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KNO scaling function of modified negative binomial distriburion

Noriaki NAKAJIMA, Minoru BIYAJIMA and Naomichi SUZUKI[1]

Department of Physics, Shinshu University, Matsumoto 390, Japan
[1]Matsusho Gakuen Junior College, Matsumoto 390-12, Japan
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Abstract

We investigate the KNO scaling function of the modified negative binomial distribution (MNBD), because this MNBD can explain the oscillating behaviors of the cumulant moment observed in e^+e^- annihilations and in hadronic collisions. By using a straightforward method and the Poisson transform we derive the KNO scaling function from the MNBD. The KNO form of experimental data in e^+e^- collisions and hadronic collisions are analyzed by the KNO scaling function of the MNBD and that of the negative binomial distribution (NBD). The KNO scaling function of the MNBD describes the data as well as that of the NBD.

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1 Introduction

Recently it has been found that cumulant moments of the multiplicity distributions both in e^+e^- annihilations and hadronic collisions show prominent oscillatory behaviors when plotted as a function of their order q [1]. In [1] this behavior was attributed to the QCD-type of branching processes apparently taking place in those reactions. However, in [2, 3] we have shown that the same behavior of the moments emerges essentially from the modified negative binomial distribution (MNBD) (which actually describes the data much better than the negative binomial distribution (NBD) [4]). This distribution can be derived from the pure birth process with the initial condition given by the binomial distribution [4][1].

In this paper we shall derive the KNO scaling function of the MNBD both by the straightforward method (i.e., proceeding to the limit of large multiplicities n and large average multiplicities $\langle n\rangle$ while keeping the scaling variable $z = n/\langle n\rangle$ finite and

[1]It is interesting to mention here that it comes also from the concept of purely bosonic sources as presented recently in [6].

fixed) and by using the Poisson transform. Using this KNO scaling function we shall analyze the observed multiplicity distributions in e^+e^- annihilations and in hadronic collisions.

2 KNO scaling function

Let us remind that the MNBD is given by the following function [2, 3, 4, 5]

$$P(0)=\left[\frac{1+r_1}{1+r_2}\right]^N;$$
$$P(n)=\frac{1}{n!}\left(\frac{r_1}{r_2}\right)^N\left(\frac{r_2}{1+r_2}\right)^n\sum_{j=1}^{N}{}_NC_j\frac{\Gamma(n+j)}{\Gamma(j)}\left(\frac{r_2-r_1}{r_1}\right)^j\frac{1}{(1+r_2)^j}, \tag{1}$$

where N (the number of excited hadrons) is an integer, r_1 is real and $r_2>0$

$$r_1=\frac{1}{2}\left(C_2-1-\frac{1}{N}\right)\langle n\rangle-\frac{1}{2},$$
$$r_2=\frac{1}{2}\left(C_2-1-\frac{1}{N}\right)\langle n\rangle-\frac{1}{2}. \tag{2}$$

Moreover we have the generating function of the MNBD

$$\Pi(u)=\sum_{n=0}P(n)u^n=[1-r_1(u-1)]^N[1-r_2(u-1)]^{-N}. \tag{3}$$

On the other hand, the NBD has the following form

$$P(n)=\frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\left(\frac{k}{\langle n\rangle}\right)^k\left(1+\frac{k}{\langle n\rangle}\right)^{-(n+k)},$$

where $k>0$. Its corresponding KNO scaling function is the gamma distribution

$$\Psi(z)=\frac{k^k}{\Gamma(k)}e^{-kz}z^{k-1}. \tag{4}$$

2.1 The straightforward method

Traditionally the KNO scaling function is derived from the multiplicity distribution $P(n)$ multiplied by the corresponding mean multiplicity $\langle n\rangle$ by going to the large multiplicity n and large mean multiplicity $\langle n\rangle$ limit while keeping their ratio, $z=\lim_{n,\langle n\rangle\to\infty}n/\langle n\rangle$ fixed. In our case, starting from Eq. (1) we arrive at the following function

$$\Psi(z)\equiv\lim_{n,\langle n\rangle\to\infty}\langle n\rangle P(n)$$
$$=\left(\frac{r_1'}{r_2'}\right)^N e^{-\frac{\langle n\rangle}{r_2'}z}\sum_{j=1}^{N}{}_NC_j\frac{1}{\Gamma(j)}\left(\frac{r_2'-r_1'}{r_1'}\right)^j\left(\frac{\langle n\rangle}{r_2'}\right)^j z^{j-1}. \tag{5}$$

The parameters r_1' and r_2' in Eq. (5) are given by

$$r_1'=\frac{1}{2}\left(C_2-1-\frac{1}{N}\right)\langle n\rangle$$
$$r_2'=\frac{1}{2}\left(C_2-1+\frac{1}{N}\right)\langle n\rangle. \tag{6}$$

which are slightly different from r_1, r_2 given by Eq. (1), because $\langle n\rangle \gg 1$. It should be noticed that the normalization of Eq. (5) differs now from the unity,

$$\int_0^{\infty} \Psi(z)dz = 1 - \left(\frac{r_1'}{r_2'}\right)^N, \tag{7}$$

where the second term corresponds to the term $\langle n\rangle P(0)$ in Eq. (1).

2.2 The Poisson transform

The KNO scaling function $\Psi(z,t)$ can be also obtained by using the inverse Poisson transform [7] in which it is related to the distribution function $P(n,t)$ as

$$P(n,t) \underset{\text{Poisson trans.}}{\overset{\text{inverse Poisson trans.}}{\rightleftarrows}} \Psi(z,t)$$

As a result we obtain in this approach that

$$P(n,t) = \int_0^{\infty} \frac{(\alpha\omega)^n}{n!} e^{-\alpha\omega} \Psi\left(\frac{\omega}{\langle n\rangle/\alpha}, t\right) \frac{d\omega}{\langle n\rangle/\alpha}, \tag{8}$$

$$\begin{aligned}\Psi\left(\frac{\omega}{\langle n\rangle/\alpha}, t\right) &= \frac{1}{2\pi} e^{\alpha\omega} \frac{\langle n\rangle}{\alpha} \int_{-\infty}^{\infty} e^{-ix\omega} \sum_{n=0}^{\infty} \left(\frac{ix}{\alpha}\right)^n P(n,t)dx \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sz} \Pi\left(1-\frac{s}{\langle n\rangle}, t\right) ds, \end{aligned} \tag{9}$$

where $\Pi(u,t)$ is the generating function of $P(n,t)$. These equations hold also in the stationary function ($t=0$).

Using now the generating function Eq. (3), $\Psi(z)$ is given by the following inverse Laplace transform

$$\Psi(z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sz} \left(\frac{r_1}{r_2}\right)^N \sum_{j=0}^{N} {}_N C_j \left(\frac{r_2-r_1}{r_1}\frac{\langle n\rangle}{r_2}\right)^j \left(s+\frac{\langle n\rangle}{r_2}\right)^{-j} ds. \tag{10}$$

Then we arrive at the KNO scaling function for the MNBD[2]

$$\begin{aligned}\Psi(z) &= \left(\frac{r_1'}{r_2'}\right)^N \delta(z-\epsilon) + \left(\frac{r_1'}{r_2'}\right)^N e^{-\frac{\langle n\rangle}{r_2'}z} \sum_{j=1}^{N} {}_N C_j \frac{1}{\Gamma(j)} \left(\frac{r_2'-r_1'}{r_1'}\right)^j \left(\frac{\langle n\rangle}{r_2'}\right)^j z^{j-1} \\ &= \left(\frac{r_1'}{r_2'}\right)^N \delta(z-\epsilon) + \left(\frac{r_1'}{r_2'}\right)^N \frac{r_2'-r_1'}{r_1'}\frac{\langle n\rangle}{r_2'} e^{-\frac{\langle n\rangle}{r_2'}z} L_{N-1}^{(1)}\left(-\frac{r_2'-r_1'}{r_1'}\frac{\langle n\rangle}{r_2'}z\right), \end{aligned} \tag{11}$$

where $L_n^{(\alpha)}(x)$ is the Laguerre's polynomial. In Eq. (11) the first term corresponds to the constant term in Eq. (10) (, or $\langle n\rangle P(0)$). Because the normalization is now just the unity, in what follows we shall use this function as the KNO scaling function of the MNBD in the analysis of data.

[2]In the actual analysis the delta function term occurring here will be approximated by the following expression

$$\delta(z-\epsilon) = \lim_{\alpha\to\text{large}} \sqrt{\frac{\alpha}{2\pi}} \exp[-\alpha(z-\epsilon)^2/2].$$

3 Analysis of experimental data

We shall investigate now the applicability of the MNBD as presented in its KNO form (i.e., using Eq. (11)) to the description of the observed multiplicity distributions in e^+e^- annihilations[8]-[13] and in hadronic collisions[14,15]. Table I shows obtained parameters C_2, N and the corresponding values of the $\chi^2_{\rm min}$. Moreover it shows other parameters r_1' and r_2' calculated from C_2 and N according to Eq. (6).

The corresponding KNO scaling functions for our MNBD are compared with the KNO form of the above mentioned multiplicity distributions in Fig. 1(a) - 1(f) and Fig. 2(a) - 2(g).

Eq. (11)	$\sqrt{s}$ [GeV]	N	C_2	r_1'	r_2'	$\chi^2_{\rm min}$/NDF
TASSO	14	31	1.101 ± 0.003	0.320 ± 0.020	0.620 ± 0.031	10.8 / 11
TASSO	22	90	1.099 ± 0.003	0.497 ± 0.027	0.622 ± 0.031	3.2 / 12
HRS	29	90	1.081 ± 0.002	0.450 ± 0.012	0.593 ± 0.019	11.1 / 12
TASSO	34.8	35	1.094 ± 0.001	0.445 ± 0.017	0.833 ± 0.029	8.7 / 16
TASSO	43.6	90	1.091 ± 0.002	0.602 ± 0.024	0.770 ± 0.028	11.3 / 17
AMY	50	20	1.083 ± 0.004	0.268 ± 0.033	1.080 ± 0.042	2.5 / 17
AMY	57	70	1.082 ± 0.002	0.582 ± 0.024	0.828 ± 0.029	6.5 / 18
AMY	60	48	1.087 ± 0.004	0.591 ± 0.037	0.902 ± 0.038	3.8 / 18
SLD	91.2	19	1.091 ± 0.001	0.401 ± 0.011	1.499 ± 0.014	34.7 / 22
L3	91.2	11	1.095 ± 0.003	0.043 ± 0.031	1.933 ± 0.058	9.6 / 21
DELPHI	91.2	12	1.091 ± 0.001	0.081 ± 0.011	1.853 ± 0.025	17.0 / 23
OPAL	91.2	11	1.092 ± 0.002	0.012 ± 0.021	1.957 ± 0.045	4.5 / 25
OPAL	133	11	1.093 ± 0.004	0.024 ± 0.047	2.152 ± 0.076	7.2 / 22

Eq. (4)	$\sqrt{s}$ [GeV]	k	$\chi^2_{\rm min}$/NDF
TASSO	14	9.325 ± 0.28	18.7 / 12
TASSO	22	10.26 ± 0.11	15.2 / 13
HRS	29	12.34 ± 0.30	36.9 / 13
TASSO	34.8	10.81 ± 0.15	42.5 / 17
TASSO	43.8	11.40 ± 0.22	41.3 / 18
AMY	50	11.75 ± 0.74	4.4 / 18
AMY	57	12.12 ± 0.40	17.9 / 19
AMY	60	11.26 ± 0.60	8.1 / 19
SLD	91.2	11.02 ± 0.07	199.6 / 23
L3	91.2	10.57 ± 0.18	9.8 / 22
DELPHI	91.2	11.00 ± 0.10	21.0 / 24
OPAL	91.2	10.86 ± 0.18	4.4 / 26
OPAL	133	10.30 ± 0.51	6.7 / 23

Table I (a)

Eq. (11)	$\sqrt{s}$ [GeV]	N	C_2	r_1'	r_2'	χ^2_{min}/NDF
FNAL	23.9	90	1.251 ± 0.007	1.020 ± 0.033	1.114 ± 0.034	26.1 / 12
ISR	30.4	7	1.187 ± 0.005	0.233 ± 0.027	1.738 ± 0.035	15.2 / 15
FNAL	38.8	90	1.265 ± 0.005	1.302 ± 0.032	1.416 ± 0.033	44.2 / 14
ISR	44.5	6	1.198 ± 0.005	0.189 ± 0.024	2.203 ± 0.034	5.2 / 17
ISR	52.6	9	1.205 ± 0.004	0.599 ± 0.026	2.017 ± 0.034	4.8 / 19
ISR	62.2	90	1.194 ± 0.005	1.246 ± 0.037	1.398 ± 0.038	21.8 / 18
UA5	200	4	1.264 ± 0.011	0.150 ± 0.118	5.500 ± 0.156	7.6 / 29
UA5	546	4	1.275 ± 0.004	0.368 ± 0.060	7.718 ± 0.243	54.2 / 45
UA5	900	4	1.301 ± 0.009	0.908 ± 0.024	9.808 ± 0.257	78.2 / 52

Eq. (5)	$\sqrt{s}$ [GeV]	k	χ^2_{min}/NDF
FNAL	23.9	4.19 ± 0.10	74.1 / 13
ISR	30.4	5.27 ± 0.14	18.8 / 16
FNAL	38.8	3.86 ± 0.07	130.7 / 15
ISR	44.5	4.92 ± 0.15	7.9 / 18
ISR	52.6	4.72 ± 0.10	33.8 / 20
ISR	62.2	5.25 ± 0.13	55.5 / 19
UA5	200	3.79 ± 0.16	7.6 / 30
UA5	546	3.60 ± 0.06	52.7 / 46
UA5	900	3.26 ± 0.09	62.8 / 53

Table I (b)

Table I The parameters of the KNO scaling function of the MNBD used in the analysis compared with those of the NBD for the charged multiplicity in : (a) e^+e^- collisions and (b) hadronic collisions.

4 Summary and Discussion

The KNO scaling function of the MNBD is obtained and is applied to the analysis of the observed multiplicity distribution in e^+e^- annihilations and hadronic collisions. The data are also analysed by the gamma distribution which is the KNO scaling function of the NBD. As is seen from Table I , the χ^2_{min} values for the MNBD fit are almost equivalent to those for the NBD fit in both reactions. The result shows to be similar to the case of the analysis of the cumulant moments in hadronic collisions[3]. On the other hand it should be noticed that the MNBD described the data of the cumulant moments much better than the NBD in e^+e^- annihilations[2].

In order to know the stochastic structure of the MNBD in detail, we discuss the following point : The solution obtained from the branching equation of the pure birth process with the immigration under the initial condition of the binomial distribution

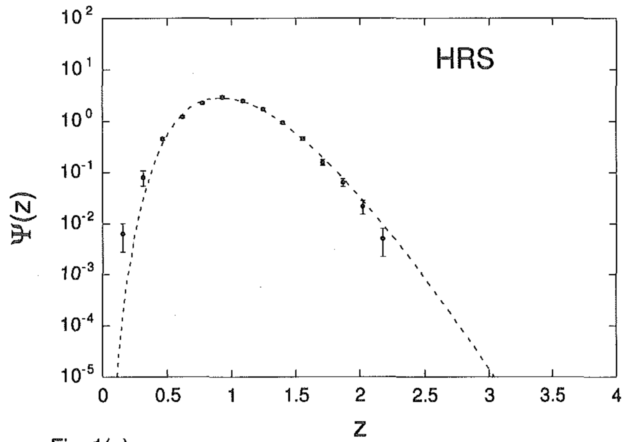

Fig. 1(a)

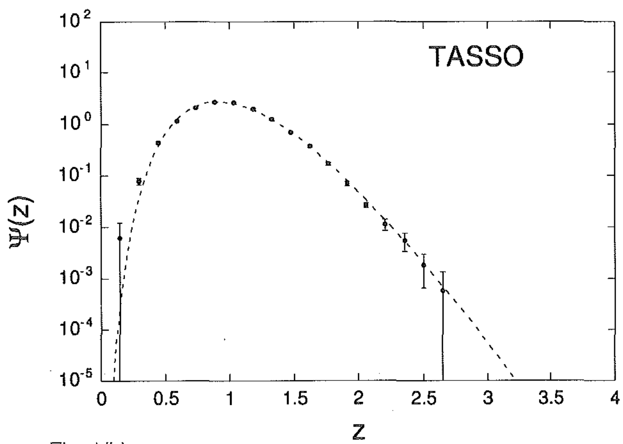

Fig. 1(b)

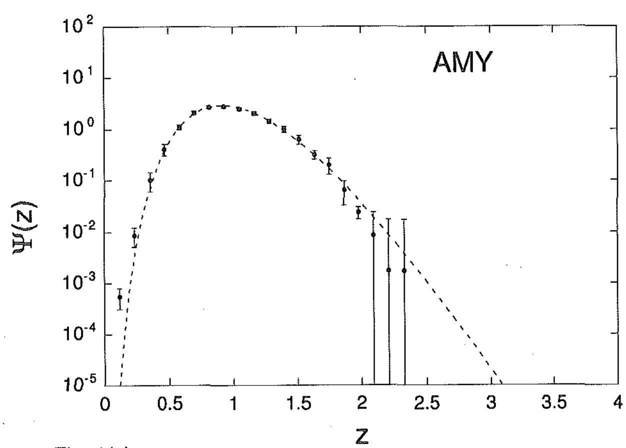

Fig. 1(c)

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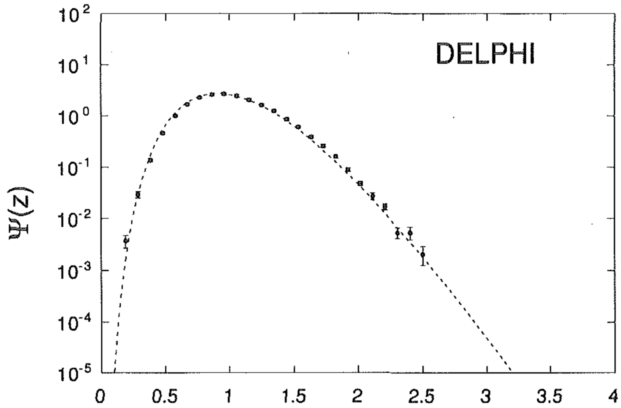

Fig. 1(d) z

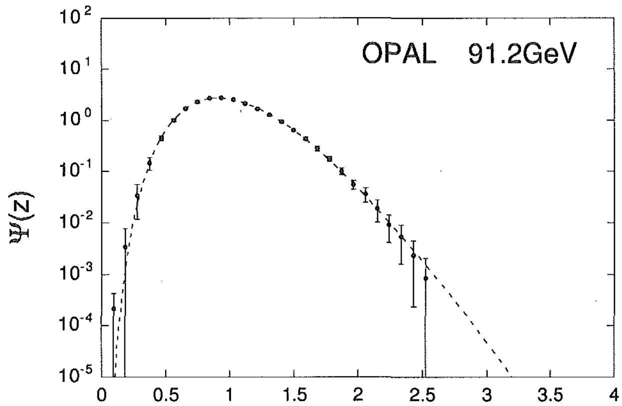

Fig. 1(e) z

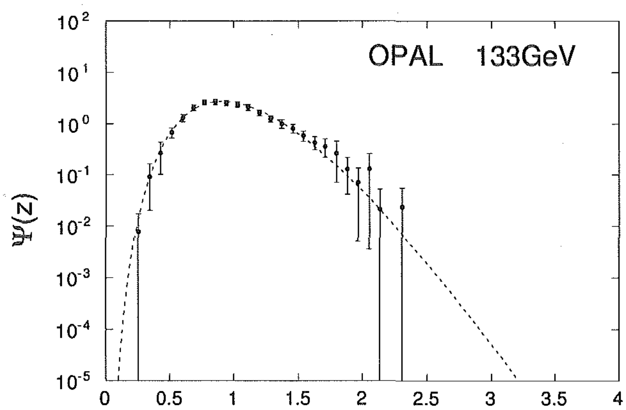

Fig. 1(f) z Nakajima et al., KNO scaling ...

Fig. 1 The KNO form of the charged multiplicity in e^+e^- collisions. The full circles are obtained from data of the following Collaborations: HRS[8], TASSO[9], AMY[10], DELPHI[11] and OPAL[12,13], respectively, in Fig. 1(a)-1(f). The broken line is the KNO scaling function obtained from the MNBD.

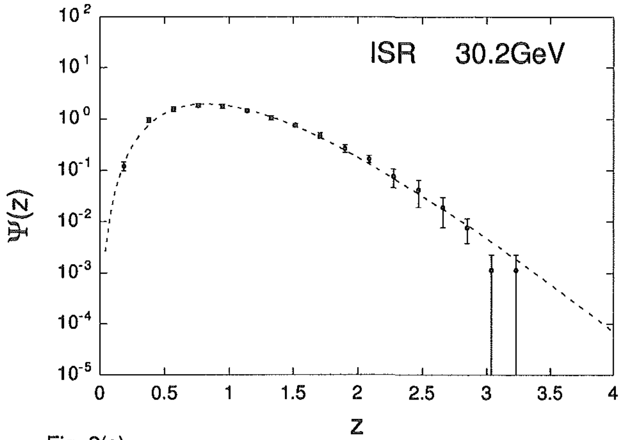

Fig. 2(a)

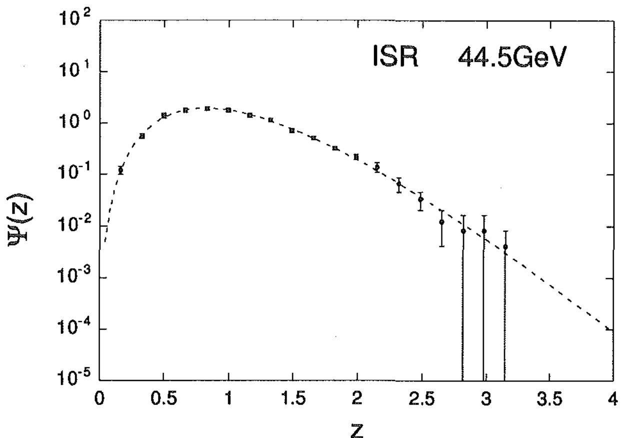

Fig. 2(b)

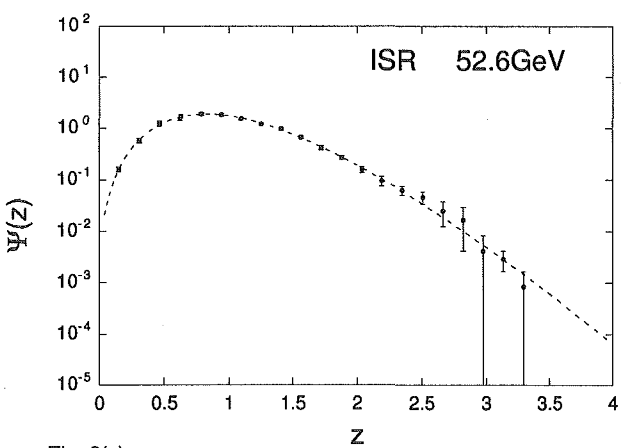

Fig. 2(c)

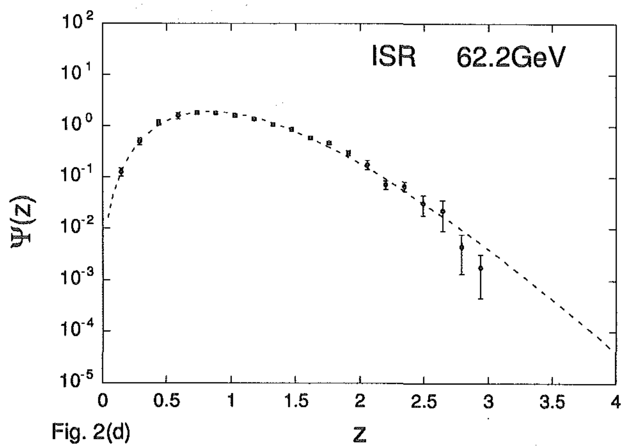

Fig. 2(d)

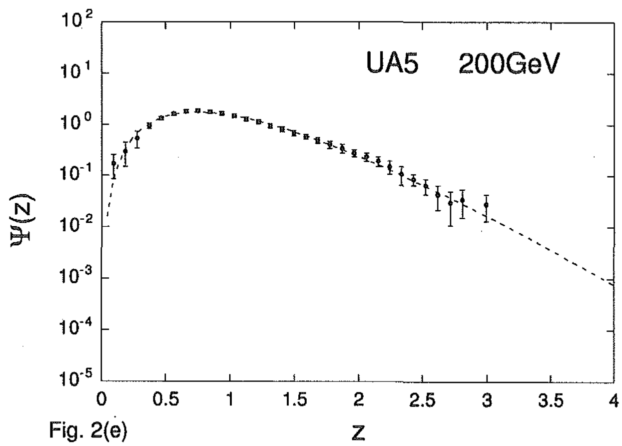

Fig. 2(e)

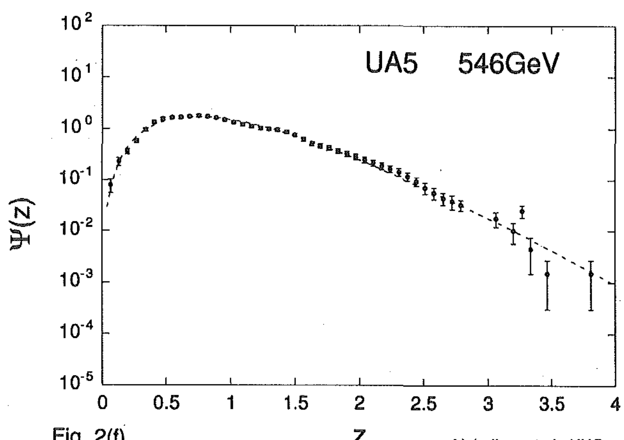

Fig. 2(f)

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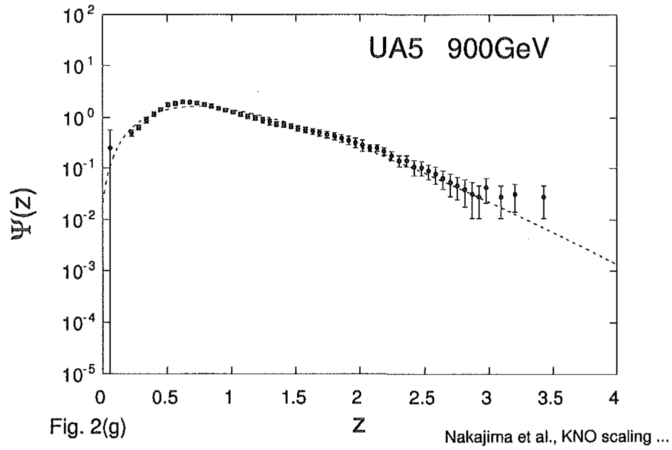

Fig. 2 The KNO form of the charged multiplicity in hadronic collisions. The full circles are obtained from data of the following Collaborations: ISR[14] and UA5[15] collaborations, respectively, in Fig. 2(a)-(b). The broken line is the KNO scaling function obtained from the MNBD.

[3,5] is one of the extensions of both the MNBD and the NBD. Its generating function is given as

$$\Pi(u)=\sum_{n=0} P(n)u^n=[1-r_1(u-1)]^N[1-r_2(u-1)]^{-k-N}. \tag{12}$$

where k is the immigration rate, and

$$r_1=\frac{\langle n\rangle}{k}\left\{1-\sqrt{\frac{k+N}{N}\left[-k\left(C_2-1-\frac{1}{\langle n\rangle}\right)+1\right]}\right\}$$

$$r_2=\frac{Nr_1+\langle n\rangle}{k+N}. \tag{13}$$

The MNBD is obtained by neglecting the power k in $\Pi(u)$. The generating function of the NBD is given by neglecting the power N in Eq. (12). The physical meaning of the immigration term k may be interpreted as a possible contribution from gluons.

Using Eq. (9), we have directly the KNO scaling function for Eq. (12)

$$\Psi(z)=\left(\frac{r_1}{r_2}\right)^N e^{-\frac{\langle n\rangle}{r_2}z}\sum_{j=0}^{N} {}_NC_j\frac{1}{\Gamma(k+j)}\left(\frac{r_2-r_1}{r_1}\right)^j\left(\frac{\langle n\rangle}{r_2}\right)^{k+j} z^{k+j-1}$$

$$=\frac{\Gamma(N+1)}{\Gamma(N+k)}\left(\frac{r_1}{r_2}\right)^N\left(\frac{\langle n\rangle}{r_2}\right)^k z^{k-1}e^{-\frac{\langle n\rangle}{r_2}z}L_N^{(k-1)}\left(-\frac{r_2-r_1}{r_1}\frac{\langle n\rangle}{r_2}z\right). \tag{14}$$

This function becomes the KNO scaling function of the MNBD (Eq. (11)) when $k\to 0$, and reduces to the gamma distribution (Eq. (4)) if $N=0$.

In concrete application, we have confirmed that the discrete distribution of Eq. (14) cannot explain the oscillating behaviors of the cumulant moment observed in e^+e^- annihilations and in hadronic collisions much better than the MNBD (Eq. (1)). However, we are expecting at present that Eq. (14) will become to be useful in analyses of data of some reactions at higher energies, since it has stochastic characteristics of the MNBD and the NBD.

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