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On the Temporally Global Existence of the Solution for a Nonlinear System of Mainly Parabolic Equations

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Abstract

A discussion is made on the temporally global problem of a nonlinear mainly parabolic system of partial differential equations which is a model of partial differential equations with factors depending on the time process of unknown functions.

The notation used below is conventional, $H^{2+\alpha}(\bar{\Omega}), H^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T), C^{1+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)$, etc., denoting Hölder spaces.

1 Introduction

We consider the following system of partial differential equations:

$$(1.1)_1 \quad \begin{cases} \dfrac{\partial}{\partial t}\psi(x,t)=\mu\phi^{+m}\dfrac{\partial^2}{\partial x^2}\psi(x,t)+b_0\psi(x,t)^2 \\ \dfrac{\partial}{\partial t}\phi(x,t)=\phi\psi, (0\leq x\leq l(<\infty), t\geq 0\ ;\ \mathrm{I}=[0,l]) \end{cases} \quad (1.1)_2$$

$$(1.2) \quad \psi(x,0)=\psi_0(x)(\geq 0)(\in H^{2+\alpha}(I), \alpha\in(0,1)), \phi(x,0)=1$$

$$(1.3) \quad \psi(0,t)=\psi(l,t)=0(t\geq 0), \text{ (accompanied by compatibility conditions)},$$

where μ, m and b_0 are positive constants and $\phi(x,t)$ is obviously expressed as

$$(1.4) \quad \phi(x,t)=\exp\left\{\int_0^t\psi(x,\tau)d\tau\right\}.$$

Now, without proof, we give:

Theorem 1 (Temporally local existence) *For some* $T\in(0,\infty)$, *there exists a unique solution* $(\phi(x,t), \psi(x,t))$ *for* (1.1)-(1.2)-(1.3) *belonging to* $C^{1+\alpha,1+\frac{\alpha}{2}}(I_T)\times H^{2+\alpha,1+\frac{\alpha}{2}}(I_T)(I_T=I\times[0,T])$. *Moreover,* $\psi(x,t)\geq 0$.

2 The case of $m > b_0$

Let $(\phi,\psi)\in C^{1+\alpha,1+\frac{\alpha}{2}}(I_T)\times H^{2+\alpha,1+\frac{\alpha}{2}}(I_T)$ satisfy (1.1)-(1.2)-(1.3) $(0\leq t\leq T)$. First, we divide both sides of $(1.1)_1$ by ϕ^m and integrate them in x over I, having

$$(2.1)\qquad \int_I \frac{\psi_t}{\phi^m}dx=\int_I\left[\left(\frac{\psi}{\phi^m}\right)_t+m\frac{\psi^2}{\phi^m}\right]dx=\mu\int_I \psi_{xx}dx+b_0\int_I\frac{\psi^2}{\phi^m}dx$$

$$=\mu\psi_x|_{x=0}^l+b_0\int_I\frac{\psi^2}{\phi^m}dx,\ (\psi_x|_{x=0}^l\leq 0),$$

$$(2.1)'\qquad \int_I\left(\frac{\psi}{\phi^m}\right)_t dx+(m-b_0)\int_I\frac{\psi^2}{\phi^m}dx-\mu\psi_x|_{x=0}^l=0.$$

Next, by integrating both sides of $(2.1)'$ in t over $[0, T]$, we obtain an equality

$$(2.2)\qquad \int_I\frac{\psi}{\phi^m}dx+(m-b_0)\int_0^t\int_I\frac{\psi^2}{\phi^m}dx-\mu\int_0^t\psi_x|_{x=0}^l dt=\int_I\psi_0\,dx=\|\psi_0\|_{L^1(I)},$$

$$\left(i.e., \int_I\frac{\psi}{\phi^m}\leq\|\psi_0\|_{L^1(I)}\right).$$

Now, multiplying both sides of $(1.1)_1$ by ϕ^{-m} and noting that, by virtue of $\psi\in H^{2+\alpha,1+\frac{\alpha}{2}}(I_T)$,

$$(2.3)\qquad \int_0^t\psi_{xx}(x,\tau)d\tau=\frac{\partial}{\partial x^2}\int_0^t\psi(x,\tau)d\tau,$$

we have,

$$(2.4)\qquad \mu\frac{\partial^2}{\partial x^2}\left[\int_0^t\psi(x,\tau)d\tau\right]=\frac{\psi}{\phi^m}(x,t)-\psi_0(x)+(m-b_0)\int_0^t\frac{\psi^2}{\phi^m}(x,\tau)d\tau.$$

Hence, on the basis of the theory of integral equations in a single independent variable ([8]), follows an equality,

$$(2.5)\qquad \mu\int_0^t\psi(x,\tau)d\tau=-\int_I G(x,\xi)\left[\frac{\psi}{\phi^m}(\xi,t)-\psi_0(\xi)+(m-b_0)\int_0^t\frac{\psi^2}{\phi^m}(\xi,\tau)d\tau\right]d\xi,$$

or

$$(2.5)'\qquad \mu\int_0^t\psi(x,\tau)d\tau+\int_I G(x,\xi)\left[\frac{\psi}{\phi^m}(\xi,t)+(m-b_0)\int_0^t\frac{\psi^2}{\phi^m}(\xi,\tau)d\tau\right]d\xi,$$

$$=\int_I G(x,\xi)\,\psi_0(\xi)\,d\xi,$$

where $G(x,\xi)$ is the Green function for the problem

$$(2.6)\qquad \frac{d^2u}{dx^2}(x)=-f,\ u(0)=u(l)=0,$$

being defined by

$$(2.7)\qquad G(x,\xi)=\begin{cases}(l-x)\dfrac{\xi}{l}\,(0\leq\xi\leq x\leq l)\\ \dfrac{x}{l}(l-\xi)\,(0\leq x\leq\xi\leq l).\end{cases}$$

From $(2.5)'$ are derived the following estimates,

$$(2.8)\qquad \begin{cases}0\leq\mu\int_0^t\psi(x,\tau)d\tau\leq\int_I G(x,\xi)\,\psi_0(\xi)d\xi\left(\leq\frac{l}{4}\|\psi_0\|_{L^1(I)},\leq\frac{l}{8}|\psi_0|_I^{(0)}\right),\\ \phi(x,t)=\exp\{\int_0^t\psi(x,\tau)d\tau\}\leq\exp\{\mu^{-1}\int_I G(x,\xi)\psi_0(\xi)d\xi\}.\end{cases}$$

Now, multiplying both sides of $(1.1)_1$ by $\phi^{-m}\psi$ and integrating them in x over I and in t over $[0, T]$, we have

$$(2.9)\qquad \int_0^t\int_I\frac{\psi}{\phi^m}\psi_t\,dxdt=\mu\int_0^t\int_I\psi_{xx}\,\psi dxdt+b_0\int_0^t\int_I\frac{\psi^2}{\phi^m}dxdt.$$

Next, taking into account the equality,

$$(2.9)' \qquad \frac{\psi}{\phi^m}\psi_t=\frac{1}{2}\left(\frac{\psi^2}{\phi^m}\right)_t+\frac{m}{2}\frac{\psi^3}{\phi^m},$$

we rewrite (2.9), obtaining

$$(2.10) \qquad \frac{1}{2}\int_I \frac{\psi^2}{\phi^m}dx+\mu\int_0^t\int_I \psi_x^2\,dxdt=\frac{1}{2}\int_I \psi_0^2\,dx+\left(b_0-\frac{m}{2}\right)\int_0^t\int_I \frac{\psi^3}{\phi^m}dxdt.$$

The case of $b_0-\frac{m}{2}\leq 0$ is easier to treat than that of $b_0-\frac{m}{2}>0$.

(i) Case of $b_0-\frac{m}{2}\leq 0$.

We differentiate both sides of $(1.1)_1$ in x (which is permissible by the strength of the smoothness of ϕ and ψ), and, moreover, after multiplying them by ψ_x, integrate them over I.

$$(2.11) \qquad \int_I \psi_{xt}\psi_x dx=\mu\int_I (\phi^m\,\psi_{xx})_x\,\psi_x\,dx+b_0\int_I (\psi^2)_x\,\psi_x dx$$
$$=\mu[\phi^m\,\psi_{xx}\,\psi_x]_{x=0}^{l}-\mu\int_I \phi^m\,\psi_{xx}^2\,dx+2b_0\int_I \psi\psi_x^2\,dx,$$

where we note $\psi_{xx}(0,t)=\psi_{xx}(l,t)=0$ by means of the boundary condition on ψ and (1.1). From (2.11) we have easily an equality,

$$(2.12) \qquad \frac{1}{2}\int_I \psi_x(x,t)^2dx+\mu\int_0^t dt\int_I \phi^m\psi_{xx}^2dx=\frac{1}{2}\int_I \psi_0'(x)^2+2b_0\int_0^t dt\int_I \psi\psi_x^2dx.$$

By virtue of (2.10) and the inequality

$$(2.13) \qquad \psi(x,t)^2\leq l\int_I \psi_x^2\,dx,\ \text{i.e.,}\ (|\psi(\cdot,t)|_I^{(0)})^2\leq l\int_I \psi_x^2dx,$$

from (2.12) we derive,

$$(2.14) \qquad \frac{1}{2l}(|\psi(\cdot,t)|_I^{(0)})^2\leq\frac{1}{2}\int_I \psi_0'(x)^2dx+2b_0|\psi|_{I_T}^{(0)}\int_0^t\int_I \psi_x^2dxdt$$
$$\leq\frac{1}{2}\int_I \psi_0'(x)^2dx+2b_0|\psi|_{I_T}^{(0)}\frac{1}{2\mu}\int_I \psi_0^2\,dx,\ (0\leq t\leq T).$$

Finally, for $|\psi|_{I_T}^{(0)}$ it holds that

$$(2.15) \qquad (|\psi|_{I_T}^{(0)})^2\leq lA_1+\frac{b_0 l}{\mu}A_0|\psi|_{I_T}^{(0)},$$
$$(A_0=\|\psi_0\|_{L^2(I)}^2,\ A_1=\|\psi_0'\|_{L^2(I)}^2).$$

Hence follows an a priori estimate of $|\psi|_{I_T}^{(0)}$,

$$(2.16) \qquad |\psi|_{I_T}^{(0)}\leq\frac{1}{2}\left[\frac{b_0 l}{\mu}A_0+\left\{\left(\frac{b_0}{\mu}A_0\right)^2+4lA_1\right\}^{\frac{1}{2}}\right],\ (0\leq T\leq\infty).$$

By (2.8), (2.12), and (2.16) the asymptotic property of $|\psi(\cdot,t)|_I^{(0)}$, which is shown below, is easily to be proved,

$$(2.17) \qquad |\psi(\cdot,t)|_I^{(0)}\leq C_0\exp^{-C_1 t}(t\geq 0),$$

where C_0 and C_1 are constants depending on $\|\psi_0\|_{L^1(I)}$, $\|\psi_0'\|_{L^2(I)}$, etc.

(ii) Case of $m>b_0>\frac{m}{2}$.

Our discussion of this case is based upon equalities (2.10) and (2.12), which hold also for $b_0 \in (\frac{m}{2}, m)$. The 2nd term of the right-hand side of (2.10) is estimated by means of (2.2) as

$$\left(b_0-\frac{m}{2}\right)\int_0^t\int_I \frac{\psi^3}{\phi^m}dxdt \leq \left(b_0-\frac{m}{2}\right)\int_0^t\left[|\psi|_I^{(0)}(\tau)\int_I \frac{\psi^2}{\phi^m}(x,\tau)dx\right]d\tau. \tag{2.18}$$

The 2nd term of the left-hand side of (2.10) is bounded from below in the following way,

$$\frac{\mu}{l}\int_0^t |\psi^2|_I^{(0)}(\tau)d\tau \leq \mu\int_0^t\left[\int_I \psi_x^2(x,\tau)dx\right]d\tau. \tag{2.19}$$

Thus, by (2.2), (2.10), (2.18), and (2.19), from (2.12) follows an estimate for ψ,

$$|\psi^2|_I^{(0)}(t) \leq l\,\|\psi_0'\|_{L^2(I)}^2+4b_0 l\,|\psi|_{I_t}^{(0)}\cdot\mu^{-1}\left[\frac{1}{2}\|\psi_0\|_{L^1(I)}^2 + \left(b_0-\frac{m}{2}\right)\int_0^t|\psi|_I^{(0)}(\tau)\left(\int_I\frac{\psi^2}{\phi^m}dx\right)d\tau\right], \tag{2.20}$$

i.e.,

$$f(t)^2 \leqq B_1+f(t)\left[B_2+B_3\int_0^t f(\tau)g(\tau)d\tau\right]_1 \quad (0\leqq t\leqq T), \tag{2.20$'$}$$

where

$$\begin{cases} B_1=l\,\|\psi_0'\|_{L^2(I)}^2,\ B_2=2b_0 l\mu^{-1}\|\psi_0\|_{L^2(I)}^2,\ B_3=4b_0 l\mu^{-1} \\ f(t)=|\psi|_I^{(0)}(t),\ g(t)=\int_I\frac{\psi^2}{\phi^m}(x,t)dx. \end{cases} \tag{2.20$''$}$$

From (2.20)$'$ we have easily an a priori estimate for ψ,

$$|\psi|_{I_T}^{(0)} \leq \sqrt{B_1}+B_2 e^{B_1B_2B_0}(\sqrt{B_1}B_3B_0+B_3), \tag{2.21}$$
$$\left(B_0=(m-b_0)^{-1}\|\psi_0\|_{L^1(I)}\left(\geqq\int_0^t g(\tau)d\tau\right)\right).$$

[Put $[\cdots]_1=Q(t)$, and transform (2.20)$'$ into a differential inequality in $Q(t)$.]

Thus, we have:

Theorem 2 *(i) In the case of* $\frac{m}{2}\geq b_0>0$, *there exists a temporally global solution* (ϕ,ψ) *of* (1.1)-(1.2)-(1.3) *belonging to* $C^{1+\alpha,1+\frac{\alpha}{2}}(I_T)\times H^{2+\alpha,1+\frac{\alpha}{2}}(I_T)$ *for an arbitrary* $T\in(0,\infty)$. *Moreover, estimates* (2.16) *and* (2.17) *hold. (ii) In the case of* $m>b_0>\frac{m}{2}$ *there exists a temporally global solution* (ϕ,ψ) *of* (1.1)-(1.2)-(1.3) *belonging to* $C^{1+\alpha,1+\frac{\alpha}{2}}(I_T)\times H^{2+\alpha,1+\frac{\alpha}{2}}(I_T)$ *for an arbitrary* $T\in(0,\infty)$. *Moreover, the estimate* (2.21) *holds.*

3 The case of $b_0\geq m$

Let $(\phi,\psi)\in C^{1+\alpha,1+\frac{\alpha}{2}}(I_T)\times H^{2+\alpha,1+\frac{\alpha}{2}}(I_T)$ satisfy (1.1)-(1.2)-(1.3). Now, we define $u(x,t)$ by

$$u(x,t)=\psi(x,t)+\frac{k}{2}\left(x-\frac{l}{2}\right)^2 \quad (k,\ \text{positive constant}). \tag{3.1}$$

Then, $u(x,t)$ satisfies,

(3.2)
$$\begin{cases} u_t(x,t)(=\psi_t)=\mu\phi^m u_{xx}+b_0\psi^2-k\mu\phi^m, \\ u(x,0)=\psi_0+\frac{k}{2}\left(x-\frac{l}{2}\right)^2\left(\le|\psi_0|_l^{(0)}+\frac{kl^2}{8}=p(k,|\psi_0|_l^{(0)})\right), \\ u(0,t)=u(l,t)=\frac{kl^2}{8}(t\ge 0). \end{cases}$$

Seeing that $b_0\psi^2-k\mu\phi^m\le b_0\psi^2-k\mu$, if $b_0\psi^2-k\mu<0$, then we can estimate $u(x,t)$ and $\psi(x,t)$ by the maximum principle as follows:

(3.3)
$$0\le\psi(x,t)\le u(x,t)\le|\psi_0||_l^{(0)}+\frac{kl^2}{8}=p(k,|\psi_0|_l^{(0)}).$$

Therefore, by the use of reductio ad absurdum, we conclude that, if $|\psi_0|_l^{(0)}(=c_0)$ is such that $b_0p(k,c_0)^2-k\mu<0$, then

(3,4)
$$0\le\psi(x,t)\le|\psi_0|_l^{(0)}+\frac{kl^2}{8}=p(k,c_0),(b_0\psi^2-k\mu\le b_0p(k,c_0)-k\mu).$$

Here, we express k by p and c_0, i.e.,

(3.5)
$$k=\frac{8}{l^2}(p-c_0).$$

Then, from $b_0p^2-k\mu=b_0p^2-\frac{8\mu}{l^2}(p-c_0)<0$ we have,

(3.6)
$$0\le c_0<p-\frac{b_0l^2}{8\mu}p^2=p\left(1-\frac{b_0l^2}{8\mu}p\right)\le\frac{2\mu}{b_0l^2},$$

which implies that, if $c_0=|\psi_0|_l^{(0)}<\frac{2\mu}{b_0l^2}$, then exists p_0 such that

(3.7)
$$0\le c_0=|\psi_0|_l^{(0)}<p_0-\frac{b_0l^2}{8\mu}p_0^2\le\frac{2\mu}{b_0l^2},$$

and then, moreover,

(3.8)
$$0\le\psi(x,t)\le p_0\,(\text{c.f. }(3.3)).$$

Thus we have:

Theorem 3 *In the case of* $b_0\ge m$, *under the condition* (3.6), *there exists a temporally global solution* (ϕ,ψ) *of* (1.1)-(1.2)-(1.3) *belonging to* $C^{1+\alpha,1+\frac{\alpha}{2}}(I_T)\times H^{2+\alpha,1+\frac{\alpha}{2}}(I_T)$ *for an arbitrary* $T\in(0,\infty)$. *Moreover, the estimate* (3.8) *holds, where* p_0 *satisfies* (3.7).

In a near future we shall treat partial differential equations of the type

$$u_t(x,t)=\phi\left(u,\int_0^t udt\right)u_{xx}+\psi\left(u,\int_0^t udt\right)$$

from a standpoint of the blowup-nonblowup problem.

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