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Vector Analysis on Sobolev Spaces

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Abstract

(∞-p)-forms on a k-th Sobolev space $W^k(X)$, X a compact (spin) manifold, is defined by using Sobolev duality. Integrals of (∞-p)-forms on a cube in $W^k(X)$ are defined without using measure. It is shown that exterior differentiability of an (∞-p)-form is astrong constraint and an exterior differentiable (∞-p)-form is always globally exact. As a consequence, the exterior differential operator d is not nilpotent when acting on the space of (∞-p)-forms. Stokes' Theorem for the integrals of (∞-p)-forms is also shown.

Introduction

Analysis on infinite dimensional spaces together with its geometric applications, has been treated mostly by using probabilistic methods (*e.g.*[3], [6], [7]). But more classical analysis related to the geometry of infinite dimensional spaces seems not so well developed. In this paper, we define an (∞-p)-form on U, an open set of k-th Sobolev space $W^k(X)$ over a compact (spin) manifold X to be a smooth map f from U to $\Lambda^p W^k(X)$, the k-th Sobolev space of alternating functions (spinors) on p-th direct product $X_{\times\cdots\times}X$ of X. Then treat differential and integral calculuses of (∞-p)-forms The outline of the paper is as follows ; In sect. 1, we fix the Sobolev metric of $W^k(X)$ by apointing a non degenerate 1-st order selfadjoint elliptic (pseudo) differential operator D on X. By using spectral eta and zeta functions of D and $|D|$, we define virtual dimension n^- of $W^k(X)$ and volumes of cubes (powers of $d$et $|D|$) in $W^{-\ell-\alpha}(X)$. Some calculations related to these quantities are also done. In sect. 2, integrals of a function f on a cube in $W^{-\ell-\alpha}(X)$ is defined in the spirit of Riemannian integral. It is shown some complete continuity of f is necessary (and sufficient) to the existence of the integral. Then ∞-forms are introduced. (∞-p)-forms and their exterior differntial are defined in sect. 3. By the definition of (∞-p)-forms, if f is an (∞-p)-form, its Frechět differential $\hat{d} f$ can be viewed as a map from U to the algebra of bounded linear operators on $W^k(X)$ (with parameters). The exterior differentiability condition is the trace class condition of $\hat{d} f$ and the exterior differntial df is defined to be $tr\ \hat{d} f$.

Examples show some renormalized exterior differential may defined and will be useful. In sect. 4. local and global exactness of exterior differentiable (∞-p)-forms are shown. As a consequence, the exterior differential operator d is not nilpotent as an operator acting on the space of (∞-p)-forms. So we can expect some spaces of (∞-p)-forms provides geometric examples of Kerner's higher gauge theory ([5]). In sect. 5, we define (formal) boundary of a cube and integrals on the boundary. Then in sect. 6, the last section, we derive some kinds of Stokes' Theorem.

In this paper, we do not discuss Clifford aspects of (∞-p)-forms. Global problems related to the analysis and geometry of mapping spaces are alos not discussed. Some parts of detailed proofs (and definitions) are omitted. These will appeare elsewhere (cf. [1]).

1. Virtual dimension of a Sobolev space

Let X be a compact (spin) manifold with a fixed Riemannian metric. E a Hermitian vector bundle over X and $L^2(X)$ is the Hilbert space of sections of E. We denote L^2 -metric of $f \in L^2(X)$ by $\|f\|$. It is fixed by the Riemannian metric of X. We take a non degenerate 1-st order elfadjoint elliptic (pseudo) differential operator D acting on the section of E and fix the k-th Sobolev metric $\|f\|_k$ of f by

$$\|f\|_k = \|D^k f\|.$$

The k-th Sobolev space of sections of E is denoted by $W^k(X)$. By Sobolev' imbedding Theorem, $W^k(X)$ is contained in the space of continuous section of E if $k > d/2$, d is the dimension of X.

Since X is compact, D can be written as

$$Df = \sum \lambda (f, e_\lambda) e_\lambda, \quad \{e_\lambda\} \textit{ is an o. n. -basis of } L^2(X).$$

Then, to set

$$e_{\lambda,k} = sgn\lambda |\lambda|^{-k} e_\lambda,$$

$\{e_{\lambda,k}\}$ becomes an O. N. -basis of $W^k(X)$.

By using spectral decomposition of D, we define operators G, the Green operator of D, $|D|$, $D_\pm$ and ϵ by

$$Gf = \sum \lambda^{-1}(f, e_\lambda) e_\lambda, \ |D| = \sum |\lambda| (f, e_\lambda) e_\lambda,$$

$$D_\pm = 1/2(|D| \pm D), \ \epsilon = G|D|.$$

The spectral eta function $\eta_D(s)$ of D and $\zeta_{|D|}(s)$ of $|D|$ are defined by

$$\eta_D(s) = \sum sgn\lambda |\lambda|^{-s}, \ \zeta_{|D|}(s) = \eta_{D^2}(s/2) = \sum |\lambda|^{-s}.$$

It is know ([2], [4], [9] cf. [8])

(i) These functions are continued meromorphically on the whole complex plane with possible poles at $s = d$, d-1, ... with the order at most 1.

(ii) They are holomorphic at $s = 0$.

Definition 1. *We say* $\zeta_{|D|}(0) = n^-$ *to be the virtual dimension of* $W^k(X)$ (*with respect to*

D).

We also define the determinant $det|D|$ of $|D|$ and $det\ D$ of D by

$$det\,|D|=\exp(-\zeta_{|D|}'(0)),$$

$$det\ D=\exp(\pi\sqrt{-1}\,\zeta_{D\_}(0))\,det\,|D|,\ \ \zeta_{D\_}(0)=1/2(n^{-}-\eta_D(0)),$$

(cf. [10]). Then we have

$$det(tD)=t^{n^-}det\ D,\ t>0,$$

$$det\,|D^k|=(det\,|D|)^k.$$

Originally, n^- may not be an integer. But later the necessity of integrity of n^- will be shown. But this is not restrictive. Because we have

(1) $$\zeta_{|D+mI|}(0)=n^-+\sum_{j=1}^{d}(-1)m^j/jc_j,$$

$$c_j=\underset{s=j}{res}\ \eta_D(s),\quad j\equiv 1\ mod.2,\ c_j=\underset{s=j}{res}\ \zeta_{|D|}(s),\quad j\equiv 0\ mod.2.$$

Precisely saying, the right hand side is the analytic continuation of the left hand side for small $|m|$.

We can also derive continuation formula for $\zeta_{|D+mI|}'(0)$, which is necessary to the definition of the determinant bundle of a mapping space.

Virtual dimension is used to the definition of determinant of $g\in Map(\mathrm{X},\mathrm{G})$, G acts on the fibre of E as a subgroup of U(N). In this case, to write $g=\exp(2\pi\mathrm{i}h)$, we can define det g by

$$det\ g=\exp\Big(2\pi\mathrm{i}\int_{\mathrm{X}}trhdx/(n^-/\mathrm{N}\,vol\mathrm{X})\Big),$$

because we may regularize $trI=n^-$, I is the identity of $\mathrm{L}^2(\mathrm{X})$. This definition of $det\ g$ depends on the choice of h. But if g is homotopic to an element in $Map(\mathrm{X},\mathrm{SU(N)})$, then does not depend on the choice of h. In this case, we denote this determinant by $det_D g$. It is shown

(2) $$det_D\,gh=det_D g\ det_D h,\quad det_D g=1\ if\ g\in Map(\mathrm{X},\mathrm{SU(N)}).$$

2. Integrals on a cube in a Sobolev space

In $\mathrm{W}^{-\ell-\alpha}(\mathrm{X})$, $\alpha>d/2$, we set

$$\mathrm{Q}(\ell,t)=\{\textstyle\sum c_n\mathrm{e}_n\,\|\,c_n|\leqq|t\lambda_n|^\ell\},$$

$$\mathrm{Q}(\ell,t,+)=\{\textstyle\sum c_n\mathrm{e}_n\,|\,0<c_n\leqq|t\lambda_n|^\ell\},\ t>0.$$

For simple, we assume $\ell\neq 0$, and set

(3) $$vol(\mathrm{Q}(\ell,\mathrm{t}))=(2t)^{\ell n^-}|det\ D|^\ell,\ \ vol(\mathrm{Q}(\ell,t,+))=t^{\ell n^-}|det\ D|^\ell.$$

Let s be in $\mathrm{I}=[0,1]$ with the binary expansion $0.\,s_1\ldots s_n\ldots$ Then we define a subset D(s) of $\mathrm{Q}(\ell,t)$ by

(4) $$\mathrm{D}(s)=\{\textstyle\sum c_n\mathrm{e}_n\,|-|t\lambda_n|^\ell\leqq c_n\leqq 0,\ if\ s_n=0,\ 0\leqq c_n\leqq|t\lambda_n|^\ell,\ if\ s=1\}.$$

By definition $\mathrm{Q}(\ell,t)=\mathrm{U}_{s\in\mathrm{I}}\,\mathrm{D}(s)$. For a function $f(x)$ on $\mathrm{Q}(\ell,t)$, we define functions f^- and f_- on I by

$$f^{-}(s)=\sup_{x\in D(s)} f(x), \quad f_{-}(s)=\inf_{x\in D(s)} f(x).$$

Then the integrals $\int_I f^{-}dsvol(Q(\ell,t))$ and $\int_I f_{-}dsvol(Q(\ell,t))$ are upper and lower Riemannian sums of $f(x)$ with respect to the partition $\{D(s)\}$ of $Q(\ell,t)$.

We assume for $(s^1,\dots,s^{m-1})\in I^{m-1}$, the partition $D(s^1,\dots,s^{m-1})$ of $Q(\ell,t)$ has been defined to be $\{\sum c_n e_n \mid a_n<c_n<b_n\}$. Then for $s^m=0.\ s_1^m s_2^m\ldots\in I$, we set

(4)' $$D(s^1,\dots,s^m)=\{\sum c_n e_n \mid a_n\leqq c_n\leqq a_n+1/2(b_n-a_n),\ if\ s_n^m=0,$$
$$a_n+1/2(b_n-a_n)\leqq c_n\leqq b_n,\ if\ s_n^m=1\}.$$

The functions $f^{-}(s^1,\dots,s^m)$ and $f_{-}(s^1,\dots,s^m)$ are defined to be

$$f^{-}(s^1,\dots,s^m)=\sup_{x\in D(s^1,\dots,s^m)} f(x),$$
$$f_{-}(s^1,\dots,s^m)=\inf_{x\in D(s^1,\dots,s^m)} f(x).$$

Lemma 1. *f^{-} and f_{-} are continuous if f is continuous by the topology of* $W^{\ell-\alpha}(X)$, $\alpha>d/2$.

Proof If $s=0.\ s_1 s_2\ldots$ and $s'=0.\ s_1' s_2'\ldots$ satisfy $|s-s'|<2^{-m}$, then $s_1=s_1',\dots,s_m=s_m'$. Therefore, by the definition of $D(s)$, we have

$$\sup_{x\in D(s)} (\inf_{y\in D(s')} \|x-y\|_{-\ell-\alpha}{}^2)<(2t)^{\ell}\sum_{n>m}|\lambda_n|^{-\alpha}$$

Hence if $\alpha>d/2$, we get $\lim_{|s-s'|\to 0} \sup_{x\in D(s)} (\inf_{y\in D(s')} \|x-y\|_{-\ell-\alpha}{}^2)=0$. This shows the continuities of $f^{-}(s)$ and $f_{-}(s)$. Higher dimensional cases are similarly proved.

On the other hand, since

$$\lim_{m\to\infty} \sup_{x,y\in D(s^1,\dots,s^m)} \|x-y\|_{-\ell-\alpha}=0,\ if\ \alpha>d/2,$$

we have

$$\lim_{m\to\infty} \sup|f^{-}(s^1,\dots,s^m)-f_{-}(s^1,\dots,s^m)|=0,$$

if f is continuous by the topology of $W^{-\ell-\alpha}(X)$, $\alpha>d/2$. Therefore we obtain

Theorem 1 *If $f(x)$ is continuous by the topology of* $W^{-\ell-\alpha}(X)$, $\alpha>d/2$, *then*

(5) $$\lim_{m\to\infty}\int_{I^m} f^{-}d^m s=\lim_{m\to\infty}\int_{I^m} f_{-}d^m s$$

Definition 2. *Let f be a (real valued) function of $Q(\ell,t)$. Then we say f is integrable on $Q(\ell,t)$ if (5) is hold and define $\int_{Q(\ell,t)} f(x)dx$ by*

(6) $$\int_{Q(\ell,t)} f(x)dx=\lim_{m\to\infty}\int_{I^m} f^{-}d^m svol(Q(\ell,t)).$$

Integrals on $Q(\ell,t,+)$ are similarly defined.

Note. In the above definition of the integral, we used special division of $Q(\ell,t)$. But this is for simplicity and we can define integral by using more arbitrary division of $Q(\ell,t)$.

Example. Let $f(x)$ be

(7) $$f(x)=\sum x_n^2\lambda_n^{-2k}, \quad x=\sum x_n e_n\in Q(\ell,t,+).$$

Then we have

$$f^-(s^1,\dots,s^m)=t^{2(\ell-k)}\sum_n\sum_m((2^{m-1}s_n^1+2^{m-2}s_n^2+\cdots+s_n^m+1)/2^m)^2\lambda_n^{2(\ell-k)},$$
$$f_-(s^1,\dots,s^m)=t^{2(\ell-k)}\sum_n\sum_m((2^{m-1}s_n^1+2^{m-2}s_n^2+\cdots+s_n^m)/2^m)^2\lambda_n^{2(\ell-k)}$$

Hence we have

$$\int_{1^m}f^-d^ms=t^{2(\ell-k)}\sum_n\left\{1/2^m\sum_{s_n^i=1,0}((2^{m-1}s_n^1+\cdots+s_n^m+1)/2^m)^2\right\}\lambda_n^{2(\ell-k)}$$
$$=t^{2(\ell-k)}\sum_n\left\{1/2^m\sum_{j=1}^{2m}(j/2^m)\right\}\lambda_n^{2(\ell-k)}$$
$$=t^{2(\ell-k)}\sum_n\{(2^m+1)(2^{m+1}+1)/6\cdot 2^{2m}\}\lambda_n^{2(\ell-k)},$$
$$\int_{1^m}f_-d^ms=t^{2(\ell-k)}\sum_n\{(2^m-1)(2^{m+1}-1)/6\cdot 2^{2m}\}\lambda_n^{2(\ell-k)}.$$

Therefore we have

(8) $$\int_{Q(\ell,t,+)}f(x)dx=1/3\cdot t^{2(\ell-k)}\zeta_D(2(k-\ell))(det\ D)^\ell.$$

(8) shows $f(x)$ is integrable if $k\text{-}\ell>d/2$ and not integrable if $k\text{-}\ell=d/2$.

There is an alternative way to the computation of $\int_{Q(\ell,t)}f(x)dx$. We set

$$Q(\ell,t,N)=\left\{\sum_{n\leqq N}c_n e_n\,|-|t\lambda_n|^\ell\leqq c_n\leqq|t\lambda_n|^\ell,\ 1<n<N\right\},$$
$$Q(\ell,t,\infty-N)=\left\{\sum_{n\geqq N+1}c_n e_n\,|-|t\lambda_n|^\ell\leqq c_n\leqq|t\lambda_n|^\ell,\ n>N+1\right\}.$$

By definition $Q(\ell,t)=Q(\ell,t,N)\times Q(\ell,t,\infty-N)$. We denote $x=(x_N,x_{\infty-N})\in Q(\ell,t)$, where $x_N\in Q(\ell,t,N)$ and $x_{\infty-N}\in Q(\ell,t,\infty-N)$. Let f be a function on $Q(\ell,t)$. Then we set

$$f^{-N}(x_N)=\sup_{y\in Q(\ell,t,\infty-N)}f(x_N,y),\ f_{-N}(x_N)=\inf_{y\in Q(\ell,t,\infty-N)}f(x_N,y)$$

Then if f is continuous by the topology of $W^{-\ell-\alpha}(X)$, $\alpha>d/2$, we have

(9) $$\int_{Q(\ell,t)}f(x)dx=\lim_{N\to\infty}\int_{Q(\ell,t,N)}f^{-N}(x_N)d^Nx\,|t\lambda_1|^{-\ell}\quad|t\lambda_N|^{-\ell}vol(Q(\ell,t))$$

For example, for the function (7) $\int_{Q(\ell,t,+)}f(x)dx$ is computed as follows:

$$\int_{Q(\ell,t,N)}f(x)dx$$
$$=\lim_{n\to\infty}\sum_{n=1}^N\int_0^{(t\lambda_1)^\ell}\cdots\int_0^{(t\lambda_N)^\ell}x_n^2\,d^Nx\,|t\lambda_1|^{-\ell}\cdots|t\lambda_N|^{-\ell}vol(Q(\ell,t))$$
$$=\lim_{n\to\infty}\sum_{n=1}^N 1/3\prod_{j\neq n}(t\lambda_1)^\ell(t\lambda_n)^3(t\lambda_1)^{-\ell}\cdots(t\lambda_N)^{-\ell}vol(Q(\ell,t))$$
$$=1/3\ t^{2(\ell-k)}\zeta_D(2(k-\ell))(det\ D)^\ell.$$

We denote the cylindrical measure (of the domain $\{\sum c_n e_n||c_n|\leqq 1\}$) by $d^\infty v$. Then by the above discussions, we may define

(10) $$\bigwedge_\lambda de_\lambda=(det\ D)^{-d/2}\ d^\infty v,\ \bigwedge_\lambda de_{\lambda,k}=(det\ D)^{-k}\bigwedge_\lambda de_\lambda$$
$$=(det\ D)^{-k-d/2}\ d^\infty v.$$

So we may consider an infinite form to be a scalar function multiplied by $(det\ D)^{-k}$. Here we may idetify $de_{\lambda,k}$ and $e_{\lambda-k}$. Because to define the function $e_{\lambda,k}$ on U, an open set of $W^k(X)$, by

$$e_{\lambda,k}(x)=x_\lambda, \quad x=\sum x_\lambda e_\lambda,$$

we have $e_{\lambda,k}(x+ty)=x_\lambda+ty_\lambda=e_{\lambda,k}(x)+t(e_{\lambda-k},y)$. Hence the Frechêt differential $d^\wedge e_{\lambda,k}(x)(=de_{\lambda,k})$ is equal to $e_{\lambda,-k}\in W^{-k}(X)$.

3. $(\infty-p)$-forms on a domain in $W^k(X)$

Let U be an open set of $W^k(X)$. A p-form on U is a smooth map from U to $\Lambda^p W^{-k}(X)$, the $(-k)$-th Sobolev space of alternating functions (sections) on $\overbrace{X\times\cdots\times X}^{p}$. Since an $(\infty-p)$-form should be the dual of p-forms, we define

Definition 3. *An $(\infty-p)$-form f on U is a smooth map from U to $\Lambda^p W^{-k}(X)$, the k-th Sobolev space of alternating functions (sections) on $\overbrace{X\times\cdots\times X}^{p}$.*

We fix the duality between $u\in\Lambda^q W^{-k}(X)$ and $f\in\Lambda^p W^k(X)$ by

(11) $$\langle u,f\rangle=(G^k_{x_1}\times\cdots\times G^k_{x_p}u,\ D^k_{x_1}\times\cdots\times D^k_{x_p}f),$$

where (,) means the inner product of $L^2(\overbrace{X\times\cdots\times X}^{p})$ determined by the product metric.

Definition 4. *The wedge product of a p-form u and an $(\infty-q)$-form f is defined as follows:*

(12) $$u\wedge f=0, \quad if\ q>p,$$
$$(u\wedge f)(x_1,\ldots,x_{p-q})=\langle u(x_{p-q+1},\ldots,x_p),\ f(x_{x_{p-z+1}},\ldots,x_p\ ;\ x_1,\ldots,x_{p-q})\rangle \quad if\ q<p$$
$$u\wedge f=\langle u,f\rangle\Lambda^\infty e_{\mu-k}, \quad if\ q=p.$$

$f\wedge u$ is defined to be $(-1)^{P(n^- -q)}\, u\wedge f$. Then, since it must be

$$f\wedge u=(-1)^{P(n^- -q)}\, u\wedge f=(-1)^{P(n^- -q)+(n^- -q)P}\, f\wedge u,$$

$2p(n^- -q)$ must be an even integer for any integer p, q. Hence we obtain

Proposition 1. *The virtual dimension should be an integer when considering $(\infty-p)$-forms.*

By (12), we may write

$$(e_{n_1,-k}\wedge\cdots\wedge e_{n_p,-k})\wedge(e_{n_1,k}\wedge\cdots\wedge e_{n_p,k})=\Lambda^\infty e_{\mu-k}.$$

So to set $A=\{n_1,\ldots,n_p\}n_1<\ldots<n_p$, we denote

$$e_{n_1,k}\wedge\cdots\wedge e_{n_p,k}=\Lambda^{\infty-\{n_1,\cdots,n_p\}}\ e_{\mu,-k}=\Lambda^{\infty-A}e_{\mu,-k}.$$

Then an $(\infty-p)$-forms f has the coordinate expression

$$f=\sum_{n_1<\cdots<n_p} f_{n_1\cdots n_p}\Lambda^{\infty-\{n_1,\cdots,n_p\}}de_{\lambda,k}.$$

Its formal exterior derivation contain infinite sums.

In coordinate free notation, the Frechet differential $d^\wedge f$ of f is a map from U to $W^{-k}(X)\otimes W^k(X)$, because U is an open set of $W^k(X)$. Since $W^{-k}(X)\otimes W^k(X)$ is a subspace of $B(W^k(X))$, the algebra of bounded linear operators on $W^k(X)$,we set

$$d^\wedge f(x)(x_1,\ldots,x_p)=d^\wedge f(x,x_1)(x_2,\ldots,x_p)=\ldots$$
$$=(-1)^{p-1}d^\wedge f(x,x_p)(x_1,\ldots,x_{p-1}),$$

$d^\wedge f(x,x_1)$ is a map from U to $B(W^k(X))$.

Denifinition 4. *An $(\infty-p)$-form is said to be exterior differntiable if $d^{\wedge}f(x, x_1)$ is a map from U to the ideal of trace class operators. In this case, we define the exterior differential df of f by*

(13) $$df(x)(x_1, \ldots, x_{p-1}) = (-1)^{p-1}\ tr\ d^{\wedge}f(x, x_p)(x_1, \ldots, x_{p-1}).$$

Example. Since the coordinate expression of the $(\infty-1)$-form D^{-s} is

$$D^{-s}x = \sum sgn\,\lambda|\lambda|^{-s}(x, e_{\lambda,-k})de_{\lambda,-k} = \sum sgn\,\lambda|\lambda|^{-s}(x, e_{\lambda,-k})\Lambda^{\infty-\{\lambda\}}de_{\lambda,k},$$

we get

(14) $$dD^{-s} = \eta_D(s)\Lambda^{\infty}de_{\lambda,k}.$$

Let $r(x)$ be the L^2-norm of x, then $dr^m(x)$ is equal to $mr^{m-2}(x)\sum(x, e_{\lambda,-k})de_{\lambda,-k}$. So we have

$$d(r^m(x)D^{-s}) = (mr^{m-2}(x)\sum sgn\,\lambda|\lambda|^{-s}(x, e_{\lambda,-k})^2 + r^m(x)\eta_D(s)\Lambda^{\infty}de_{\lambda,-k}.$$

Hence in the sence of analytic continuation, we have

(15) $$\lim_{s\to 0} d(r^m(x)D^{-s}) = 0, \quad if -m = \eta_D(0).$$

Similar result hold replacing D by $|D|$. Since $I(=|D|^0)$ is $\sum e_\lambda\Lambda^{\infty-\{\lambda\}}de_\lambda$, $r^{-n^-}(x)I$ is the formal extension of the volume element of the sphere. (15) shows this formal extension is renormalized closed.

In general, taking s sufficiently large, $D^{-s}f$ becomes exterior differentiable. Since $d(D^{-s}f) = D^{-s}(df)$ if f is exterior differentiable, there might exist some theory of renormalized exterior differential. This will be a problem in future.

4. Exactness of exterior differentiable $(\infty-p)$-forms

By using absolute values, we arrange the proper values of D as follows:

$|\lambda_1| \leqq |\lambda_2| \leqq \cdots\cdots.$

We denote dx_n instead of $de_{\lambda_n,k}$, for simple. Then an $(\infty-1)$-form f is written

$$f = \sum_{i=1}^{\infty} f_i\Lambda^{\infty-\{i\}}dx_n.$$

If f is extrior differentiable, then we have

(16) $$df = \left(\sum_{i=1}^{\infty}\partial f_i/\partial x_i\right)\Lambda^{\infty}dx_n.$$

Note. Formally, df is given by the right hand side of (16). Exterior differentiability (trace class assumption) is its convergence condition.

We want to get local integration of f by the following form $(\infty-2)$-form g : $g = \sum_{i=1}^{\infty} g_{i,i+1}\Lambda^{\infty-\{i,i+1\}}dx_n$. Then we get

$$dg = \partial g_{1,2}/\partial x_2\Lambda^{\infty-\{1\}}dx_n + \sum_{i=2}^{\infty}(\partial g_{i,i+1}/\partial x_{i+1} - \partial g_{i-1,i}/\partial x_{i-1})\Lambda^{\infty-\{i\}}dx_n.$$

Hence, if $dg = f$, we get

$$g_{1,2}(x) = \int_0^{x_2} f_1(x)dx_2,$$

$$g_{i,i+1}(x) = \int_0^{x_{i+1}}(f_1(x) + \partial g_{i-1,i}\ (x)/\partial x_{i-1})\partial x_{i+1},\ i \geqq 2.$$

Since we have

$$g_{2,3}(x)=\int_0^{x_3}\left(f_2+\partial/\partial x_1\int_0^{x_2}f_1(x)dx_2\right)dx_3=\int_0^{x_3}\left(f_2+\int_0^{x_2}\partial f_1/\partial x_1 dx_2\right)dx_3,$$

we obtain

(17) $$\partial g_{2,3}/\partial x_2=\int_0^{x_3}(\partial f_2/\partial x_2+\partial f_1/\partial x_1)dx_3.$$

We assume

(18) $$\partial g_{n-1,n}/\partial x_{n-1}=\int_0^{x_n}\left(\sum_{i=1}^{n-1}\partial f_i/\partial x_i\right)dx_n.$$

Then we have

$$\partial g_{n,n+1}/\partial x_{n+1}=f_n+\partial g_{n-1,n}/\partial x_{n-1}=f_n+\int_0^{x_n}\left(\sum_{i=1}^{n-1}\partial f_i/\partial x_i\right)dt,$$

$$\begin{aligned}\partial g_{n+1,n+2}/\partial x_{n+2}&=f_{n+1}+\partial g_{n,n+1}/\partial x_n\\&=f_{n+1}+\partial/\partial x_n\int_0^{x_{n+1}}\left(f_n+\int_0^{x_n}\sum_{i=1}^{n-1}\partial f_i/\partial x_i\right)dt\\&=f_{n+1}+\int_0^{x_{n+1}}\left(\sum_{i=1}^{n}\partial f_i/\partial x_i\right)dt.\end{aligned}$$

Hence (18) is hold for any n.

Since f is exterior differentiable, $\sum_{i=1}^{\infty}\partial f_i/\partial x_i$ exists. Hence $\{\sum_{i=1}^{n}\partial f_i/\partial x_i\}$ is uniformly bounded. Since integrations are done in a neigh-borhood of the origin of $W^k(X)$, $\sum g_{i,i+1}\Lambda^{\infty-\{i,i+1\}}dx_n$ converges as an element of $\Lambda^\infty W^k(X)$. Similarly, by using lexicographic linear order of the index set $\{j_1,\ldots,j_p\}$, $j_1<\ldots<j_p$, expressing an $(\infty-2)$ -form f as

(19) $$f=\sum_{J'}\sum_{i\geq j_p}f_{(J',i)}\Lambda^{\infty-\{J',i\}}dx_n,\quad \boldsymbol{J}\text{ is locally minimum, } \boldsymbol{J}'=\{j_1,\cdots,j_{p-1}\},$$

we can construct a local integration g of f in the form

(20) $$g=\sum_{J'}\sum_{i\geq j_p}f_{\{J',i,i+1\}}\Lambda^{\infty-\{J',i,i+1\}}dx_n.$$

Hence we obtain

Lemma 2. *An exterior differentiable $(\infty-p)$-form is always locally exact.* *Corollary 1. The exterior differential operator d is not nilpotent as an operator on the space of $(\infty-p)$-forms.*

Example. Let g be $\sum(1-1/2^i)x_i x_{i+1}\Lambda^{\infty-\{i,i+1\}}dx_n$. Then dg is equal to $\sum 1/2^i x_i\Lambda^{\infty-\{i\}}dx_n$. Hence d^2g is equal to $\Lambda^\infty dx_n\neq 0$.

Corollary 2. *If an $(\infty-p)$-form f is exterior differentiable, then for any natural number q, locally we can write*

(21) $f=d^qg$, *g is an $(\infty-p-q)$-form.*

Proof. By Lemma 2, locally we can write $f=dg_1$. This means g_1 is exterior differentiable. Hence we can write $g_1=dg_2$, locally. Repeating this, we have Corollary.

Note. In the local integration of f, we used exterior differentiability of f only showing the convergence of formal integration. This is not curious, because taking s sufficiently large, $D^{-s}f$ becomes exterior differentiable for any $(\infty-p)$-form f. So we have

$$D^{-s}f = dg,$$

For any (∞-p)-form. If g takes the values in the domain of D^s, then we have

$$f = D^s dg = d(D^s g).$$

So an (∞-p)-form is always formaly locally integrable.

Let u be a smooth function and f an (∞-p)-form. Then we have

$$d(uf) = du_\wedge f + udf.$$

$$d^2(uf) = -du_\wedge df + du_\wedge df + ud^2 f = ud^2 f.$$

Repeating this, we get

Lemma 3. *Let u be a smooth function and f be an (∞-p)-form, then the followings are hold*

(22) $$d^{2m}(uf) = ud^{2m}f,$$

$$d^{2m+1}(uf) = du_\wedge d^{2m}f + ud^{2m+1}f.$$

Note. Similarly, if u is a p-form, we get

(22)' $$d^{2m}(uf) = ud^{2m}f, \quad d^{2m+1}(uf) = du_\wedge d^{2m}f + (-1)^p ud^{2m+1}f.$$

Theorem 2. *Let f be an exterior differentiable (∞-p)-form on an open set U of* W^k (X). *Then for any q, there is an (∞-p-q)-form g on U such that*

(23) $$f = d^q g, \text{ on } U.$$

Proof. First we assume $q \equiv 0$, *mod*.2. Then by Lemma 2 and (22), we haveTheorem by using smooth partition of unity. If $q \equiv 1$, *mod*.2, f can be written as $f = d^{q+1} g_1$ on U. Hence we have theorem taking $g = dg_1$.

Note. Since smooth partition of unity subordinate to any locally finite open covering exists on any Sobolev manifold, this Theorem is hold on any Sobolev manifold, especially on amapping space *Map*(X, M). On the other hand, since we used partition of unity, it is unclear whether this Theorem is hold in analytic category.

Since d is not nilpotent, it is a problm that can we provide some geometric models of Kerner's higher gauge theory ([5]) by using (∞-p)-forms.

5. Boundary of a cube domain and integration on the boundary

We set $Q(\ell, t; x_n = |t\lambda_n|^\ell) = \{\sum c_n e_n | -|t\lambda_n|^\ell < c_n < |t\lambda_n|^\ell, m \neq n, c_n |t\lambda_n|^\ell\}$.
$Q(\ell, t; x_n = -|t\lambda_n|^\ell)$ is similarly defined. The volumes of $Q(\ell, t; x_n \pm |t\lambda_n|^\ell)$ are defined to be $(2|t\lambda_n|^{-\ell})\ vol(Q(\ell, t))$.

Let $f = \sum_{i=1}^{\infty} f_i \Lambda^{\infty-\{i\}} dx_n$ be an (∞-1)-form. we define the integral of f on $Q(\ell, t; x_n = \pm|t\lambda_n|^\ell)$ to be the integral of f_n on $Q(\ell, t; x_n = \pm|t\lambda_n|^\ell)$, which is defined similarly as the integral on $Q(\ell, t)$.

Lemma 4. *Let f be an (∞-1)-form such that continuous and Frechĕt differentiable by the topology of* $W^{-\ell-\alpha}$(X), $\alpha > d2$. *Then we have*

(24) $$\lim_{n,m\to\infty} \sum_{i=n}^{m} \left(\int_{Q(\ell,t;\, x_n=|t\lambda_n|^\ell)} f - \int_{Q(\ell,t;\, x_n=-|t\lambda_n|^\ell)} f \right) = 0.$$

Proof. We assume $\|d\hat{}f\| \leqq C$ on $Q(\ell, t)$. Then we have

$$\left| f_i\Big(\sum_{n \neq i} x_n e_n + |t\lambda_i|^\ell e_i\Big) - f_i\Big(\sum_{n \neq i} x_n e_n - |t\lambda_i|^\ell e_i\Big) \right| < 2Ct^\ell |\lambda_i|^{\ell-\alpha},$$

because we have

$$\left\| \Big(\sum_{n \neq i} x_n e_n + |t\lambda_i|^\ell e_i\Big) - \Big(\sum_{n \neq i} x_n e_n - |t\lambda_i|^\ell e_n\Big) \right\|_{-\ell-\alpha}$$
$$= 2|t\lambda_i|^\ell |\lambda_i|^{-\alpha} = 2t^\ell |\lambda_i|^{\ell-\alpha}.$$

Hence we obtain

$$\left| \sum_{i=n}^{m} \Big(\int_{Q(\ell,t;\, x_i=|t\lambda_i|^\ell)} f - \int_{Q(\ell,t;\, x_i=-|t\lambda_i|^\ell)} f \Big) \right|$$
$$\leqq \sum_{i=n}^{m} \left| \int_{Q(\ell,t;\, x_i=|t\lambda_i|^\ell)} f_i - \int_{Q(\ell,t;\, x_i=|t\lambda_i|^\ell)} f_i \right|$$
$$\leqq \sum_{i=n}^{m} 2Ct^\ell |\lambda_i|^{\ell-\alpha} |t\lambda_i|^{\ell-\alpha} |t\lambda_i|^{-\ell} vol(Q(\ell,\ t)) = \sum_{i=n}^{m} 2C\,|t\lambda_i|^{-\alpha} vol(Q(\ell,\ t)).$$

Since $\alpha > d2$, this last term tends to 0 when n, m tends to infinity. Therefore we obtain Lemma.

Corollary. Under the same assumption on f,

$lim_{n\to\infty} \sum_{i=1}^{m} (-1)^{i-1} \Big(\int_{Q(\ell,t;\, x_n=|t\lambda_n|^\ell)} f - \int_{Q(\ell,t;\, x_n=-|t\lambda_n|^\ell)} f \Big)$ exists.

Formally, we denote

$$\partial Q(\ell, t) = \sum_{i=1}^{\infty} (-1)^{i-1} (Q(\ell, t\,;\ x_n = |t\lambda_n|^\ell) - Q(\ell, t\,;\ x_n = -|t\lambda_n|^\ell)) \tag{25}$$

This is only a formal sum. But by Corollary of Lemma 4, tha following definition has a meaning.

Definition 5. *Let f be an $(\infty$-1)-form defined on a neighborhood of $Q(\ell, t)$. Then we define the integral of f on $\partial Q(\ell, t)$ by the following limit*

$$\int_{\partial Q(\ell,t)} f = \lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{i-1} \Big(\int_{Q(\ell,t;\, x_n=|t\lambda_n|^\ell)} f - \int_{Q(\ell,t;\, x_n=-|t\lambda_n|^\ell)} f \Big) \tag{26}$$

Although $\partial Q(\ell, t)$ is a formal sum, we have

$$\partial Q(\ell,\ t) = \partial Q(\ell,\ t,\ N) \times Q(\ell,\ t, \infty - N) + (-1)^N Q(\ell,\ t) \times \partial Q(\ell,\ t, \infty - N). \tag{27}$$

Here $Q(\ell,\ t, \infty - N)$ is defined similarly as $Q(\ell, t)$. Corollary of Lemma 4 shows

$$\int_{\partial Q(\ell,t)} f = \lim_{n\to\infty} \int_{\partial Q(\ell,t,N)\times Q(\ell,t,\infty-N)} f. \tag{28}$$

Example. Since $D^{-s} \sum sgn\,\lambda_n |\lambda_n|^{-s} x_n = \Lambda^{\infty-\{n\}} dx_n$, as an ($\infty$-1)-form, we have

$$\int_{Q(\ell,t;\, x_n=\pm|t\lambda_n|^\ell)} D^{-s}$$
$$= \int_{Q(\ell,t;\, x_n=\pm|t\lambda_n|^\ell)} sgn\,\lambda_n |\lambda_n|^{-s} (\pm |t\lambda_n|^\ell) \Lambda^{\infty-\{n\}} dx_n$$
$$= \pm(-1)^{n-1} sgn\,\lambda_N |\lambda_N|^{-s}\, |t\lambda_n|^\ell\, |2t\lambda_n|^{-\ell}\, vol(Q(\ell,\ t)).$$

Hence we get

$$\int_{\partial Q(\ell,t)} D^{-s} = \eta_D(s) vol(Q(\ell, t)) = (2t)^{\ell n^-} \eta_D(s) (det\,|D|)^\ell. \tag{29}$$

Similarly, we have

(29)' $$\int_{\partial Q(\ell,t)} |D|^{-s} = (2t)^{\ell n^-} \zeta_{|D|}(s)(det\,|D|)^{\ell}.$$

Note. (29) and (29)' show that both I $(=\sum x_n \Lambda^{\infty-\{x\}} dx_n)$ and ϵ $(=\sum sgn\lambda_n x_n \Lambda^{\infty-\{x\}} dx_n)$ are renormalized integrable on $\partial Q(\ell, t)$. Their values are given by

(30) $$\int_{\partial Q(\ell,t)} \epsilon = (2t)^{\ell n^-} \eta_D(0)(det\,|D|)^{\ell},$$

$$\int_{\partial Q(\ell,t)} I = (2t)^{\ell n^-} n^-(det\,|D|)^{\ell}.$$

6. Stokes' Theorem

Let f be an exterior differentiable $(\infty-p)$-form with the coordinate expression $\sum_J f_J \Lambda^{\infty-J} dx_n$. Then we have

$$df = \sum_K \Big(\sum_{i \notin K} sgn\{i, \boldsymbol{K}\} \partial f_{\{i,K\}}/\partial x_i \Big) \Lambda^{\infty-K} dx_n, \ \boldsymbol{K} = \{k_1, \ldots, k_{p-1}\},$$

$$sgn\{i, \boldsymbol{K}\} = 1, \ i < k, \ sgn\{i, \boldsymbol{K}\} = (-1)^q, \ k_q < i < k_{q+1},$$

$$sgn\{i, \boldsymbol{K}\} = (-1)_p, \ i > k_{p-1}.$$

Under these notations, we set

$$d^N f = \sum_K \Big(\sum_{i \notin K, i \leq N} sgn\{i, \boldsymbol{K}\} \partial f_{\{i,K\}}/\partial x_i \Big) \Lambda^{\infty-K} dx_n.$$

Then as an element of $\Lambda^{p-1} W^k(X)$, we have

(31) $$\lim_{N\to\infty} d^N f(x) = df(x) \ \ x \in Q(\ell, \ t, \ \infty - p + 1), \text{ if } \ell > -(k - d/2).$$

This convergence is uniform if df is continuous by the topology of $W^{k-\alpha}(X)$, $\alpha > 0$.

Let f be an $(\infty$-1)-form. We set

$$f^{-N}(x) = \sum \sup_{y \in Q(\ell,t,\infty-N)} f_i(x, y) \Lambda^{\infty-\{i\}} dx_n,$$

$$f_{-N}(x) = \sum \inf_{y \in Q(\ell,t,\infty-N)} f_i(x, y) \Lambda^{\infty-\{i\}} dx_n, \quad x \in Q(\ell, \ t, \ N).$$

By definitions, we have

$$df^{-N} = d^N f^{-N}, \ df_{-N} = d^N f_{-N},$$

$$\int_{Q(\ell,t;\, x_i = |t\lambda_i|^{\ell})} f^{-N} = \int_{Q(\ell,t;\, x_i = |t\lambda_i|^{\ell})} f^{-N}$$

$$\int_{Q(\ell,t;\, x_i = |t\lambda_i|^{\ell})} f_{-N} = \int_{Q(\ell,t;\, x_i = |t\lambda_i|^{\ell})} f_{-N}, \quad i > N+1.$$

Therefore we obtain

(32) $$\int_{\partial Q(\ell,t)} f^{-N}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \Big(\int_{Q(\ell,t;\, x_i = |t\lambda_i|^{\ell})} f^{-N} - \int_{Q(\ell,t;\, x_i = |t\lambda_i|^{\ell})} f^{-N} \Big)$$

$$= \int_{\partial Q(\ell,t,N)} \times Q(\ell, t, \infty - N) f^{-N} = \int_{Q(\ell,t)} df^{-N}$$

$$\int_{\partial Q(\ell,t)} f_{-N} = \int_{\partial Q(\ell,t,N) \times Q(\ell,t,\infty-N)} f_{-N} = \int_{Q(\ell,t)} df_{-N}.$$

On the other hand, if f and df both continuous by the topology of $W^{-\ell-\alpha}(X)$, $\alpha > d/2$, we have

$$\lim_{n\to\infty}\int_{\partial Q(\ell.t)} f^{-N} = \int_{\partial Q(\ell.t)} f,$$
$$\lim_{n\to\infty}\int_{Q(\ell.t)} df^{-N} = \int_{Q(\ell.t)} df.$$

Thereforewe obtain

Theorem 3. *Let f be an exterior differentiable $(\infty-1)$-form such that f and df both continuous by the topology of $W^{-\ell-\alpha}(X)$, $\alpha > d/2$, and $d^{\wedge}f$ is continuous by the topology of $W^{-\alpha}(X)$, $\alpha > d/2$. Then we have*

$$\int_{Q(\ell.t)} df = \int_{\partial Q(\ell.t)} f\left(=\lim_{n\to\infty}\int_{\partial Q(\ell.t,N)\times Q(\ell,t,\infty-N)} f\right). \tag{33}$$

Example. Since dD^{-s} is $\eta_D(s)\Lambda^{\infty}dx_n$ and $d|D|^{-s}=\xi_{|D|}(s)\Lambda^{\infty}dx_n$ by (14), we get

$$\int_{Q(\ell.t)} dD^{-s} = \eta_D(s)vol(Q(\ell, t)),$$
$$\int_{Q(\ell.t)} d|D|^{-s} = \xi_{|D|}(s)vol(Q(\ell, t)).$$

These values coincide to (30).

In general, $d^m f$ is not equal to 0 if f is an $(\infty-p)$-form and $m \leqq p$. So we want to compute $\int_{Q(\ell.t)} d^m f$ for an $(\infty-m)$-form f on some neighborhood of $Q(\ell, t)$. We assume the followings:

(i) $f, df, \ldots, d^m f$ are all continuous by the topology of $W^{-\ell-\alpha}(X)$, $\alpha > d/2$.

(ii) $d^{\wedge}f, d^{\wedge}df, \ldots, d^{\wedge}d^{m-1}f$ are all continuous by the topology of $W^{-\alpha}(X)$, $\alpha > d/2$.

Then, since we have (formally)

$$\partial(\partial Q(\ell, t, N)\times Q(\ell, t, \infty-N)) = (-1)^{N-1}\partial Q(\ell, t, N)\times\partial Q(\ell, t, \infty-N), \ldots,$$
$$\partial(\partial Q(\ell, t, N_1)\times\cdots\times\partial Q(\ell, t, N_{m-1}))\times Q(\ell, t, \infty-(N_1+\cdots+N_{m-1}))$$
$$= sgn(N_1,\ldots,N_{m-1})\partial Q(\ell, t, N_1)\times\cdots\times\partial Q(\ell, t, N_{m-1})\times$$
$$\times\,\partial Q(\ell, t, \infty-(N_1+\cdots+N_{m-1})),$$

we get

$$\int_{Q(\ell.t)} d^m f = \lim_{N_1\to\infty,\cdots,N_m\to\infty} sgn(N_1,\ldots,N_{m-1})\times \tag{34}$$
$$\times\int_{\partial Q(\ell.t,N_1)\times\cdots\times\partial Q(\ell.t,N_m)\times Q(\ell,t,\infty-(N_1+\cdots+N_m))} f,$$
$$sgn(N_1,\ldots,N_{m-1}) = (-1)^{N_2+N_4+\cdots+N_{m-1}}, \quad m\equiv 1,\ mod.\ 4,$$
$$= (-1)^{N_1+N_3+\cdots+N_{m-1}-1}, \quad m\equiv 1,\ mod.\ 4,$$
$$= (-1)^{N_2+N_4+\cdots+N_{m-1}-1}, \quad m\equiv 3,\ mod.\ 4,$$
$$= (-1)^{N_1+N_3+\cdots+N_{m-1}}, \quad m\equiv 0,\ mod.\ 4.$$

Here $Q(\ell, t, N_k)$ means $\{\sum_{n=N_1+\cdots+N_{k-1}+1}^{n=N_1+\cdots+N_k} c_n e_n \mid -|t\lambda_n|^{\ell} < c_n < |t\lambda_n|^{\ell}\}$. For simple, we set

$$\lim_{N_1\to\infty\cdots N_m\to\infty} sgn(N_1,\ldots,N_{m-1})\times$$

$$\times \int_{\partial Q(\ell.t,N_1)X\partial Q(\ell.t,N_2)X\cdots XQ(\ell.t,N_m)XQ(\ell,t,\infty-(N_1+\cdots+N_m))} f$$

$$= \int_{\partial^m Q(\ell,t)} f.$$

Then by (34), we obtain

Theorem 4. *Let* f *be an* $(\infty-m)$*-form on a neighborhood of* $Q(\ell, t)$ *stisfying the assumptions* (i) *and* (ii). *Then we have*

$$\int_{Q(\ell,t)} d^m f = \int_{\partial^m Q(\ell,t)} f. \tag{35}$$

Note. Formally, we may write

$$\partial Q(\ell, t\,;x_n=\pm|t\lambda_n|^\ell)=\sum_{i=1}^{n-1}(-1)^{i-1}(Q(\ell, t\,; x_i=|t\lambda_n|^\ell, x_n=\pm|t\lambda_n|^\ell)-$$
$$-Q(\ell, t\,; x_i=-|t\lambda_n|^\ell, x_n=\pm|t\lambda_n|^\ell))+$$
$$+\sum_{i=n+1}^{\infty}(-1)^i(Q(\ell, t\,; x_n=\pm|t\lambda_n|^\ell, x_i=|t\lambda_i|^\ell)-$$
$$-Q(\ell, t\,; x_n=\pm|t\lambda_n|^\ell, x_i=-|t\lambda_i|^\ell)).$$

Then, formally, we get

$$\partial^2 Q(\ell, t)\sum_{n=1}^{\infty}(-1)^{n-1}\Big\{\sum_{i=1}^{n-1}(-1)^{i-1}(Q(\ell, t\,; x_i=|t\lambda_i|^\ell, x_n=|t\lambda_n|^\ell)-$$
$$-Q(\ell, t\,; x_i=-|t\lambda_i|^\ell, x_n=|t\lambda_n|^\ell))+$$
$$+\sum_{i=1}^{n-1}(-1)^i(Q(\ell, t\,; x_i=|t\lambda_i|^\ell, x_n=-|t\lambda_n|^\ell)-$$
$$-Q(\ell, t\,; x_i=-|t\lambda_i|^\ell, x_n=-|t\lambda_n|^\ell))+$$
$$+\sum_{i=n+1}^{\infty}(-1)^i(Q(\ell, t\,; x_n=|t\lambda_n|^\ell, x_i=|t\lambda_i|^\ell)-$$
$$-Q(\ell, t\,; x_n=|t\lambda_n|^\ell, x_i=-|t\lambda_i|^\ell))+$$
$$+\sum_{i=n+1}^{\infty}(-1)^{i+1}(Q(\ell, t\,; x_n=-|t\lambda_n|^\ell, x_i=|t\lambda_i|^\ell)-$$
$$-Q(\ell, t\,; x_n=-|t\lambda_n|^\ell, x_i=-|t\lambda_i|^\ell))\Big\}$$

This expression is formal and we can not change the order of summation, because they are infinite sums. So we can not conclude $\partial^2 Q(\ell, t)$ is equal to 0.

Theorems 2 and 4 show integrals on $Q(\ell, t)$ may be reduced to the integrals on $Q(\ell, t, \infty-N)$, N is arbitrary large, but finite.

Note. At this stage, we still lack good theory of infinite dimensional singular chains. To get such theory and apply above results on integrals of $(\infty-p)$-forms will be a future problem.

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