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Volume formula of compact simple Lie groups and values of τ

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This paper is a correction of the volume formula of compact simple Lie groups given by Freudenthal [2]. For a compact center free simple Lie group G, Freudenthal [2] p.202 gives the volume formula of G with respect to the metric induced by the Killing form as

$$\mu_0(G) = \frac{k\, 2^r\, m!\, D^{-1/2}}{(\ell!\, \prod_{i=1}^{\ell} q_i)\, \tau}\pi^{\ell+m} = \frac{c\, 2^r\, m!\, D^{-1/2}}{\tau}\pi^{\ell+m}$$

where k=order of Weyl group, r=dimension, ℓ=rank, $m=\frac{1}{2}(r-\ell)$, $D=\det((\rho_i, \rho_j))$ ($\rho_1, \cdots, \rho_\ell$ are simple roots), $q_1, \cdots, q_\ell$ are the coefficients of the maximal root and c=order of the center of the universal covering group $\tilde{G}$ of G.(note that $k=c\,\ell!\times\prod_{i=1}^{\ell} q_i$). The value of τ is given by

$$\tau=(-1)^m \sum(\alpha_{i_1}, \alpha_{i_2})(\alpha_{i_3}, \alpha_{i_4})\cdots(\alpha_{i_{2m-1}}, \alpha_{i_{2m}})$$

where the summation runs over all permutations of the $2m$ roots $\alpha_1, \cdots, \alpha_{2m}$. To calculate the volume of a compact simple Lie group G, we need to know the value of τ. However it is hard to compute τ directly because it involves the summation over the symmetric group (even if we use computers in the case $\ell \geqq 4$).

Now, we shall correct the Freudenthal's volume formula, that is, in the proof of Freudenthal-de Vries [2], p.200, 1.2, $\mu(G)=kv(\boldsymbol{D})=kv'(\boldsymbol{D})$ shoud be change to

$$\mu(G)=k\,\ell!\,\prod q_i\, v(\boldsymbol{D})=k\,\ell!\,\prod q_i\, v'(\boldsymbol{D}).$$

(In [2], the volume of Weyl cell should be taken for that of a paralleltope. (The ratio of their volumes is $1 : \prod_{i=1}^{\ell} q_i$)). Hence the volume $\mu_0(G)$ of G must be $\mu_0(G)=\frac{k\, 2^r\, m!\, D^{-1/2}}{\tau}\pi^{\ell+m}$. Thus we have

THEOREM 1. *The natural volume $\mu_0(\tilde{G})$ of a simply connected compact simple Lie group $\tilde{G}$ is*

$$\mu_0(\tilde{G}) = \frac{c\, k\, 2^r m!\, D^{-1/2}}{\tau}\pi^{\ell+m}.$$

According to Abe-Yokota [1], the volumes of simply connected compact simple

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Lie groups are calculated, hence we can determine the value of τ from Theorem 1.

THEOREM 2. *The values* $\tau(\tilde{G})$ *of simply connected compact simple Lie groups* $\tilde{G}$ *are given as follows.*

$$\tau(SU(n)) = \frac{(n(n-1)/2)!1!2!\cdots n!}{n^{n(n-1)/2}},$$

$$\tau(Spin(2n+1)) = \frac{1!3!\cdots(2n-1)!\ n!\ (n^2)!}{(2n-1)^{n^2}},$$

$$\tau(Sp(n)) = \frac{1!3!\cdots(2n-1)!\ n!(n^2)!}{2^{n(n-2)}\ (n+1)^{n^2}},$$

$$\tau(Spin(2n)) = \frac{1!3!\cdots(2n-3)!\ (n-1)!\ n!(n(n-1))!}{2^{(n-1)^2}\ (n-1)^{n(n-1)}} \quad (n \geqq 2),$$

$$\tau(G_2) = \frac{3\ 5^2}{2^3}, \quad \tau(F_4) = \frac{2^{10}\ 5^4\ 7^2\ 11}{3^{39}}24!, \quad \tau(E_6) = \frac{5^6\ 7^3\ 11}{2^{40}\ 3^{22}}36!$$

$$\tau(E_7) = \frac{5^{11}\ 7^7\ 11^3\ 13^2\ 17}{2^6\ 3^{100}}63!, \quad \tau(E_8) = \frac{7^{15}\ 11^8\ 13^6\ 17^4\ 19^3\ 23^2\ 29}{2^9\ 3^{68}\ 5^{97}}120!.$$

Proof. It follows from the following table using Theorem 1.

$$\mu_0(SU(n)) = \frac{2^{(n-1)(2n+3)/2}\ n^{n^2/2}}{1!2!\cdots(n-1)!}\pi^{(n-1)(n+2)/2},$$

$$\mu_0(Spin(2n+1)) = \frac{2\ 2^{n(4n+5)/2}\ (2n-1)^{n(2n+1)/2}}{1!3!\cdots(2n-1)!}\pi^{n(n+1)},$$

$$\mu_0(Sp(n)) = \frac{2^{n(3n+1)}\ (n+1)^{n(2n+1)/2}}{1!3!\cdots(2n-1)!}\pi^{n(n+1)},$$

$$\mu_0(Spin(2n)) = \frac{2\ 2^{n(3n-1)}\ (n-1)^{n(2n-1)/2}}{1!3!\cdots(2n-3)!(n-1)!}\pi^{n^2},$$

$$\mu_0(G_2) = \frac{2^{26}\ 3^2\ \sqrt{3}}{5}\pi^8, \quad \mu_0(F_4) = \frac{2^{52}\ 3^{45}}{5^4\ 7^2\ 11}\pi^{28}, \quad \mu_0(E_6) = \frac{2^{134}\ 3^{29}\ \sqrt{3}}{5^5\ 7^3\ 11}\pi^{42},$$

$$\mu_0(E_7) = \frac{2^{156}\ \sqrt{2}\ 3^{111}}{5^{10}\ 7^6\ 11^3\ 13^2\ 17}\pi^{70}, \quad \mu_0(E_8) = \frac{2^{279}\ 3^{77}\ 5^{103}}{7^{14}\ 11^8\ 13^6\ 17^4\ 19^3\ 23^2\ 29}\pi^{128}.$$

References

[1] K. Abe and I. Yokota, *Volumes of compact symmetric spaces*, to appear.

[2] H. Freudenthal and H. de Vries, Linear Lie groups, Academic Press, New York, 1969.