$, $\alpha>d/2$, then*

$$\lim_{m\to\infty}\int_{I^m}\bar f d^m s=\lim_{m\to\infty}\int_{I^m}\underline{f} d^m s \tag{13}$$

**Definition 2.1** *Let $f$ be a (real valued) function of $Q(l,t)$. Then we say $f$ is integrable on $Q(l,t)$ if (13) is hold and define $\int_{Q(l,t)} f(x)dx$ by*

$$\int_{Q(l,t)} f(x)dx=\lim_{m\to\infty}\int_{I^m}\bar f d^m s\ \ \mathrm{vol}(Q(l,t)). \tag{14}$$

*Integrals on $Q(l, t, +)$ are similarly defined.*

**Note.** In the above definition of the integral, we used special division of $Q(l, t)$. But this is for simplicity and we can define integral by using more arbitrary division of $Q(l, t)$.

On exponetial calculation, because it is not too easy in 2.1, although it is essential at analysis, we use an alternative way([2]). We set

$$Q(l, t, N)=\left\{\sum_{n\leq N} c_n e_n \middle| -|t\lambda_n|^l \leq c_n \leq |t\lambda_n|^l,\ 1<n<N\right\},$$
$$Q(l, t, \infty-N)=\left\{\sum_{n\geq N+1} c_n e_n \middle| -|t\lambda_n|^l \leq c_n \leq |t\lambda_n|^l,\ n>N\right\}. \tag{15}$$

By definition $Q(l, t)=Q(l, t, N)\times Q(l, t, \infty-N)$. We denote $x=(x_N, x_{\infty-N})\in Q(l, t)$, where $x_N \in Q(l, t, N)$ and $x_{\infty-N} \in Q(l, t, \infty-N)$. Let $f$ be a function on $Q(l, t)$. Then we set

$$\bar{f}^N(x_N)=\sup_{y\in Q(l,t,\infty-N)} f(x_N, y),\ \underline{f}_N=\inf_{y\in Q(l,t,\infty-N)} f(x_N, y) \tag{16}$$

Then if $f$ is continuous by the topology of $W^{-l-a}(X)$, $a>d/2$, we have

$$\int_{Q(l,t)} f(x)dx=\lim_{N\to\infty}\int_{Q(l,t,N)} \bar{f}^N(x_N)d^N x|2t\lambda_1|^{-l}\cdots|2t\lambda_1|^{-l}\operatorname{vol}(Q(l, t)). \tag{17}$$

## 3 Gaussian integral of infinite dimension in a Sobolev space

Let $f(x)$ be

$$f(x)=\exp(-\pi\sum\lambda_n x_n^2),\ x=\sum x_n e_n \in Q(l, t),\ \lambda_n>0(n=1,2,\ldots) \tag{18}$$

Then, for the function $\int_{Q(l,t)} f(x)dx$ is computed as follows:

$$\begin{aligned}
&\int_{Q(l,t)} \exp\left(-\pi\sum_{n=1}^{\infty}\lambda_n x_n^2\right)dx\\
&=\lim_{N\to\infty}\int_{-(t\lambda_1)^l}^{(t\lambda_1)^l}\cdots\int_{-(t\lambda_N)^l}^{(t\lambda_N)^l}\exp\left(-\pi\sum_{n=1}^{N}\lambda_n x_n^2\right)d^N x|2t\lambda_1|^{-l}\cdots|2t\lambda_1|^{-l}\operatorname{vol}(Q(l, t))\\
&=\lim_{N\to\infty}\left(\prod_{n=1}^{N}\int_{-(t\lambda_n)^l}^{(t\lambda_n)^l}\exp(-\pi\lambda_n x_n^2)dx_n|2t\lambda_n|^{-l}\right)\operatorname{vol}(Q(l, t))\\
&=\lim_{N\to\infty}\left\{\prod_{n=1}^{N}\left(\frac{1}{\sqrt{\lambda_n}}-\frac{2}{\sqrt{\pi\lambda_n}}\operatorname{Erfc}(\sqrt{\pi\lambda}(t\lambda_n)^l)\right)|2t\lambda_n|^{-l}\right\}\operatorname{vol}(Q(l, t)).
\end{aligned} \tag{19}$$

Using incomplete $\Psi$-function

$$\operatorname{Erfc} x=\int_x^{\infty}e^{-u^2}du=\frac{1}{2}e^{-x^2}\Psi(1/2, 1/2; x^2),$$
$$\Psi(a, c; x)=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-xu}u^{a-1}(1+u)^{c-a-1}du \quad \operatorname{Re}\ a>0, \tag{20}$$

We have (cf.[5])

$$\int_{Q(l,t)} \exp\left(-\pi \sum_{n=1}^{\infty} \lambda_n x_n^2\right) dx$$
$$= \lim_{N\to\infty}\left\{ \prod_{n=1}^{N} \frac{1}{\sqrt{\lambda_n}} \left(1 - \frac{1}{\sqrt{\pi}} \Psi(1/2, 1/2\,;\, \pi t^{2l} \lambda_n^{2l+1})\right) |2t\lambda_n|^{-l} \right\} \mathrm{vol}(Q(l,\, t)). \tag{21}$$

In (21) we regard as

$$\lim_{N\to\infty} |2t\lambda_1|^{-l} \cdots |2t\lambda_N|^{-l}\, \mathrm{vol}(Q(l,\, t)) = 1,$$
$$\lim_{N\to\infty} \sqrt{\lambda_1} \cdots \sqrt{\lambda_N} = \sqrt{\det|D|}. \tag{22}$$

Justifications of (22) will be discussed in Appendix, so we consider only the limit

$$\lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{\sqrt{\lambda_n}} \left\{1 - \frac{1}{\sqrt{\pi}} e^{-\pi t^{2l} \lambda_n^{2l+1}} \Psi(1/2, 1/2\,;\, \pi t^{2l} \lambda_n^{2l+1})\right\}. \tag{23}$$

Generally, as the absolute sum $\sum|x|$ converges on finite value, we prove positively infinite product $\prod(1+x)$ converges on finite value. Therefore, we discuss following convergence:

$$\lim_{n\to\infty} \sum |e^{-\pi t^{2l} \lambda_n^{2l+1}} \Psi(1/2, 1/2\,;\, \pi t^{2l} \lambda_n^{2l+1})|. \tag{24}$$

Because convergence of exponetial function is too fast, we certify that $\Psi$-function converges as form of $n^{-M}$, $M$ is a const. $\Psi$-function is written as

$$\Psi(1/2, 1/2; x^2) = \frac{1}{\Gamma(1/2)} \int_0^{\infty} e^{-x^2 u} u^{-1/2} (1+u)^{-1} dt$$
$$= \frac{1}{x\Gamma(1/2)} \int_0^{\infty} e^{-t} t^{-1/2} \left(1 + \frac{t}{x^2}\right)^{-1} dt. \tag{25}$$

In this equation, $(1+t/x^2)^{-1}$ is a monotone increasing function regarding $x$. If $x \geq a > 0$, We have

$$\frac{1}{x\Gamma(1/2)} \int_0^{\infty} e^{-t} t^{-1/2} dt > \frac{1}{x\Gamma(1/2)} \int_0^{\infty} e^{-t} t^{-1/2} \left(1 + \frac{t}{x^2}\right)^{-1} dt. \tag{26}$$

Samely, if $x \geq a > 0$,

$$\frac{1}{x\Gamma(1/2)} \int_0^{\infty} e^{-t} t^{-1/2} \left(1 + \frac{t}{x^2}\right)^{-1} dt > \frac{1}{x\Gamma(1/2)} \int_0^{\infty} e^{-t} t^{-1/2} \left(1 + \frac{t}{a^2}\right)^{-1} dt. \tag{27}$$

So $\Psi$-function does not diverge, and it contributes to infinite product as form in proportional to $1/x = 1/\sqrt{\pi\lambda_n}(t\lambda_n)^l$. Therefore the convergence of infinite demensional Gaussian integral results in the convergence of the exponetial part of (24). Because $\lim_{n\to\infty} \sqrt[n]{1/\sqrt{\pi\lambda_n}(t\lambda_n)^l} = 1$. Therefore we get $(2l+1)/d > 1$, $i,e$.

$$l > \frac{d-1}{2} \tag{28}$$

as the necessary and sufficient condition to the convergence of (23) by the asymptotic distribution of $\{\lambda_n\}$([6]). The consequence that the integrablity must depend on dimension $d$ is interesting one. Since $Q(l,\, t) \subset W^{-l-a}(X)$, $a > d/2$, (28) shows

$$\lim_{n\to\infty}\int_{Q(l,t)} e^{-\pi(x,Dx)}\,dx=\frac{1}{\sqrt{\det D}} \tag{29}$$

holds if and only if $Q(l,\ t)\subset W^{k}(X)$, $k<-d+1/2$. Since (23) divergence to 0 if $l\leq\dfrac{d-1}{2}$, we may consider

$$\lim_{t\to\infty}\int_{Q(l,t)} e^{-\pi(x,Dx)}\,dx=0 \tag{30}$$

if $Q(l,\ t)\subset W^{-l-k}(X)$, $a\leq d/2$.

## Appendix

Since $e^{-\zeta'_{D}(s)}=\prod|\lambda_n|^{|\lambda_n|^{-s}}$, $s>d/2$, replacing $\lambda_n$ by $\lambda_n^{\lambda_n^{-s}}\equiv a_n(s)$, we have

$$\lim_{n\to\infty}\prod a_n(s)=e^{-\zeta'_D}(s)$$
$$\lim_{n\to\infty}\prod (a_n(s))^{-1/2}=(e^{-\zeta'_D}(s))^{-1/2}. \tag{31}$$

Analytic continuation on $s$ provides (22). There remains one problem. Since we replace $\lambda_n$ by $a_n(s)$. The infinite product

$$\prod\left(1-\frac{2}{\sqrt{\pi}}\mathrm{Erfc}(\sqrt{\pi a_n(s)}(ta_n(s))^{l})\right) \tag{32}$$

does not coverge. Therefore, we need first to consider the limit

$$\lim_{N\to\infty}\int_{-(ta_1(s))^{l}}^{(ta_1(s))^{l}}\cdots\int_{-(ta_N(s))^{l}}^{(ta_N(s))^{l}}\exp(-\pi\sum a_n(s)x_n^2)d^Nx\times \tag{33}$$

$$\left(\prod_{n=1}^{N}\left(1-\frac{2}{\sqrt{\pi}}\mathrm{Erfc}(\sqrt{\pi a_n(s)}(ta_n(s))^{l})\right)\right)^{-1}|2ta_1(s)|^{-1}\cdots|2ta_N(s)|^{-1}(e^{-2t\zeta'_D(s)})^{-1}, \tag{34}$$

which is $(e^{-2t\zeta'_D(s)})^{-1/2}$. Then we consider its analytic continuation to $s\to 0$.

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