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The Dirichlet Problem for a Certain Degenerate Parabolic Equation

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Abstract: The unique existence of the time-global solution is shown for a certain degenerate parabolic equation with a zero boundary condition.

1. Introduction

In this paper, we consider, from a classical point of view, the following degenerate parabolic equation

(1.1) $$u_t(x,\ t)=x(1-x)u_{xx}(x,\ t) \quad (0<x<1\ ,\ 0<t<\infty)$$

with boundary-initial conditions

(1.1)′ $$u(x,0)=u_0(x) \quad (0\leqq x\leqq 1),$$
(1.1)″ $$u(0,t)=u(1,t)=0 \quad (t\geqq 0),$$

[as the case may be, suitable compatible conditions are imposed].

We shall construct the solution to (1.1)-(1.1)′-(1.1)″ by means of the theory of integral equations and that of orthogonal functions. The uniqueness of the solution is shown by the use of the maximum principle. The notation used is conventional, so that we do not explain it in particular beforehand.

2. Preparation

Firstly, we search for functions $\{U(x,\ t)\}$ satisfying only (1.1)-(1.1)″. According to the well-known method of separation of variables, if $U(x,\ t)=X(x)T(t)$, then $X(x)$ and $T(t)$ satisfy

(2.1) $$x(1-x)\frac{X''(x)}{X(x)}=\frac{T'(t)}{T(t)}=-\lambda \quad (\lambda: \text{const.}),$$

whence we have for $X(x)$ and $T(t)$, resp.,

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(2.2) $$x(1-x)X'' = -\lambda X \quad (X(0) = X(1) = 0),$$

(2.2)′ $$T' = -\lambda T \quad (\text{i.e., } T(t) = c_0 e^{-\lambda t} \quad (c_0 : \text{const.})),$$

where we note that the ordinary differential equation (2.2) is an hypergeometric one with $-\alpha\beta = \lambda$, $\alpha + \beta + 1 = 0$ and $\gamma = 0$, being of the Fuchsian type with singular points at $x = 0, 1$. Thus, we have as solutions for (2.2), putting $\lambda = \lambda_n \equiv (n+1)(n+2)$ $(n = 0, 1, 2, \cdots)$,

(2.3) $$X_n(x) = \lim_{\gamma \to 0} \frac{\gamma}{-\lambda_n} F(-n-2,\ n+1,\ \gamma,\ x),$$
$$\left(\alpha = -n-2,\ \beta = n+1\ ;\ \gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)}\right).$$

More exactly speaking,

(2.3)′ $$X_n(x) = x(1-x) \cdot \sum_{k=0}^{n} \frac{(n+3)_k(-n)_k}{(k+1)!\,k!} x^k \equiv x(1-x)p_n(x) \quad (n = 0, 1, 2, \cdots).$$

Moreover, each $X_n(x)$ is to be expressed as

(2.4) $$X_n(x) = \frac{1}{(n+1)!} \frac{d^n}{dx^n} [x^{n+1}(1-x)^{n+1}],$$

which turns out to be $\int_0^x L_{n+1}(1-2t)\,dt$, where $L_{n+1}(x)$ is the Legendre polynomial of $(n+1)$-th degree, satisfying

(2.5) $$(1-x^2)\frac{d^2}{dx^2} L_{n+1}(x) - 2x\frac{d}{dx} L_{n+1}(x) + (n+1)(n+2)\, L_{n+1}(x) = 0.$$

Lemma 2.1. (i) *The functions* $\{X_n\}$ *make an orthogonal system with weight* $w(x)$ $(w(x) = x(1-x))$; (ii)

$$(p_n, p_n)_w = \int_0^1 w(x) p_n(x)^2\, dx = (X_n, X_n)_{w^{-1}} = \frac{1}{(n+1)(n+2)(2n+3)}.$$

Proof. (i) By use of integration and subtraction we have easily,

$$(\lambda_m - \lambda_n)(X_n, X_m)_{w^{-1}} = (\lambda_m - \lambda_n)(p_n, p_m)_w = 0,$$

which implies that

(2.6) $$(X_n, X_m)_{w^{-1}} = (p_n, p_m)_w = 0 \qquad (n \neq m).$$

(ii) The result is obtained after applying integration by parts several times to $(X_n, X_m)_{w^{-1}} = \int_0^1 \frac{1}{w} (D^n w^{n+1})(D^n w^{n+1})\, dx.$ (Q.E.D.)

Thus, it follows that $U_n(x, t) = \sum_{k=0}^{n} c_k e^{-\lambda_k t} X_k(x)$ $(n = 0, 1, 2, \cdots)$ $(c_k : \text{const.})$ satisfy (1.1)-(1.1)″.

3. Some lemmas

We state some lemmas before proceeding to the process of constructing the solution of (1.1)-(1.1)′-(1.1)″.

Now, we consider the integral equation below, which corresponds to the ordinary differential equation (2.2)††,

$$X(x)=\lambda\int_0^1 G(x,\xi)\frac{X(\xi)}{w(\xi)}\,d\xi \quad (0\leqq x\leqq 1), \tag{3.1}$$

where $G(x,\xi)$ is the kernel of the integral equation corresponding to the Dirichlet eigenvalue problem $y''=-\mu y\,(y(0)=y(1)=0)$, i.e.,

$$G(x,\xi)=\begin{cases}(1-x)\xi & (\xi\leqq x)\\ (1-\xi)x & (\xi\geqq x),\end{cases} \tag{3.1$'$}$$

$$\left[\text{N.B.: } \int_0^1 G(x,\xi)\,w(\xi)^{-1}\,d\xi=-(1-x)\log(1-x)-x\log x\leqq\log 2<1,\right.$$
$$\left. 0\leqq G(x,\xi)w(\xi)^{-1}\leqq 1\right].$$

The equation (3.1) is to be changed into a symmetric form

$$\tilde{X}(x)\left(\equiv\frac{X(x)}{\sqrt{w(x)}}\right)=\lambda\int_0^1\tilde{G}(x,\xi)\,\tilde{X}(x)\,d\xi=\lambda\tilde{G}\circ\tilde{X}(x), \tag{3.2}$$

where

$$\tilde{G}(x,\xi)=G(x,\xi)w(x)^{-\frac{1}{2}}w(\xi)^{-\frac{1}{2}}=\begin{cases}\sqrt{\dfrac{(1-x)\xi}{(1-\xi)x}} & (x\geqq\xi)\\ \sqrt{\dfrac{(1-\xi)x}{(1-x)\xi}} & (x\leqq\xi),\end{cases} \tag{3.2$'$}$$

[N.B.: $\tilde{G}(x,\xi)=\tilde{G}(\xi,x)$, $0\leqq\tilde{G}\leqq 1$].

Thus, $\{X_n(x)\}$ of (2.3) satisfy,

$$\frac{X_n(x)}{\sqrt{w(x)}}(=\tilde{X}_n(x))=\lambda_n\tilde{G}\circ\frac{X_n}{\sqrt{w}}(x)=\lambda_n\tilde{G}\circ\tilde{X}_n(x) \tag{3.3}$$
$$(\lambda_n=(n+1)(n+2)),\ (n=0,1,2,\cdots).$$

Lemma. 3.1. (i) $\sup_{x\in I}\int_0^1\tilde{G}(x,\xi)^2\,d\xi\leqq 1 \quad (I=[0,1])$; (ii)

$$\int_0^1(\tilde{G}(x_1,\xi)-\tilde{G}(x_2,\xi))^2\,d\xi\leqq 5|x_2-x_1|^{\frac{1}{2}}. \tag{3.4}$$

Proof.

(i) $$\int_0^1\tilde{G}(x,\xi)^2\,d\xi=\frac{1-x}{x}\int_0^x\frac{\xi}{1-\xi}\,d\xi+\frac{x}{1-x}\int_x^1\frac{1-\xi}{\xi}\,d\xi\leqq x+(1-x)=1.$$

†† See the afterword of this paper.

(ii) Let x_1 be smaller than x_2 ($x_1 < x_2$), which does not violate the generality. Now,

$$(3.5)\quad \int_0^1 (\tilde{G}(x_1,\xi) - \tilde{G}(x_2,\xi))^2 d\xi = \int_0^{x_1} [\cdots]^2 d\xi + \int_{x_1}^{x_2} [\cdots]^2 d\xi + \int_{x_2}^1 [\cdots]^2 d\xi = I_1 + I_2 + I_3.$$

We estimate I_1, taking notice of the inequalities,

$$(3.6)\qquad \sqrt{\frac{1-x_1}{x_1}} - \sqrt{\frac{1-x_2}{x_2}} \leqq \sqrt{\frac{1-x_1}{x_1} - \frac{1-x_2}{x_2}} \leqq \frac{1}{x_1}\sqrt{x_2 - x_1} \qquad (0 < x_1 < x_2 \leqq 1),$$

$$(3.7)\qquad I_1 \leqq \frac{1}{x_1^2}(x_2 - x_1) \cdot \frac{1}{1-x_1}\int_0^{x_1} \xi \, d\xi \leqq x_2 - x_1 \qquad \left(0 < x_1 \leqq \frac{1}{2}\right),$$

$$(3.7)'\qquad \begin{aligned} I_1 &\leqq \frac{1}{x_1^2}(x_2 - x_1) \cdot x_1 \int_0^{x_1} \frac{d\xi}{1-\xi} = \frac{1}{x_1}(x_2 - x_1)(-\log(1-x_1)) \\ &\leqq 2\sqrt{x_2 - x_1} \cdot \frac{\sqrt{x_2 - x_1}}{\sqrt{1-x_1}} \cdot \sqrt{1-x_1}\,(-\log(1-x_1)) \\ &\leqq 2\sqrt{x_2 - x_1} \quad \left(\frac{1}{2} \leqq x_1 < x_2 \leqq 1\right) \end{aligned}$$

Hence we have as a result,

$$(3.8)\qquad I_1 \leqq 2\sqrt{x_2 - x_1} \quad (0 \leqq x_1 < x_2 \leqq 1)$$

In the same way, for I_3 follows a similar estimate,

$$(3.8)'\qquad I_3 \leqq 2\sqrt{x_2 - x_1} \quad (0 \leqq x_1 < x_2 \leqq 1).$$

Moreover, we obtain for I_2,

$$(3.9)\qquad I_2 = \int_{x_1}^{x_2} [\cdots]^2 d\xi \leqq \int_{x_1}^{x_2} 1 \cdot d\xi = x_2 - x_1 \leqq \sqrt{x_2 - x_1},$$

because $0 \leqq G(x_1, \xi) \leqq 1$ and $0 \leqq G(x_2, \xi) \leqq 1$. As a final result, our assertion (3.4) holds. (Q.E.D.)

According to lemma 3.1, for any $f(x) \in C^0((0,1)) \cap L^2([0,1])$, we have inequalities,

$$(3.10)\qquad |\tilde{G} \circ f(x)| \leqq \|f\|_{L^2} \qquad \text{(Schwarz's inequality)},$$

$$(3.10)'\qquad |\tilde{G} \circ f(x_1) - \tilde{G} \circ f(x_2)| \leqq \sqrt{5}\,|x_2 - x_1|^{\frac{1}{4}} \|f\|_{L^2} \qquad \text{(Schwarz's inequality)}.$$

Therefore, the functions $\{\tilde{G} \circ f(x) ; \|f\|_{L^2} \leqq 1\}$ are uniformly bounded and equicontinuous, which leads us to Ascoli-Arzela's theorem of choice, so that $\tilde{G}$ is completely continuous as an operator from $L^2([0,1])$ into itself as well as one from $C^0([0,1])$ into itself (cf. Yoshida[14]).

Lemma. 3.2. *Let $y(x) \in C^0([0,1])$ satisfy, for some $\lambda \in \mathbf{R}^1$,*

$$(3.11)\qquad y(x) = \lambda G \circ \frac{y}{w}(x) \left(= \lambda\sqrt{w(x)} \cdot \tilde{G} \circ \frac{y}{\sqrt{w}}(x)\right).$$

Then it holds for $y(x)$ that

(3.11)′ $$|y(x)| \leqq C_1|\lambda|\,|y|^{(0)} w(x) \quad (C_1: \text{positive const.}).$$

Proof. We write (3.11) in a concrete form,

(3.12) $$y(x) = \lambda\int_0^1 G(x,\xi)\frac{1}{w(\xi)}y(\xi)\,d\xi$$
$$= \lambda\Big[\int_0^x \frac{1-x}{1-\xi}y(\xi)\,d\xi + \int_x^1 \frac{x}{\xi}y(\xi)\,d\xi\Big],$$ from which we have firstly,

(3.13) $$|y(x)| \leqq |\lambda|\cdot|(1-x)\log(1-x)+x\log x|_A \cdot |y|^{(0)} \qquad (0<x<1).$$

Noting that

(3.13)′ $$|\cdots|_A \leqq w(x)^\alpha|(1-x)^{1-\alpha}\cdot x^{-\alpha}\log(1-x)+x^{1-\alpha}(1-x)^{-\alpha}\log x|_B$$
$$\leqq C(\alpha)\,w(x)^\alpha, \qquad \Big(C(\alpha)=\sup_{x\in I}|\cdots|_B \leqq \frac{4}{\alpha} \quad (0<\alpha<1)\Big),$$

We easily have the inequality

(3.14) $$|y(x)| \leqq |\lambda|C(\alpha)|y|^{(0)} w(x)^\alpha \qquad (0\leqq x\leqq 1)$$

By use of the estimate (3.14) we again estimate $y(x)$ in (3.12), which brings forth another result, i.e.,

(3.15) $$|y(x)| \leqq |\lambda|^2C(\alpha)|y|^{(0)}\int_0^1 G(x,\xi)w(\xi)^{\alpha-1}\,d\xi$$
$$\leqq |\lambda|^2C(\alpha)|y|^{(0)}[(1-x)x^\alpha\{1-(1-x)^\alpha\}+x(1-x)^\alpha(1-x^\alpha)]$$
$$= |\lambda|^2C(\alpha)|y|^{(0)}w(x)[x^\alpha f_0(x,\alpha)+(1-x)^\alpha f_0(1-x,\alpha)]c,$$

where $f_0(x,\alpha)$ is a continuous function on $[0,1]$ such that (cf. [6],[12])

(3.16) $$1-(1-x)^\alpha = x\cdot f_0(x,\alpha) \quad (\alpha>0),$$

$$[\text{N.B.}: 0\leqq f_0(x,\alpha)\leqq 1,\ f_0(0,\alpha)=\alpha,\ f_0(1,\alpha)=1].$$

Thus, we have our assertion with $C_1 = \inf_{\alpha\in(0,1)}\{C(\alpha)[\cdots]c\}(\leqq 4)$.

(Q.E.D.)

Lemma. 3.3. *Let $y(x)$ and λ be the same as in Lemma 3.2. Then $y'(0)$ and $y'(1)$ exist.*

Proof. According to (3.12), $y'(x)$ has an expression

(3.17) $$y'(x) = \lambda\Big[-\int_0^x \frac{1}{1-\xi}y(\xi)\,d\xi + \int_x^1 \frac{1}{\xi}y(\xi)\,d\xi\Big] \quad (0<x<1),$$

whence we have, by (3.11)′, the boundedness of $y'(x)$, i.e.,

(3.18) $$|y'(x)| \leqq |\lambda|\Big[\int_0^x \frac{1}{1-\xi}|y(\xi)|\,d\xi + \int_x^1 \frac{1}{\xi}|y(\xi)|\,d\xi\Big]$$
$$\leqq |\lambda|^2C_1|y|^{(0)}\Big[\int_0^x \xi\,d\xi + \int_x^1(1-\xi)\,d\xi\Big] \leqq \frac{1}{2}\lambda^2C_1|y|^{(0)}, \qquad (0<x<1).$$

Next, for x and x' such that $0 < x' < x < 1$, it holds that

$$(3.19)\qquad |y'(x) - y'(x')| = |\lambda| \cdot \left| -\int_{x'}^{x} \frac{1}{1-\xi} y(\xi)\, d\xi - \int_{x'}^{x} \frac{1}{\xi} y(\xi)\, d\xi \right|$$
$$\leqq |\lambda|^2 C_1 |y|^{(0)} \left[\int_{x'}^{x} \xi\, d\xi + \int_{x'}^{x} (1-\xi) d\xi \right]$$
$$= |\lambda|^2 C_1 |y|^{(0)} |x - x'|,$$

which implies the existence of $\lim_{x\to+0} y'(x)$ and $y'(0)$, at the same time guaranteeing their equality. The same holds for $x=1$. (Q.E.D.)

Lemma. 3.4. *Let $y(x)$ and λ be as in Lemma 3.2. Then* (i) *$y(x)$ has a form,*

$$(3.20)\qquad y(x) = w(x)h(x), \quad (h(x) \in C^2((0,1)) \cap C^0([0,1]),$$

where

$$(3.20)'\qquad h(x) = \begin{cases} y(x)w(x)^{-1} & (0 < x < 1) \\ y'(0)\ (x=0), -y'(1) & (x=1). \end{cases}$$

Moreover, (ii) *λ is positive for $y(x) \not\equiv 0$, and* (iii) *$y \in C^2([0,1])$.*

Proof. (i) The assertion is obvious by the strength of lemmas 3.2 and 3.3, as seen below,

$$(3.21)\qquad \begin{cases} y'(0) = \lim_{x\to+0} \dfrac{y(x) - y(0)}{x} = \lim_{x\to+0} \dfrac{y(x)}{x(1-x)} = \lim_{x\to+0} h(x), \\ y'(1) = -\lim_{x\to1-0} h(x). \end{cases}$$

(ii) From (3.11) we have, for a small number $\varepsilon (> 0)$,

$$(3.22)\qquad \int_{\varepsilon}^{1-\varepsilon} y''(x)y(x)dx = y'(x)y(x)\Big|_{x=\varepsilon}^{1-\varepsilon} - \int_{\varepsilon}^{1-\varepsilon} y'(x)^2 dx$$
$$= -\lambda \int_{\varepsilon}^{1-\varepsilon} \frac{y(x)^2}{w(x)} dx.$$

Since $y(x)$ belongs to $C^1([0,1])$ by lemmas 3.2 and 3.3, letting ε tend towards 0, we see that

$$(3.22)'\qquad \int_0^1 y(x)^2 dx = \lambda \int_0^1 \frac{y'(x)^2}{w(x)} dx,$$

which implies the positivity of λ.
(iii) Remark that $y''(x) = -\lambda h(x)$. (Q.E.D.)

Lemma. 3.5. *The system of functions $\{\tilde{X}_n = \sqrt{w}\, p_n\}$ is complete in $\mathscr{F} = \{\sqrt{w} f\,;\ f \in C^0([0,1])\}$.*

Proof. It suffices to demonstrate that Parseval's equality holds for an arbitrary $\sqrt{w}f\in\mathscr{F}$, i.e.,

$$(3.23)\qquad \int_I(\sqrt{w}f)^2dx=\int_I wf^2dx$$
$$=\sum_{k=0}^{\infty}\Big(\int_I\sqrt{w}f\tilde{X}_n dx\Big)^2=\sum_{k=0}^{\infty}\Big(\int_I wf p_n dx\Big)^2,$$

where $\tilde{X}_n$ and p_n $(n=0,1,2,\cdots)$ are normalized respectively with weight 1 and weight w. Now, according to Bernstein ([2]), the sequence of polynomials $\{P_n(x)\}$ defined by

$$(3.24)\qquad P_n(x)=\sum_{k=0}^{n}\binom{n}{k}f\Big(\frac{k}{n}\Big)x^k(1-x)^{n-k}\qquad(n=0,1,2,\cdots)$$

converges uniformly to $f(x)$ on $[0,1]$. Each $P_n(x)$ is to be expressed as

$$(3.24)'\qquad P_n(x)=\sum_{k=0}^{n}C_k^{(n)}p_n(x)\qquad(n=0,1,2,\cdots).$$

For an arbitrary $\varepsilon(>0)$, there exists $N\in\mathbf{N}$ such that

$$(3.25)\qquad |f(x)-P_n(x)|=\Big|f(x)-\sum_{k=0}^{n}C_k^{(n)}p_n(x)\Big|<\varepsilon\qquad(\forall n\geqq N).$$

It is obvious by Bessel's inequality and the orthogonality of $\{p_n\}$ with weight w that

$$(3.26)\qquad 0\leqq\int_I wf^2dx-\sum_{k=0}^{n}\Big(\int_I wf p_k dx\Big)^2=\int_I w\cdot\Big|f-\sum_{k=0}^{n}\Big(\int_I wf p_k dx\Big)p_k\Big|^2dx$$
$$\leqq\int_I w\Big|f(x)-\sum_{k=0}^{\infty}C_k^{(n)}p_k(x)\Big|^2dx<\frac{\varepsilon^2}{6},$$

which shows the validity of (3.23). (Q.E.D.)

Lemma 3.6. *If, for some $\lambda(\neq0)$ such that $\lambda\neq\lambda_n$ $(n=0,1,2,\cdots)$, $y(x)$ satisfies* (3.11), *then $y(x)\equiv0$.*

Proof. Assume that $y(x)\not\equiv0$. Then, by lemma 3.4, $y(x)$ has a form $y(x)=w(x)h(x)$ $(h\in C^0([0,1]),\ h\not\equiv0)$, Next, we have easily

$$(3.27)\qquad \begin{cases}\lambda_n^{-1}(\tilde{y},\tilde{X}_n)=(\tilde{y},\tilde{G}\circ\tilde{X}_n)=(\tilde{G}\circ\tilde{y},\tilde{X}_n)=\lambda^{-1}(\tilde{y},\tilde{X}_n),\ (\tilde{y}=yw^{-\frac{1}{2}}),\\ (\lambda^{-1}-\lambda_n^{-1})(\tilde{y},\tilde{X}_n)=0,\\ (\tilde{y},\tilde{X}_n)=(wh,p_n)=0.\end{cases}$$

Putting $f=h$ in (3.23), we obtain $\int_I wh^2dx=0$, which implies that $y(x)=w(x)h(x)=0$ (contradiction). (Q.E.D.)

Corollary of lemma 3.6 *The set of eigenvalues of $\tilde{G}$ and that of eigenfunctions of $\tilde{G}$ are equal to $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\tilde{X}_n\}_{n=0}^{\infty}$, respectively.*

Proof. By the lemma, it is obvious. (Q.E.D.)

Lemma. 3.7. *The system of* $\{\tilde{X}_n\}_{n=0}^{\infty}$ *is complete in* $C^0([0,1])$.

Proof. Let $f(x)$ belong to $C^0([0,1])$, being such that $(f, \tilde{X}_n)=0$ $(n=0,1,2,\cdots)$. Then we have,

$$(3.28)\qquad 0=(f,\tilde{X}_n)=\lambda_n(f,\tilde{G}\circ\tilde{X}_n)=\lambda_n(\tilde{G}\circ f,\tilde{X}_n)$$
$$=\lambda_n\left(\tilde{G}\circ\frac{f}{\sqrt{w}},p_n\right)\qquad(n=0,1,2,\cdots),$$

where we note that $p_n(x)$ is a polynomial of n-th degree for each n. Thus, it follows that

$$(3.29)\qquad G\circ\frac{f}{\sqrt{w}}(x)\ (\in C^0([0,1]))\equiv 0,$$

which implies that

$$(3.29)'\qquad 0=\frac{d^2}{dx^2}\left(G\circ\frac{f}{\sqrt{w}}\right)=-\frac{f}{\sqrt{w}}\qquad(0<x<1).$$

Therefore, from $f\in C^0([0,1])$ follows that $f(x)=0$ $(0\leqq x\leqq 1)$. (Q.E.D.)

Lemma. 3.8. (A. A. Markov's Inequality, [5],[8]). *Let* $Q_n(x)$ *be a polynomial of* n*-th degree, satisfying*

$$(3.30)\qquad |Q_n(x)|\leqq M\qquad(0\leqq x\leqq 1).$$

Then it holds that

$$(3.30)'\qquad |Q_n'(x)|\leqq 2n^2M\qquad(0\leqq x\leqq 1).$$

Proof. See [5],[8], etc. (Q.E.D.)

4. Main theorem.

Here we shall construct the solution of (1.1)-(1.1)′-(1.1)″, giving a certain condition upon the initial data $u_0(x)$. Firstly, we define a formal series solution $U(x,t)$ for our problem by

$$(4.1)\qquad U(x,t)=\sum_{n=0}^{\infty}\left(\int_I u_0(x)X_n(x)w(x)^{-1}dx\right)e^{-\lambda_n t}X_n(x)$$
$$=\sum_{n=0}^{\infty}a_ne^{-\lambda_n t}X_n(x),$$

$(\lambda=(n+1)(n+2)$; X_n's are each normalized with weight $w^{-1})$.

Now, we apply to our problem the following Hilbert-Schmidt expansion theorem, where the kernel $\tilde{G}(x,t)$ is completely continuous as an operator from $C^0([0,1])$ into itself (cf.[14]):

Hilbert-Schmidt Expansion Theorem ([1], [13], [14], etc.).

For any function $f \in C^0([0,1])$ *the series* $\sum_{n=0}^{\infty}(\tilde{G}\circ f, \tilde{X}_n)\tilde{X}_n$ *converges to* $\tilde{G}\circ f(x)$ *in an absolute and uniform way, where*

$$(4.2)\qquad \begin{cases} \tilde{G}\circ f(x) = \int_I \tilde{G}(x,\xi)f(\xi)\,d\xi, \\ (\tilde{G}\circ f, \tilde{X}_n) = \int_I (G\circ f)(\xi)\tilde{X}_n(\xi)\,d\xi. \end{cases}$$

If we put $t=0$ in $U(x,t)$, then we have formally for $U(x,0)=u_0(x)$,

$$(4.3)\qquad u_0(x) = \sum_{n=0}^{\infty}(u_0, w^{-1}X_n)X_n(x) = w(x)^{\frac{1}{2}}\sum_{n=0}^{\infty}(u_0, w^{-\frac{1}{2}}\tilde{X}_n)\tilde{X}_n(x).$$

We seek a sufficient condition on u_0 for the equality (4.3) to hold. Since, under the assumption $u_0 \in C^2([0,1])$, u_0 satisfies

$$(4.4)\qquad u_0'' = -(-u_0''),\ u_0(0) = u_0(1) = 0 \ (\text{whence } u_0 \in w\cdot C^0([0,1])),$$

we can express u_0 as

$$(4.5)\qquad u_0(x) = \int_I G(x,\xi)(-u_0''(\xi))d\xi = w(x)^{\frac{1}{2}}\int_I \tilde{G}(x,\xi)(-w^{\frac{1}{2}}u_0'')(\xi)d\xi.$$

Thus, we have:

Lemma. 4.1. *If we assume that*

$$(4.6)\qquad \begin{cases} u_0(x) \in C^0([0,1]) \cap C^2((0,1)) \quad (u_0(0) = u_0(1) = 0), \\ w(x)^{\frac{1}{2}}u_0''(x) \in C^0([0,1]), \end{cases}$$

then $u_0(x)$ *is to be expanded absolutely and uniformly on* $[0,1]$ *as below,*

$$(4.7)\qquad u_0(x) = w(x)^{\frac{1}{2}}\sum_{n=0}^{\infty}(\tilde{G}\circ(-w^{\frac{1}{2}}u_0''), \tilde{X}_n)\,\tilde{X}_n(x) = \sum_{n=0}^{\infty}(u_0, w^{-1}X_n)X_n(x),$$

where we note that

$$(4.7)'\qquad (\tilde{G}\circ(-w^{\frac{1}{2}}u_0''), \tilde{X}_n) = -(w^{\frac{1}{2}}u_0'', \tilde{G}\circ\tilde{X}_n) = -\left(w^{\frac{1}{2}}u_0'', \frac{\tilde{X}_n}{\lambda_n}\right)$$
$$= -\left(u_0'', \frac{X_n}{\lambda_n}\right) = -\left(u_0, \frac{X_n''}{\lambda_n}\right) = \left(u_0, \frac{X_n}{w}\right)(\equiv a_n).$$

Lemma. 4.2. *For each* $X_n(x)$, *normalized with weight* w^{-1}, *it holds that*

$$(4.8)\qquad |X_n|^{(0)} \leqq \sqrt{\lambda_n} = \sqrt{(n+1)(n+2)}.$$

Proof. From the equation $X_n'' + \lambda_n w^{-1} X_n = 0$ we easily obtain

$$(4.9)\qquad \int_I (X_n')^2\,dx = \lambda_n \int_I w^{-1} X_n^2 dx = \lambda_n,$$

whence $|X_n|^{(0)} \leqq \int_I (X_n')^2\,dx = \sqrt{\lambda_n}$. (Q.E.D.)

Lemma. 4.3. *For the series* (4.7) *we have an estimate*

$$(4.10)\qquad \left|\sum_{n=k}^{l} (u_0, w^{-1} X_n) X_n(x)\right| \leqq \left(\sum_{n=k}^{l} b_n^2\right)^{\frac{1}{2}} \cdot \left(\sum_{n=k}^{l} \frac{1}{\lambda_n}\right)^{\frac{1}{2}} \qquad (k \leqq l).$$

Proof.

$$(4.11)\qquad \begin{aligned}(u_0, w^{-1} X_n) X_n &= (\tilde{G} \circ (-w^{\frac{1}{2}} u_0''), \tilde{X}_n) X_n = (-w^{\frac{1}{2}} u_0'', \tilde{G}\tilde{X}_n) X_n \\ &= (-w^{\frac{1}{2}} u_0'', \tilde{X}_n)\,\lambda_n^{-1} X_n = b_n \lambda_n^{-1} X_n, \quad (n = 0, 1, 2, \cdots), \\ &(b_n \equiv (-w^{\frac{1}{2}} u_0'', \tilde{X}_n) = a_n \lambda_n),\end{aligned}$$

where we note that

$$(4.11)'\qquad \begin{cases} \sum_{n=0}^{\infty} b_n^2 = (w^{\frac{1}{2}} u_0'', w^{\frac{1}{2}} u_0'') \quad \text{(Parseval's equality)}, \\ \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1. \end{cases}$$

By (4.11),

$$\begin{aligned}\left|\sum_{n=k}^{l} (u_0, w^{-1} X_n) X_n(x)\right| &\leqq \sum_{n=k}^{l} |b_n| \cdot |\lambda_n^{-1}| \cdot |X_n|^{(0)} \\ &\leqq \sum_{n=k}^{l} |b_n| \lambda_n^{-\frac{1}{2}} \leqq \left(\sum_{n=k}^{l} b_n^2\right)^{\frac{1}{2}} \left(\sum_{n=k}^{l} \frac{1}{\lambda_n}\right)^{\frac{1}{2}}.\end{aligned}$$

(Q.E.D.)

Lemma. 4.4. *Under the assumption* (4.6) *on* u_0, *the series* $U(x, t)$ *of* (4.1) *converges absolutely and uniformly on* $[0,1] \times [0, \infty)$. *Moreover, for* $t > 0$, $U_x(x, t)$, $U_{xx}(x, t)$ *and* $U_t(x, t)$ *exist, being expressed respectively as below,*

$$(4.12)\qquad \begin{cases} U_x(x, t) = \sum_{n=0}^{\infty} a_n X_n'(x) e^{-\lambda_n t}, \\ U_{xx}(x, t) = \sum_{n=0}^{\infty} a_n X_n''(x) e^{-\lambda_n t}, \\ U_t(x, t) = -\sum_{n=0}^{\infty} a_n \lambda_n X_n(x) e^{-\lambda_n t}, \end{cases}$$

where, for an arbitrary positive t_0, *each convergence is absolute and uniform on* $[0,1] \times [t_0, \infty)$, *which implies that* U_x, U_{xx} *and* U_t *are continuous there.*

Proof. It is generally well known that, if $\{f_n(x) \in C^1([0,1])\}$ converges to $f(x)$ on [0, 1] and $\{f'_n(x)\}$ does to $g(x)$ uniformly on [0,1], then $f'(x)=g(x)$. Also, for $t>0$, $e^{-\lambda_n t} < l!(1+\lambda_n^l t^l)^{-1}(\forall l \in \mathrm{N})$. By virtue of these relations and lemmas 3.8, 4.1~4.4, our assertion is obtained. (Q.E.D.)

Main Theorem. *Under the assumption* (4.6) *on* u_0, *there exists a unique solution* $u(x,t)$ *of* (1.1)-(1.1)′-(1.1)″, *belonging to* $C^0([0,1]\times[0,T])\cap C^{2,1}([0,1]\times(0,T])$ *for an arbitrary* $T\in(0,\infty)$. *Moreover,*

$$(4.13)\qquad \begin{cases} |u(x,t)|\leqq|u_0|^{(0)}\quad, & |u_x(x,t)|\leqq A_1\sum_{n=0}^{\infty}b_n^2=A_1D_0,\\ |u_{xx}(x,t)|\leqq A_2D_0\quad, & |u_t(x,t)|\leqq A_3D_0.\end{cases}$$

Proof. It is obvious that $u(x,t)=U(x,t)$ in (4.1) with the condition (4.6) given upon u_0, satisfies (1.1)-(1.1)″. It remains for us to demonstrate (i) the continuity of $u(x,t)$ at $t=0$, (ii) the uniqueness of the solution, and (iii) the inequality in (4.13).

(i) The sequence $\{A_n(t)=e^{-\lambda_n t}\}$ is positive and monotonically decreasing in n. Putting $B_n(x)=a_nX_n(x)$, we express $u(x,t)-u_0(x)$ as follows:

$$(4.14)\qquad \begin{aligned} u(x,t)-u_0(x)&=\sum_{n=0}^{\infty}[A_n(t)B_n(x)-A_n(0)B_n(x)]\\ &=\sum_{n=0}^{N}(A_n(t)-1)B_n(x)+\sum_{n=N+1}^{\infty}(A_n(t)-1)B_n(x).\end{aligned}$$

Since $|A_n(t)-1|<1$ and $\sum_{n=0}^{\infty}B_n(x)$ converges absolutely and uniformly in x, for an arbitrary $\varepsilon(>0)$ there exists $N\in\mathrm{N}$ such that

$\left|\sum_{n=N+1}^{\infty}(A_n(t)-1)B_n(x)\right|<\varepsilon$. Nextly, it holds that

$$(4.15)\qquad \left|\sum_{n=0}^{N}(A_n(t)-1)B_n(x)\right|\leqq(1-A_N(t))\sum_{n=0}^{N}|B_n(x)|\leqq(1-A_N(t))M_0,$$

$$\left(M_0=\sup_{x\in I}\sum_{n=0}^{\infty}|B_n(x)|+1<\infty\right).$$

For $M_0^{-1}\varepsilon$, there exists $t_0(>0)$ such that $0\leqq 1-A_N(t)<M_0^{-1}\varepsilon\,(0\leqq t\leqq t_0)$, Thus we have,

$$(4.15)'\qquad |u(x,t)-u_0(x)|<2\varepsilon,\ (0\leqq t\leqq t_0(\varepsilon,N(\varepsilon))).$$

(ii) We assume that $u(x,t)\not\equiv 0$. Now, put $v(x,t)=e^{-\mu t}u(x,t)(\mu>0)$. Then $v(x,t)$ satisfies

$$(4.16)\qquad \begin{cases} v_t=e^{-\mu t}u_t-\mu e^{-\mu t}u=w(x)v_{xx}-\mu v,\\ v(x,0)=u_0(x),\ v(0,t)=v(1,t)=0.\end{cases}$$

If $v(x,t)$ takes its maximum as a positive number at $(x_1,t_1)\in\Omega_T=(0,1)\times(0,T](0<T<\infty)$, then $v_t(x_1,t_1)\geqq 0$, $v_{xx}(x_1,t_1)\leqq 0$. Therefore,

$$(4.17)\qquad 0 \leqq v_t(x_1, t_1) - w(x_1)v_{xx}(x_1, t_1) = -\mu v(x_1, t_1) < 0$$

which is contradictory. Similarly $v(x,t)$ does not take its minimum as a negative number in Ω_T. As a conclusion, we have

$$(4.18)\qquad \begin{cases} \min[0, \min u_0(x)] \leqq v(x,t) \leqq \max[0, \max u_0(x)], \\ |v(x,t)| \leqq |u_0|^{(0)} \quad (0 \leqq t \leqq T), \\ |u(x,t)| \leqq \inf_{\mu>0} e^{\mu t}|u_0|^{(0)} = |u_0|^{(0)} \quad (0 \leqq t \leqq T). \end{cases}$$

We note that the third inequality also holds for $u(x,t) \equiv 0$. Hence, if there are two functions $u_1(x,t)$ and $u_2(x,t)$ satisfying (1.1)-(1.1)′-(1.1)″, then $|u_1(x,t) - u_2(x,t)| = |(u_1 - u_2)(x,0)|^{(0)} = 0$.

(iii) The four inequalities are obvious by (4.18), and by lemmas 3.8 and 4.1~4.4, where we note that $X_n(x)$ is a polynomial of $(n+2)$-th degree for each n and that the following inequalities hold,

$$(4.19)\qquad \begin{cases} 2(n+2)^2 e^{-\lambda_n t} \leqq A_1,\ 4(n+2)^4 e^{-\lambda_n t} \leqq A_2, \\ \lambda_n e^{-\lambda_n t} \leqq A_3\ (A_1, A_2, A_3 \text{ are positive constants}). \end{cases}$$

(Q.E.D.)

Afterword: Strictly speaking, there may be need to make some comment on the correspondence relation between the differential equation (2.2) and the integral equation (3.1). However, we only make a remark that it plays an important role that the differential equation (2.2) is of the Fuchsian type with singular points at $x=0, 1$. In this paper, we have made discussions on the basis of the integral equation (3.1), almost independently of the differential equations (2.2). Lastly, we add, for reference, that, if $X(x)$ satisfies (2.2), then $X'(x)=O(\log x)\,(0<x\leqq 1/2)$ and $X'(x)=O(\log(1-x))\,(1/2\leqq x<1)$, which is derived directly from (2.2), and, accordingly, that $X(x)$ is to be expressed in the form of (3.1).

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