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A Memoir on the Spatially Spherosymmetric Solution for a Nonlinear Parabolic Equation

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Abstract : The paper discusses the temporal behavior of the spatially spherosymmetric solution for a nonlinear parabolic equation, which is a sense related with that of the spatially spherosymmetric solution for the 3-dimentional commpressible Burgers' equation.

As for the notation used below, it is conventional, $H^{2+\alpha}(\bar{\Omega})$, $H^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)$, etc., denoting Hölder spaces.

1 Introduction

The partial differential equation to be considered is:

$$\frac{\partial}{\partial t}\psi(x,t)=\phi(x,t)^{-2}\mu\Delta_n\psi(x,t)+\psi(x,t)^2 \quad (|x|\leq l(>0), t\geq 0) \tag{1.1}$$

$$\left(x=(x_1,\cdots.x_n),\ \Delta_n=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right),$$

$$\psi(x,0)=\psi_0(x)\ (\in H^{2+\alpha}(\bar{\Omega}),\ \Omega=\{x|\ |x|\leq l\}, 0<\alpha<1), \tag{1.2}$$

$$\psi(x,t)|_{|x|=l}=0(t\geq 0),\ \text{(accompanied by compatibility conditions)}, \tag{1.3}$$

where μ is a positive constant and $\phi(x,t)$ is defined by

$$\phi(x,t)=\exp\left\{\int_0^t\psi(x,\tau)d\tau\right\}. \tag{1.4}$$

Now, without proof, we give two theorems.

Theorem 1 (Temporally local existence). *For some* $T\in(0,\infty)$, *there exists a unique solution* $\psi(x,t)$ *for* (1.1)-(1.2)-(1.3) *belonging to* $H^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)(\Omega_T=\Omega\times[0,T])$. [*We note that, if* $\psi_0(x)\geq 0$, *then* $\psi(x,t)\geq 0$.]

Theorem 2 (Spatial spherosymmetry). *If* $\psi(x,t)\in H^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)$ $(0<T<\infty)$ *satisfies* (1.1)-(1.2)-(1.3) *with* $\psi_0(x)=\tilde{\psi}_0(|x|)$, *then* $\psi(x,t)$ *has a form* $\tilde{\psi}(|x|,t)$,

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where $\tilde{\psi}_0(r)$ *depends only on* r *and* $\tilde{\psi}(r, t)$ *does only on* r *and* t. *Moreover,* $\tilde{\psi}(r, t)$ *satisfies*

$$\frac{\partial}{\partial t}\tilde{\psi}(r, t) = \tilde{\phi}(r, t)^{-2}\mu\left[\frac{\partial^2}{\partial r^2}\tilde{\psi}(r, t) + \frac{n-1}{r}\frac{\partial}{\partial r}\tilde{\psi}(r, t)\right] + \tilde{\psi}(r, t)^2 \quad (1.5)$$

$$(0 \le r \le l, 0 \le t \le T), \quad \left(\frac{\tilde{\psi}_r}{r}\bigg|_{r=0} \equiv \tilde{\psi}_{rr}(0, t)\right),$$

$$\tilde{\psi}(r, 0) = \tilde{\psi}_0(r), \quad \tilde{\psi}_r(0, t) = \tilde{\psi}(l, t) = 0, \quad (1.6)$$

(*together with compatibility conditions*). ($\tilde{\phi}(r, t) = \exp[\int_0^t \tilde{\psi}(r, \tau)d\tau]$.) [*In the discussion of the* 3-*dimentional compressible Burgers' equation, the case* $n=5$ *matters.*]

2 Blow-up result

In this section, we show that the solution $\psi(x, t)$ for (1.1)-(1.2)-(1.3) with

$$\psi(x, 0) = \tilde{\psi}_0(|x|) \ (\ge 0, \not\equiv 0) \quad (2.1)$$

blows up in a finite time under a certain condition on $\tilde{\psi}_0(r)$.

First, we define $J^{(n)}(r)$ $(n = 1, 2, \cdots)$ by

$$J^{(n)}(r) = C_n r^{\frac{n}{2}} Z_{\frac{n}{2}-1}(\beta_n r), \quad (2.2)$$

$$\left(C_n(>0), \text{ const. such that } \int_0^l J^{(n)}(r)dr = 1\right),$$

where $Z_{\frac{n}{2}-1}(s)$ is the Bessel function of order $\frac{n}{2}-1$ for each n and $\beta_n l$ is the first zero-point of $Z_{\frac{n}{2}-1}(s)$ $(s>0)$. For example, $J^{(1)}(r) = C_1\cos\frac{\pi}{2l}r$, $J^{(2)}(r) = C_2 r J_0(\beta_2 r)$ $(J_0(\beta_2 l) = 0)$, etc. Next, let $\psi(x, t)$ belong to $H^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}_T)$ for some $T \in (0, \infty)$ and satisfy (1.1)-(1.2)-(2.1). Then, by theorem 2, $\psi(x, t)$ is expressed as $\psi(x, t) = \tilde{\psi}(|x|, t)$. Moreover, $\tilde{\psi}(r, t)$ $(r=|x|)$ satisfies (1.5)-(1.6). Multiplying both sides of (1.5) by $\tilde{\phi}(r, t)^2 \cdot J^{(n)}(r)$ and integrating them in r over $[0, l]$, we have easily,

$$\int_0^l \tilde{\phi}^2\frac{\partial}{\partial t}\tilde{\psi}(r, t)J^{(n)}(r)dr = -\mu\beta_n\int_0^l \tilde{\psi}(r, t)J^{(n)}(r)dr + \int_0^l \tilde{\phi}^2\tilde{\psi}(r, t)^2 J^{(n)}(r)dr, \quad (2.3)$$

$$\left(\text{N.B.}: \frac{d^2}{dr^2}J^{(n)}(r) - (n-1)\frac{d}{dr}\left(\frac{J^{(n)}(r)}{r}\right) = -\beta_n J^{(n)}(r)\right).$$

By the strength of the equality $\tilde{\phi}_t = \tilde{\phi}\tilde{\psi}$, it is seen that

$$\int_0^l \tilde{\phi}^2\frac{\partial}{\partial t}\tilde{\psi}(r, t)J^{(n)}(r)dr = \frac{d}{dt}\int_0^l \tilde{\phi}^2\tilde{\psi}(r, t)J^{(n)}(r)dr - 2\int_0^l \tilde{\phi}^2\tilde{\psi}(r, t)^2 J^{(n)}(r)dr. \quad (2.4)$$

Thus, it holds that

$$\frac{d}{dt}\int_0^l \tilde{\phi}^2\tilde{\psi}(r, t)J^{(n)}(r)dr = -\mu\beta_n\int_0^l \tilde{\psi}(r, t)J^{(n)}(r)dr + 3\int_0^l \tilde{\phi}^2\tilde{\psi}(r, t)^2 J^{(n)}(r)dr$$

$$\ge -\mu\beta_n\int_0^l \tilde{\phi}^2\tilde{\psi}(r, t)J^{(n)}(r)dr$$

$$+3\frac{\{\int_0^l \tilde{\phi}^2\tilde{\psi}(r,t)J^{(n)}(r)dr\}^2}{\int_0^l \tilde{\phi}^2 J^{(n)}(r)}, \tag{2.5}$$

$$(\text{N.B.: } \tilde{\phi}^2\tilde{\psi}J^{(n)} = \tilde{\phi}(J^{(n)})^{\frac{1}{2}}\tilde{\phi}\tilde{\psi}(J^{(n)})^{\frac{1}{2}},\ \tilde{\phi}\geq 1,\ J^{(n)}\geq 0\,(\not\equiv 0)).$$

Let $y_n(t)$ denote $\int_0^l \tilde{\phi}^2\tilde{\psi}J^{(n)}dr$ (>0). Then from (2.5) follows an inequality,

$$\frac{d}{dt}y_n(t) \geq -\mu\beta_n y_n(t) + 3y_n(t)^2\left(2\int_0^t y_n(\tau)\,d\tau+1\right)^{-1}, \tag{2.6}$$

$$\left(\text{N.B.: }\frac{d}{dt}\int_0^l \tilde{\phi}^2 J^{(n)}dr = \int 2\tilde{\phi}^2\tilde{\psi}J^{(n)}dr = 2y_n,\ \int_0^l \tilde{\phi}(r,0)^2 J^{(n)}dr = 1\right).$$

Hereafter, we write $y_n(t)$, $J^{(n)}(r)$ and β_n simply $y(t)$, $J(r)$ and β, respectively. Defining $Q(t)$ by

$$Q(t) = 2\int_0^t y(\tau)\,d\tau + 1 \ (\geq 1), \tag{2.7}$$

we rewrite (2.6) in the following way:

$$\frac{Q}{2}\frac{d}{dt}\dot{Q} \geq -\mu\beta\dot{Q}\frac{Q}{2} + \frac{3}{4}\dot{Q}^2 \left(\dot{Q} = \frac{d}{dt}Q\right), \tag{2.8}$$

$$Q(0) = 1,\ \dot{Q}(0) = 2y(0)(>0) \tag{2.9}$$

According to the relation $\frac{d}{dt}\dot{Q} = \dot{Q}\frac{d}{dQ}\dot{Q}$ (N.B.: $\dot{Q} = 2y > 0$), from (2.8) we have an inequality

$$\frac{d}{dQ}\dot{Q} \geq -\nu + \frac{3}{2}\frac{\dot{Q}}{Q} \quad (\nu = \mu\beta), \tag{2.10}$$

or, what is the same,

$$\frac{d}{dQ}\dot{Q} - \frac{3}{2}\frac{\dot{Q}}{Q} \geq -\nu. \tag{2.11}$$

Hence,

$$Q^{-\frac{3}{2}}\dot{Q} - Q(0)^{-\frac{3}{2}}\dot{Q}(0) = Q^{-\frac{3}{2}}\dot{Q} - 2y(0) \geq -\nu\int_{Q(0)=1}^{Q} Q^{-\frac{3}{2}}dQ = 2\nu(Q^{-\frac{1}{2}}-1). \tag{2.12}$$

Thus, we have

$$\dot{Q} \geq (2y(0)-2\nu)\,Q^{\frac{3}{2}} + 2\nu Q > (2y(0)-2\nu)\,Q^{\frac{3}{2}}, \tag{2.13}$$

$$Q(0) = 1, \tag{2.14}$$

which leads us to the assertion that, if $y(0)$ satisfies

$$y(0)\left(=\int_0^l \tilde{\psi}_0(r)J(r)\,dr\right) > \nu = \mu\beta, \tag{2.15}$$

then

$$Q(t) > \frac{1}{\{1-(y(0)-\nu)\,t\}^2}\ \left(0 \leq t < \frac{1}{y(0)-\nu}\right). \tag{2.16}$$

From the discussion made above, we have:

Theorem 3. *The solution* $\psi(x, t)$ *for* (1.1)-(1.2)-(1.3)-(2.1) *blows up in a finite time under the condition* (2.15).

3 Global existence

Here, we give a theorem which asserts the temporally global existence of a unique solution for our problem (1.1)-(1.2)-(1.3)-(2.1) under an additional condition on $\tilde{\psi}_0(r)$. (In the case of $\tilde{\psi}_0(r) \leq 0$), the existence of such a solution is obvious.)

Theorem 4. *There exists a unique temporally global solution* $\psi(x, t)$ *of the problem* (1.1)-(1.2)-(1.3)-(2.1) *belonging to* $H^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}_T)$ *for an arbirary* $T \in (0,\infty)$ *under the condition on* $\tilde{\psi}_0(r)$,

$$|\tilde{\psi}_0|_I^{(0)} = |\psi_0|_{\Omega}^{(0)} < \frac{\nu n}{6l^2} (= B_0), \ (I = [0, l]). \tag{3.1}$$

Moreover, it holds that

$$|\tilde{\psi}(\cdot, t)|_I^{(0)} = |\psi(\cdot, t)|_{\Omega}^{(0)} \leq \frac{2B_0}{1 + 4B_0 t}, \quad (t \geq 0). \tag{3.2}$$

Outline of the proof. Let $\psi(x, t)$ satisfy (1.1)-(1.2)-(1.3)-(2.1)-(3.1) in $\bar{\Omega}_T$ for some $T \in (0,\infty)$. (N.B.: ψ has the form $\psi(x, t) = \tilde{\psi}(|x|, t)$.) Now, we define $w(x,t;a,k)$ (or, simply, $w(x,t)$) by

$$w(x,t;a,k) = (1 + ak\mu t)\,\psi(x, t) = (1 + ak\mu t)\,\tilde{\psi}(|x|, t) \tag{3.3}$$

(a and k, positive constants).

Then $w(x,t)$ satisfies

$$w_t(x,t) = \tilde{\phi}(|x|,t)^{-2}\mu\Delta_n w(x,t) + \frac{w(x,t)^2 + ak\mu w(x,t)}{1 + ak\mu t}, \tag{3.4}$$

$$w(x,0) = \psi(x, 0) = \tilde{\psi}_0(|x|), \tag{3.5}$$

$$w(x,t)|_{|x|=l} = \psi(x, t)|_{|x|=l} = \tilde{\psi}(l,t) = 0 \ (0 \leq t \leq T). \tag{3.6}$$

Moreover, $\hat{w}(x,t)$ defined by

$$\hat{w}(x,t) = w(x,t) + \frac{k}{2n}|x|^2$$

satisfies

$$\hat{w}_t (= w_t) = \tilde{\phi}^{-2}\Delta_n \hat{w} + \left\{\frac{w^2 + ak\mu w}{1 + ak\mu t} - k\mu\tilde{\phi}^{-2}\right\}_A, \tag{3.7}$$

$$\hat{w}(x,0) = \tilde{\psi}_0(|x|) + \frac{k}{2n}|x|^2, \quad \hat{w}(x,t)_{|x|=l} = \frac{k}{2n}l^2. \tag{3.8}$$

After a somewhat lengthy calculation concerning (3.7)-(3.8), on the basis of the maximum principle, we obtain a sufficient condition on $|\tilde{\psi}_0|_I^{(0)}$ under which $|\hat{w}|_{\Omega_T}^{(0)}$ is

bounded from above by known quantities and the term $\{\cdots\}_A$ is smaller than 0. As a final result, we have our assertion.
Q.E.D

4 Concluding remark

The blowup-nonblowup problem of the spatially spherosymmetric solution for the 3-dimentional compressible Burgers' equation

$$v_t(x,t) = \frac{\mu}{\rho(x,t)}\left(\Delta + \frac{1}{3}\nabla\cdot \mathrm{div}\right)v(x,t) - (v\cdot\nabla)v(x,t), \tag{4.1}$$

$$\rho_t(x,t) + \mathrm{div}(\rho(x,t)v(x,t)) = 0 \tag{4.2}$$

is closely related to our problem (1.1)-(1.2)-(1.3)-(2.1) ($n=5$) in a technical sense, although the former is more complicated and more difficult than the latter. Our discussion above will be useful in treating the former problem, whose settlement consists in estimating $\rho(x,t) = \tilde{\rho}(|x|,t)$ in an *a priori* way.

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