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L^2-stability Theory of Linear Difference Schemes with variable Coefficients

by

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1. Introduction.

We shall treat here the stability problem for the difference scheme

$$(1.1) \qquad u(t+k,\ x)=S_h u(t,\ x),$$

approximating the Cauchy problem

$$(1.2) \qquad D_t u=p(X,\ D)u \ \left(D_t=-i\frac{\partial}{\partial t},\ D=\left(-i\frac{\partial}{\partial x_1},\cdots\cdots,-i\frac{\partial}{\partial x_n}\right)\right)$$
$$\text{with } u(o,\ x)=u_o(\in L^2(R^n_x)).$$

In the finite difference schemes the amplitude matrix is the symbol $\sigma(S_h)=S(x,\ \omega)_{\omega=h\xi}$ of the operator S_h, the elements of which are trigonometric polynomials in $\omega=h\xi$. The von Neumann condition is the statement that the modulus of eigenvalues of $S(x,\ \omega)$ do not exeed 1 and, as is well known, this condition is necessary for the stability of the scheme. Many schemes used in practice of numerical analysis are stable under the von Neumann condition, while there is an instable scheme satisfying the von Neumann condition. Then, it is quite natural to ask what class of schemes are stable only under the von Neumann condition.

The aim of this paper is to give an answer for the above question, that is to state that the schemes of Kreiss's class (defined in the beginning of the section 3) having the property (*) are L^2-stable. These schemes satisfy necessarily the von Neumann condition. The property (*) is given in the section 3 and we shall give comments on it in relation to the so called Courant-Friedrichs-Lewy condition in the section 4. From the observation there we can see that the property (*) is not restrictive for schemes approximating a hyperbolic system. Uniformly diagonalizable schemes approximating a hyperbolic system belong to this class.

The stability of the schemes of Kreiss's class with constant coefficients and some criterion were completely discussed by H. O. Kreiss ([7]). Our stability statement is a prolongation of Kreiss's to the variable coefficients case.

The proof of the above statement will be done in the algebra $\{S^0_{\lambda_h}\}$ of pseudo-differential operators with a special basic weight function $\lambda_h(\xi)$. As for definitions of $\lambda_h(\xi)$ and $\{S^m_{\lambda_h}\}$, see the section 2 of this paper or § 2 of [5] which origioate from Kumano-go [9], [10]. The study of stability often meets the problen of decomposition of a non-negative symbol into a finite sum of squares ([12], [17]), but this problem can be passed over by means of the Friedrichs part of the operator with non-negative symbol. The Friedrichs part gives us the symmetrization of operator P_h with Hermitian symmetric symbol (see Remark 2.5 - Proposition 2.7) as well as an approximation to P_h. It should be noted that the approximation is an O (h)-approximation in our difference calculus. By virtue of these properties of the Friedrichs part the stability discussion here is more direct and simpler than in § 5 of [5]. Especially, the stability of the Lax-Wendroff scheme with Hermitian symmetric coefficients (already known in the paper [13]) can be derived within our method of treatment. The idea of dissipation formerly discussed by several authors ([8], [14]) is unnecessary for the linear schemes with C^∞-coefficients.

One of the algebraic criterion as was discussed by H. O. Kreiss, will be in the variable coefficients case the assertion that the inequality

$$|S(x, \omega)^j| \leqq C$$

holds where C is independent of x, ω and j. For finite difference schemes this is a necessary condition for the L^2-stability ([15], [16] In the variable coefficients case the problen whether the L^2-stability of S_h can be reduced to such a simple criterion or not, remains open. Nevertheless, our discussion shows that new schemes (not necessarily diagonalizable) may be stable. we propose such an example in the section 5. we can see leastways that the stability of many classical schemes known hitherto can be derived by the unified treatment developped here (See also Remark 3. 7.)

For clarification of our method we treat here the L^2-stability of linear schemes with C^∞-coefficients, however, one can develop it to the C^2-coefficients case through the approximation theory due to H. Kumano-go and M. Nagase ([11]).

2. Preliminaries.

Notations used here are same as in [6].

In the following we give a brief survey concerning on families of pseudo-differential operators with a parameter h $(0<h<1)$. Let $\alpha=(\alpha_1, \cdots\cdots, \alpha_n)$ be a multi-integer of $\alpha_j \geqq 0$. we put $|\alpha|=\alpha_1+\cdots\cdots+\alpha_n$, $\alpha!=\alpha_1!\cdots\cdots\alpha_n!$ and $(\partial/\partial\xi)^\alpha=(\partial/\partial\xi)^{\alpha_1}\cdots\cdots(\partial/\partial\xi_n)^{\alpha_n}$.

Definition 2. 1. A family $\lambda_h(\xi)$ of real valued C^∞-functions defined on R^n_ξ is called a basic weight function, when there exist positive constants A_0, A_α (independent of h) such that

i) $1 \leqq \lambda_h(\xi) \leqq A_o<\xi>$

(2.1) and

ii) $|\lambda_h^{(\alpha)}(\xi)| \leqq A_\alpha \lambda_h(\xi)^{1-|\alpha|}$

for any α, where $<\xi> = (1+|\xi|^2)^{1/2}$ and $\lambda^{(\alpha)}{}_h(\xi) = (\partial/\partial\xi)^\alpha \lambda^h(\xi)$.
Here and hereafter we shall use the special basic weight function defined by $\lambda_h(\xi) = <\varsigma_h>$, where $\varsigma_h(\xi) = (h^{-1}\sin h\xi_1, \cdots\cdots, h^{-1}\sin h\xi_n)$.
It is easily verified that

(2.2) $\quad h \leqq (n+1)^{1/2}\lambda_h(\xi)^{-1}.$

For applications we shall often use $\tilde{\lambda}_h(\xi) = \lambda_{h/2}(\xi)$. But once a scheme is given, the basic weight function is fixed.

Definition 2. 2. i) A family of C^∞-symbols $p_h(x, \xi)$ in $R^n_x \times R^n_\xi$ is called of class $\{S^m_{\lambda_h}\}$ $(-\infty<m<\infty)$, when there exist constants $C_{\alpha,\beta}$ (independent of h) such that

(2.3) $\quad |p^{(\alpha)}{}_{h,(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\lambda_h(\xi)^{m-|\alpha|}$

for any α, β where $p^{(\alpha)}{}_{h,(\beta)}(x, \xi) = D^\beta{}_x \partial^\alpha{}_\xi\, p_h(x, \xi)$ $\quad (D_{xj} = -i\dfrac{\partial}{\partial x_j})$.

ii) The set of all symbols $p_h(x, \xi)$ such that $h^{-1}p_h \in \{S^{m+1}_{\lambda_h}\}$ is denoted by $\{\mathring{S}^m_{\lambda_h}\}$ and the set of all symbols $p_h(x, \xi)$ such that $h^{-1}p^{(\alpha)}{}_h \in \{S^{m+1-|\alpha|}_{\lambda_h}\}$ for any α $(\neq 0)$ is denoted by $\{\tilde{S}^m_{\lambda_h}\}$.

iii) A family of linear operators $P_h: \mathscr{S} \to \mathscr{S}$ is called a pseudo-differential operator of class $\{S^m_{\lambda_h}\}$ with symbol $p_h(x, \xi)$ when there exists a symbol $p_h(x, \xi)$ *of class* $\{S^m_{\lambda_h}\}$ such that

(2.4) $\quad P_h u = p_h(X, D)\, u(x) = \int e^{ix\xi} p_h(x, \xi)\, \hat{u}(\xi)\, đ\xi$

for $u \in S$, where $đ\xi = (2\pi)^{-n}d\xi$ and $\hat{u}(\xi) = \int e^{-ix\xi}\, u(x)\,dx$. We denote

(2.4) briefly by $P_h = p_h(X, D) \in \{S^m_{\lambda_h}\}$, or $\sigma(P_h) = p_h(x, \xi)$.

It is evident that $\{S^{m_2}_{\lambda_h}\} \subset \{S^{m_1}_{\lambda_h}\}$ for $m_2 \leqq m_1$. We set $\{S^{-\infty}_{\lambda_h}\} = \bigcap_m \{S^m_{\lambda_h}\}$, $\{S^\infty_{\lambda_h}\} = \bigcup_m \{S^m_{\lambda_h}\}$. The classes $\{\mathring{S}^0_{\lambda_h}\}$, $\{\tilde{S}^0_{\lambda_h}\}$ are important for difference calculus. The several examples of P_h introduced in § 2 of [6] will give us a good view and we omitt to recall them here.

Furthermore we shall use the algebra of operators of class $\{S^m_{\lambda_h}\}$ and the Calderón-Vaillancourt theorem. (see Theorem 3. 1, its corollaries and Lemma 3. 8 in [6]). We recall here only the commutator lemma which is a corollary of the asymptotic expansion formula.

Lemma 2. 3. For a scalar valued function $q_h(\xi)$ $(\in\{\tilde{S}^0_{\lambda h}\})$ and $p_h(x,\xi)$ $(\in\{S^m_{\lambda h}\})$ we get

(2 . 5) $$[q_h, p_h]=q_h(D)\ p_h(X,D)-p_h(X,D)\ q_h(D)\in\{\mathring{S}^{m-1}_{\lambda h}\}.$$

Now we turn to the equation (1. 1), (1. 2). The hyperbolicity of (1. 2) means the following (2. 6), (2. 7), (2. 8).

(2 . 6) $p(x,\xi)\in S^1_{<\xi>}$ (Hörmander class of order 1) and is homogeneous of degree 1 in ξ,

(2 . 7) $\mu_j(x,\xi)$ $(j=1,\cdots\cdots,d)$, eigenvalues of $p(x,\xi)$ are real,

(2 . 8) there exists a uniform diagonalizer $N(x,\xi)$ for large$<\xi>$, i. e.$N(x,\xi)$ $p(x,\xi)=\mathscr{D}(x,\xi)\ N(x,\xi)$

for $|\xi|\geqq M$, where$|\det \mathrm{N}(x,\xi)|\geqq c_0$ for some positive constant c_0. The stability calculus may be done indepentently of these assumptions (2. 6) - (2. 8), however, according to the well known Lax's equivalence theorem it is natural to expect that the L^2-wellposedness of the Cauchy problem (1. 2). This is the reason why we set these assumptions and we may forget them in the inference of the stability calculus.

Let the symbol of the scheme S_h in (1. 1) be

(2 . 9) $$\sigma(S_h)=q_h(\xi)\ I+Q_h(x,\xi),$$

where I is the identity $d\times d$ matrix, $q_h(\xi)$ is a real scalar valued function of $\{\tilde{S}^0_{\lambda h}\}$ and $Q_h(x,\xi)$ is a $d\times d$ matrix valued function of $\{\mathring{S}^0_{\lambda h}\}$. These schemes are restricted in the sense that $q_h(D)$ operates in the same fasion for each component of $u(t,x)$ in (1. 1). In the latter part of the following section we shall treat the stability of schemes with diagonal matrix valued function $q_h(\xi)$. In this case we shall impose on $q_h(\xi)$

(2 .10) $$I-q_h(\xi)\in\{\mathring{S}^0_{\lambda h}\}.$$

(See Theorem 3. 6 and Remark 3. 7.) Q_h contains a real parameter $\tau(=k/h)$ which is constant in the stability calculus and therefore we omitt to write it explicitly except in the section 4. From the elementary observation for the discretization of (1. 2), it is seen that the symbol $\sigma(S_h)$ given in (2. 9) (or (2. 9) with (2. 10)) is the general representation of the two-step difference scheme. (See also Remark 3. 7) Evidently we see that $\sigma(S_h)\in\{\tilde{S}^0_{\lambda h}\}$: We often use the following symbol $\sigma(\check{S}_h)$ in place of $\sigma(S_h)$:

$$\sigma(\check{S}_h)=q_h(\xi)\ I+Q_h(x,\xi)\ (1-\chi_h(\xi)),$$

where χ $(\in C_o(R^n_\xi))$ is identically 1 in some neighborhood of $\zeta=o$ and $\chi_h(\xi)=\chi(\zeta_h(\xi))$. In the stability calculus we may neglect $O(h)$ quantity and then we may regard $\sigma(\check{S}_h)$ as the symbol of S_h through the cutting-off principle. (see Lemma 5. 4 in [6]) Also we see that $\sigma(\check{S}_h)\in\{\tilde{S}^0_{\lambda h}\}$.

We recall the Friedrichs part of operator and several propositions relating to it.

For further clarification [3], [10] should be referred to.

Let $q(\sigma)$ be an even and C^∞—function satisfying that $q(\sigma)\geqq 0$, supp $q(\sigma)\subset\{\sigma:|\sigma|\leqq 1\}$ *and* $\int q^2(\sigma)\,d\sigma=1$. We define $F(\xi,\zeta)$ by

$$F(\xi,\zeta)=q((\xi-\zeta)\,\lambda_h(\xi)^{-1/2})\,\lambda_h(\xi)^{-n/4}$$

and double symbol $p_{h,F}(\xi,x',\xi')$ by

$$p_{h,F}(\xi,x',\xi')=\int F(\xi,\zeta)\,p_h(x',\zeta)\,F(\xi',\zeta)\,d\zeta$$

for $p_h\in\{S^m_{\lambda_h}\}$.

Definition 2. 4. The operator $p_{h,F}$ called the Friedrichs part of P_h is defined by

$$\widehat{P_{h,F}u}\,(\xi)=\int e^{-ix'\xi}\,\{\int e^{ix'\xi'}\,p_{h,F}(\xi,x',\xi')\,\hat{u}(\xi')\,d\!\!{}^-\xi'\}\,dx'.$$

Let $p_{h,F}(x,\xi)$ denote the simplified symbol of $P_{h,F}$. Then we get the asymptotic expansion

(2.11) $\quad p_{h,F}(x,\xi)\sim p_h(x,\xi)+\sum_{|\beta|=1}\psi_{h,\beta}(\xi)\,p_{h,(\beta)}(x,\xi)+\sum_{|\alpha+\beta|\geqq 2}\psi_{h,\alpha,\beta}(\xi)\,p_{h,}{}^{(\alpha)}{}_{(\beta)}(x,\xi)$,

where $\psi_{h,\beta}(\xi)\in\{S^{-1}_{\lambda_h}\}$, $\psi_{h,\beta}(\xi)\,p_{h,(\beta)}(x,\xi)\in\{S^{m-1}_{\lambda_h}\}$, $\psi_{h,\alpha,\beta}(\xi)\in\{S^{(|\alpha|-|\beta|)/2}_{\lambda_h}\}$ and $\psi_{h,\alpha,\beta}p_{h,}{}^{(\alpha)}{}_{(\beta)}(x,\xi)\in\{S^{m-|\alpha+\beta|/2}_{\lambda_h}\}$.

The following remark is essential.

Remark. 2. 5. If $p_h(x,\xi)$ is independent of x and $p_h(\xi)\in\{\tilde{S}^0_{\lambda_h}\}$, then the terms with subscripts $\beta(\neq 0)$ in the right hand side of

(2.11) vanish and therefore we have

(2.12) $\quad p_{h,F}(\xi)-p_h(\xi)\in\{\mathring{S}^{-1}_{\lambda_h}\}$.

The following propositions are well known.

Proposition 2. 6. If $p_h(x,\xi)$ $(\in\{S^m_{\lambda_h}\})$ is Hermitian symmetric, then we have

(2.13) $\quad (P_{h,F}u,v)=(u,P_{h,F}v)$

for u, v ∈ &. Especially, if $p_h(x,\xi)$ *is non-negative, then we have*

(2.14) $\quad (P_{h,F}u,u)\geqq 0$.

Proposition 2. 7. (a sharp form of Gårding's inequality). If $p_h(x,\xi)$ $(\in\{S^m_{\lambda_h}\})$ is Hermitian symmetric and satisfies the inequality $p_h(x,\xi)\geqq c\lambda_h(\xi)^m$, then we have

(2.15) $\quad Re(P_hu,u)\geqq c\|u\|^2_{\lambda_h,m}-C\|u\|^2_{\lambda_h,(m-1)/2}$

for some constant C, where $\|\cdot\|_{\lambda_h,s}$ denotes the Sobolev norm with respect to the basic weight function $\lambda_h(\xi)$.

3. Stability of Schemes of Kreiss's Class.

We begin with the definition of schemes of Kreiss's class.

Definition 3. 1. The scheme S_h is called of Kreiss's class if there exists an Hermitian symbol $H_h(x, \xi)$ for large $\zeta_h(\xi)$ such that

(I) $|\partial^\alpha_\xi D^\beta x H_h(x, \xi)| \leqq C_{\alpha,\beta}\, \lambda_h(\xi)^{-|\alpha|}$,

(II) $0 < c \leqq H_h(x, \xi) \leqq C$

and

(III) $H_h(x, \xi) - \sigma(S_h)^* H_h(x, \xi)\ \sigma(S_h) \geqq 0$

holds there.

Remark 3. 2. For the constant coefficients case $H_h(x, \xi)$ (if it exist) is independent of x and is of the form $H_h(\xi) = H(\omega)_{\omega=h\xi}$ in general for classical finite difference schemes. (see Kreiss [7]). This is the reason why the schemes defined in Definition 3. 1 are called of Kreiss's class. From (III) we see that schemes of Kreiss' s class satisfy necessarily the von Neumann condition for large $\zeta_h(\xi)$.

The schemes we shall treat here are of the form (2. 9). These are restricted in the sense that $q_h(\xi)$ (the term independent of the equation (1. 2)) is a real scalar valued function, i. e. $q_h(D)$ works in the same fashon for each component of $u(t, x)$ in (1. 1). Further we assume for $q_h(\xi)$ that

(*) $1 - q^2{}_h(\xi) \geqq 0$.

Then, our stability statement is the following theorem.

Theorem 3. 3. Let S_h be of Kreiss's class and of the form (2. 9) with the property (*). Then S_h is L^2-stable.

Proof. For clarification of the method of our proof, first the proof will be done in the special case when we may put $H_h(x, \xi) = I$ for $(x, \xi) \in R^n_x \times R^n_\xi$ and secondly the general case will be treated.

(i) The case when $H_h(x, \xi) = I$.

We have

$$\| S_h u \|^2 - \| u \|^2 = ((I - S_h^\# S_h)\ u,\ u)$$

$$(3.1) \qquad = Re\ ((I - S_h^* \circ S_h)\ u,\ u) + O(h) \| u \|^2,$$

where $A \circ B$ (called the symbolic product of A and B) denotes the operator with symbol $\sigma(A)\ \sigma(B)$. We have applied in (3. 1) the asymptotic expansion formula of pseudo-differential operators. Since the taking of the Friedrichs part is linear operator, we see

$$(I-S_h^* \circ S_h)_{\mathrm{F}} = (1-q^2{}_h(D))_{\mathrm{F}} + R_{h,\mathrm{F}}(X, D),$$

where $\sigma(R_h) = q_h(\xi)Q_h(x, \xi) + Q_h^*(x, \xi)\ q_h(\xi) + Q_h^*(x, \xi)\ Q_h(x, \xi) \in \{\mathring{S}^0_{\lambda_h}\}$. We get here by applying (2. 12) to $p_h(\xi) = 1-q^2{}_h(\xi)$,

(3 . 2) $$(1-q^2{}_h(D))_{\mathrm{F}} - (1-q^2{}_h(D) \in \{\mathring{S}^{-1}_{\lambda_h}\}$$

and, by applying (2. 11) to $p_h = R_h(x, \xi)$,

(3 . 3) $$R_{h,\mathrm{F}}(X,D) - R_h(X, D) \in \{\mathring{S}^{-1}_{\lambda_h}\}.$$

Then we have

$$\begin{aligned}(I-S_h^* \circ S_h) &= (I-S_h^* \circ S_h)_{\mathrm{F}} \\ &\quad + (I-S_h^* \circ S_h) - (I-S_h^* \circ S_h)_{\mathrm{F}} \\ &= (I-S_h^* \circ S_h)_{\mathrm{F}} + (1-q^2{}_h(D)) - (1-q^2{}_h(D))_{\mathrm{F}} \\ &\quad + (R_h - R_{h,\mathrm{F}}).\end{aligned}$$

Since $((I-S_h^* \circ S_h)_{\mathrm{F}} u, u) \geqq 0$ by (2. 14), we get by using (3. 2) and (3. 3)

$$\| S_h u \|^2 - \| u \|^2 \leqq O(h) \| u \|^2.$$

Thus the proof in the case (i) is completed.

Before the proof in the general case we mention a corollary.

Corollary 3. 4. The Hermitian symmetric scheme of the form (2. 9) with the property (*) is L²-stable.

In this case the proof above works well also for diagonal matrix valued function $q_h(\xi)$. The stability of the Lax-Wendroff scheme with Hermitian symmetric coefficients (already known in [13]) can be derived within our method of treatment as follows.

The Lax-Wendroff scheme L_h has the symbol:

(3 . 4) $$\sigma(L_h) = I + i\tau h p_h - (1/2)\, \tau^2 h^2 p^2{}_h - D_h,$$

where $p_h = h^{-1}\ (A(x) \sin h\xi_1 + B(x) \sin h\xi_2)$, $A(x)$ and $B(x)$ are real symmetric matrix and $D_h = 2\tau^2\ (A^2(x) \sin^4 (h\xi_1/2) + B^2(x) \sin^4 (h\xi_2/2))$. If $A(x)$ and $B(x)$ are functions of $B(R^2{}_x)$, we see that $\sigma(L_h)$ is of the form (2. 9) by setting $\tilde{\lambda}_h = (1+4h^{-2}\sum_{j=1}^{2}\sin^2 (h\xi_j/2))^{1/2}$. Since the von Neumann condition is satisfied for τ such that

(3 . 5) $$\tau^2 A^2 \leqq (1/8)\ I \text{ and } \tau^2 B^2 \leqq (1/8)\ I,$$

L_h is of Kreiss's class. Then, the proof in the case (i) gives an alternative one for the L²-stability of L_h.

(ii) The general case.

In order to reduce the main part of the proof to that in the case (i) we make the followilg device.

Assume that (I) and (II) hold for $|\zeta_h(\xi)|\geqq M$. Choose positive numbers M_j ($j=0, 1, 2$) such that $M<M_0<M_1<M_2$. These may be arbitrarily large but fixed. We prepare non-negative scalar valued functions $\varphi_h(\xi)$, $\psi_h(\xi)$ and $\psi_h(\xi)$ as follws. Let φ, ψ and ψ be C^∞-functions such that

$$\begin{cases}\varphi(\zeta)=0 & \text{for } |\zeta|\leqq M_0\\ \varphi(\zeta)=1 & \text{for } |\zeta|\geqq M_1,\end{cases}\quad \begin{cases}\psi(\zeta)=1 & \text{for } |\zeta|\leqq M_1\\ \psi(\zeta)=0 & \text{for } |\zeta|\geqq M_2\end{cases}$$

and
$$\begin{cases}\chi(\zeta)=1 & \text{for } |\zeta|\leqq M_2\\ \chi(\zeta)=0 & \text{for large } \zeta,\end{cases}$$

and set $\varphi_h(\xi)=\varphi(\zeta_h(\xi))$, $\psi_h(\xi)=\psi(\zeta_h(\xi))$, $\chi_h(\xi)=\chi(\zeta_h(\xi))$.

Define $\check{H}_h(x, \xi)=H_h(x, \xi)\varphi^2{}_h(\xi)+\psi^2{}_h(\xi)I$. Then we see by the definition of $\varphi_h(\xi)$, $\psi_h(\xi)$ that

(3.6) $$0<c_0\leqq \check{H}_h(x, \xi)\leqq C_0 \text{ for any } (x, \xi)\in R^n_x\times R^n_\xi.$$

Therefore we see by proposition 2. 6 that there exists a positive number C' such that

(3.7) $$Re(\check{H}_h(X, D)u, u)\geqq c_0\|u\|^2-C'\|u\|^2_{\lambda_h,-1/2}.$$

We construct a new norm $\|.\|_{G_h}$ equivalent to the original L^2-norm as follows. Put $K_h(x, \xi)=H_h(x, \xi)^{1/2}$and

(3.8) $$R_h(x,\xi)=\sigma((K_h(X, D)\varphi_h(D))^\#)-K_h(x, \xi)\varphi_h(\xi).$$

Note that $K_h(x, \xi)\varphi_h(\xi)$ is well defined and $K_h\varphi_h\in\{S^0_{\lambda_h}\}$.
Then we know by the asymptotic expansion formula that

(3.9) $$R_h(x, \xi)\in\{S^{-1}_{\lambda_h}\}.$$

Therefore we have

$$\sigma((K_h\varphi_h)^\#)\sigma(K_h\varphi_h)=H_h(x, \xi)\varphi_h(\xi)^2+R'_h(x, \xi),$$

where $R'_h(x, \xi)=R_h(x, \xi)\sigma(K_h\varphi_h)\in\{S^{-1}_{\lambda_h}\}$.
Further put

$$R''_h(x, \xi)=\sigma((K_h\varphi_h)^\#(K_h\varphi_h))-\sigma((K_h\varphi_h))^\#)\sigma(K_h\varphi_h).$$

Then by using the asymptotic expansion formula again we see

$$R''_h(x, \xi)\in\{S^{-1}_{\lambda_h}\}.$$

Now define the operator G_h by

(3.10) $$G_h=(K_h\varphi_h)^\#(K_h\varphi_h)+\psi^2_h(D)+C''\lambda_h(D)^{-1},$$

where C″ will be determined below and the new norm by

(3.11) $$\|\cdot\|_{G_h}=(G_hu, u)^{1/2}.$$

This new nerm is equivalent to the original L^2-norm. For, we

get $$Re(((K_h\varphi_h)^{\#}(K_h\varphi_h)+\psi_h^2(D))u,\ u) = Re(\check{H}_h u,\ u)+Re(R'''_h u,\ u),$$

where $R'''_h = R'_h + R''_h \in \{S_{\lambda_h}^{-1}\}$, and furthermore get by (3. 7)

$$Re(((K_h\varphi_h)^{\#}(K_h\varphi_h)+\psi_h^2(D))u,\ u) \geqq c_0\|u\|^2 - C''\|u\|^2_{\lambda_h,-1/2}$$

for some positive constant C''.

Then we shall calculate $\|\check{S}_h u\|^2_{G_h} - \|u\|^2_{G_h}$ for $\check{S}_h = q_h(\xi)+Q_h(x,\ \xi)(I-\chi_h(\xi))$.

We have

$$\begin{aligned}\|\check{S}_h u\|^2_{G_h} - \|u\|^2_{G_h} &= (G_h\check{S}_h u,\ \check{S}_h u)-(G_h u,\ u)\\ &= (K_h\varphi_h\check{S}_h u,\ K_h\varphi_h\check{S}_h u)-(K_h\varphi_h u,\ K_h\varphi_h u)\\ &\quad - Re(((\psi_h^2(D)+C''\lambda_h(D)^{-1})(I-S_h^*\circ S_h)u,\ u)+O(h)\|u\|^2\end{aligned}$$

and further get by noting the facts that the commutators $[\psi_h,\ q_h]$, $[\varphi_h,\ Q_h]$ belong to $\in\{\mathring{S}^0_{\lambda_h}\}$,

$$\begin{aligned}&\|\check{S}_h u\|^2_{G_h} - \|u\|^2_{G_h}\\ &= (K_h\check{S}_h\varphi_h u,\ K_h\check{S}_h\varphi_h u)-(K_h\varphi_h u,\ K_h\varphi_h u)\\ &\quad - Re((\psi_h^2(D)+C''\lambda_h(D)^{-1})(I-S_h^*\circ S_h)u,\ u)+O(h)\|u\|^2.\end{aligned} \tag{3.12}$$

Let J_1, J_2 denote

$$J_1 = (K_h\check{S}_h\varphi_h u,\ K_h\check{S}_h\varphi_h u)-(K_h\varphi_h u,\ K_h\varphi_h u)$$

and

$$J_2 = -((\psi_h^2(D)+C''\lambda_h(D)^{-1})(I-\check{S}_h^*\circ\check{S}_h)u,\ u),\ \text{respectively.}$$

In order to estimate J_1 we define Z_h by

$$\begin{aligned}&\sigma(Z_h)=K_h(x,\ \xi)\sigma(\check{S}_h)K_h(x,\ \xi)^{-1} &&\text{for}\ |\varsigma_h(\xi)|\geqq M_0\\ \text{and}\quad &\sigma(Z_h)=q_h(\xi) &&\text{for}\ |\varsigma_h(\xi)|\leqq M_1.\end{aligned} \tag{3.13}$$

Then $Z_h$is well defined and of the form (2. 9) and $Z_h\in\{\tilde{S}^0_{\lambda_h}\}$.

Noting that both $\sigma(\check{S}_h)$ and $\sigma(Z_h)$ are of the same form (2. 9), we get through the matrix equation $K_h\sigma(\check{S}_h) = \sigma(Z_h)K_h$

$$K_h\check{S}_h \equiv K_h\circ\check{S}_h = Z_h\circ K_h \equiv Z_h K_h, \tag{3.14}$$

where $A\equiv B$ means that the operator $A-B$ is an $O(h)\cdot L^2$-bounded operator. By (3. 14) we get

$$\begin{aligned}J_1 &= (Z_h v_h,\ Z_h v_h)-(v_h,\ v_h)\\ &= -Re((I-Z^{\#}_h Z_h)v_h,\ v_h)+O(h)\|u\|^2,\end{aligned}$$

where $v_h = K_h \varphi_h u$. Further by using the asymptotic expansion formula we get

$$J_1 = -Re((I - Z_h^* \circ Z_h) v_h, v_h) + O(h) \| u \|^2.$$

Now we see from the assumption (III) (in Definition 3. 1) that

$$\begin{aligned}\sigma(I - Z_h^* \circ Z_h) &= I - \sigma(Z_h)^* \sigma(Z_h) \\ &= K_h^{-1/2}(H_h - \sigma(\check{S}_h)^* H_h \sigma(\check{S}_h)) K_h^{-1/2} \geqq 0.\end{aligned}$$

Thus, through the similar discussion to the case (i) we get

$$(3.15) \qquad J_1 \leqq O(h) \| u \|^2.$$

The estimation of J_2 is much easy. Because of the equality $\sigma(I - \check{S}_h^* \check{S}_h) = 1 - q_h^2(\xi) + \tilde{Q}_h(x, \xi)(1 - \chi_h(\xi))$, where $\tilde{Q}_h \in \{\mathring{S}_{\lambda_h}^0\}$, we see that

$(\psi_h^2(D) + C'' \lambda_h(D)^{-1}) \tilde{Q}_h(X, D)$ is $O(h) \cdot L^2$-bounded operator.

Therefore we have only to estimate

$$(\psi_h^2(D) + C'' \lambda_h(D)^{-1})(I - q_h^2(D)) u, u),$$

that is equal to

$$((I - q_h^2(D)) w_h, w_h) + C''((I - q_h^2(D)) y_h, y_h),$$

where $w_h = \psi_h(D) u$ and $y_h = \lambda_h(D)^{-1/2} u$.
By the property (*) we have from (2. 12)

$$(3.16) \qquad J_2 \leqq O(h) \| u \|^2.$$

Combining (3. 15) with (3. 16) we get

$$\| \check{S}_h u \|_{G_h}^2 - \| u \|_{G_h}^2 = O(h) \| u \|^2.$$

Through the cutting-off principle S_h is also L^2-stable. *Q. E. D.*

Remark 3. 5. In the constant coefficients case H_h and K_h are independent of x and so is $\sigma(Z_h)$. Then, the proof above works well also for diagonal matrix valued function $q_h(\xi)$ $(\in \{\tilde{S}_{\lambda_h}^0\})$. This is the well known Kreiss stability theorem. Since $K_h(x, \xi) q_h(\xi) K_h(x, \xi)^{-1}$ depends on x in general, the above discussion can not be expected to work well for diagonal matrix $q_h(\xi)$.

Theorem 3. 6. Let S_h be of Kreiss's class and of the form (2. 9) (2. 10) with the property (*). Then S_h is L^2-stable.

Proof. We have only to observe to what extent $K_h(x, \xi) q_h(\xi) K_h(x, \xi)^{-1}$ in the representation of $\sigma(Z_h)$ is violated from scalar valued function. Put $K_h(x, \xi) = (k_{ij})$, $q_h(\xi) = (q_i \delta_{ij})$ and $K_h(x, \xi)^{-1} = (\tilde{k}_{ij})$. Then we have

$$\sigma_{ij} = \text{the } (ij)\text{-element of } K_h(x,\ \xi)q_h(\xi)K_h(x,\ \xi)^{-1}$$
$$= \sum_{k=1}^{n} \sum_{l,l'=1}^{n} q_l q_{l'}\ \bar{k}_{kl}\bar{\tilde{k}}_{li}k_{kl'}\ \tilde{k}_{l'j}$$

and further get, by applying the indentity

$$q_l q_{l'} = 1-(1-q_l)-q_l(1-q_{l'}),$$
$$\sigma_{ij} = \sum_{k=1}^{n} \sum_{l,l'=1}^{n} \bar{k}_{kl}\bar{\tilde{k}}_{ki}k_{kl'}k_{l'j}$$
$$-\sum_{k=1}^{n} \sum_{l,l'=1}^{n} (1-q_l)\bar{k}_{kl}\bar{\tilde{k}}_{li}k_{kl'}k_{l'j}$$
$$-\sum_{k=1}^{n} \sum_{l,l'=1}^{n} q_l(1-q_{l'})\bar{k}_{kl}\bar{\tilde{k}}_{li}k_{kl'}k_{l'j}.$$

The lst term in the right hand side is Kronecker's δ_{ij}.
The 2nd and 3rd term belog to $\{\mathring{S}^0_{\lambda_h}\}$ by the condition (2. 10). Therefore we can see that $\sigma(Z_h)$ is rewrittin in the form

$$\sigma(Z_h) = I + Q_h(x,\ \xi),\ \text{where } Q_h(x,\ \xi)\in\{\mathring{S}^0_{\lambda_h}\}.$$

Other part of the proof (ii) of Theorem 3. 3 works well here also.

Remark 3. 7. The classical finite difference scheme has the symbol of the form $\sum_{\alpha,\text{finite}} A_\alpha(x)e^{i\alpha h\xi}(\alpha\xi=\sum_{j=1}^{n}\alpha_j\xi_j)$ with relation $\sum_\alpha A_\alpha(x)=I$.
Then we have

$$\sigma(S_h)=\sum_\alpha A_\alpha(x)e^{i\alpha h\xi}=I+Q_h(x,\ \xi),\ \text{where } Q_h=\sum_\alpha A_\alpha(x)\ (e^{i\alpha h\xi}-1).$$

Since we know

$$e^{i\alpha h\xi}-1=\sum_{j=1}^{n}\ \sin(h\xi_j/2)b_j(h\xi) \qquad (b_j(\omega)\in B(R^n{}_\omega)),$$

we get $Q_h(x,\ \xi)\in\{\mathring{S}^0_{\tilde{\lambda}_h}\}$ by setting the basic weight function $\tilde{\lambda}_h=\lambda_{h/2}$.

This shows that the class of schemes of the form (2. 9) with (2. 10) is satisfactorily wide one.

4. Comments on the Property (*) and the τ-dependence of Stability.

For given T (>0) the approximate solution u $(t,\ x)$ is calculated through the j-times iteration of the operator S^h ($T=jk$) from u $(0,\ x)$ $=u_0$. Now let S_h be a finite difference scheme, *i. e.* $S_h=\sum_{\alpha,\text{finite}} A_\alpha(x)T^\alpha{}_h$, where $T_h^\alpha=T_{1,h}^{\alpha_1}\cdots T_{n,h}^{\alpha_n}$ and $T_{j,h}\ u(x)=u$ $(x+he_j)$. Then (1. 1) shows that the values of u $(t+k,\ x)$ in the neighborhood with center x_0 are determined from the values of $u(t,\ x)$ in the finite number of neighborhoods with centers $x_0+\sum_{j=1}^{n}\alpha_j he_j$. Therefore, from the law of finite speed of propagation for the solution of the hyperbolic system, we see that for large k/h $u(T, x)$ can not approximate the solution of (1. 2) in any topology. This fact was originally pointed out by R. Courant, K. Friedrichs and H. Lewy ([1]). We recall it in the following proposition.

Proposition 4. 1(the C-F-L condition). Let the finite difference scheme S_h approximate the hyperbolic system having the property of finite speed of propagation, Then, for the L^2-stability of S_h it is necessary that there exists a positive bound τ_0 such that $k/h \leqq \tau_0$.

Proof. It was sketched in the above. The strict proof will be completed on the basis of the Lax equivalence theorem and the L^2-wellposedness of (1. 2) and it is left to the reader.

On the other hand we get the following proposition.

Proposition 4. 2. If the von Neumann condition is satisfied for small τ, then the property (*) holds.

Proof. If the property (*) do not hold, then there exist h_0 and ξ_0 such that $1-q_{h_0}^2(\xi_0)<0$. Tending τ to zero in (2. 9) for this pair (h_0, ξ_0) contradicts the von Neumann condition for S_h.

Now, we know that difference schemes with consistency are obtained by putting $\tau=k/h$ in (2. 9). Then from the observation above mentioned we see that the property (*) is not restrictive for schemes approximating a hyperbolic system.

5. An Example of stable Difference Scheme.

We propose here a new difference scheme approximating a hyperbolic system. Its stability will be justified by our theorem in the section 3.

Let the hyperbolic system be such that

(5 . 1) $D_t u = A(x) D_x u$ $(n=1,\ d=2)$,

where $u=\begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}$, $A(x)=(a_{ij}(x))$, $a_{ij}(x)\in B(R^1)$, *all* $a_{ij}(x)$ are real valued and $a_{21}=0$,

and let the scheme S_h approximating the system (5. 1) be the following:

$$(5\,.\,2) \qquad \sigma(S_h)=\begin{pmatrix} \Delta_1, & i\tau a_{12}(x)\sin h\xi \\ 0, & \Delta_2 \end{pmatrix}$$

where $\Delta_1=\cos h\xi+i2\tau a_{11}(x)\sin(h\xi/2)$, $\Delta_2=\cos h\xi+i2\tau a_{22}(x)\sin(h\xi/2)$.

This scheme is of the form (2. 9) by setting the basic weight function $\tilde{\lambda}_h(\xi)=\lambda_{h/2}(\xi)$ and not necessarily diagonalizable even if $A(x)$ is diagonalizable. The von Neumann condition is

(5.3) $\tau^2 a^2_{11}(x) \leqq 1/2$ and $\tau^2 a^2_{22}(x) \leqq 1/2$.

In order to see that S_h is of Kreiss's class for well selected τ, we shall calculate $H_h - \sigma(S_h)^* H_h \sigma(S_h)$ for $H_h = \begin{pmatrix} 1, & f_h \\ f_h, & d \end{pmatrix}$ where f_h and d will be determined later.

We have

$$H_h - \sigma(S_h)^* H_h \sigma(S_h)$$
$$(5.4) \qquad = \begin{pmatrix} 1-|\Delta_1|^2, & f_h(1-\bar{\Delta}_1\Delta_2) - \bar{\Delta}_1 i\tau a_{12}(x)\sin h\xi \\ \bar{f}_h(1-\Delta_1\bar{\Delta}_2) + \Delta_1 i\tau a_{12}(x)\sin h\xi, & \Lambda \end{pmatrix},$$

where $\Lambda = d(1-|\Delta_2|^2) - \tau^2 a^2_{12}(x)\sin^2 h\xi$
$+ i\tau a_{12}(x) f_h \Delta_2 \sin h\xi - i\tau a_{12}(x)\bar{f}_h\bar{\Delta}_2 \sin h\xi.$

Then we get

$$\det(H_h - \sigma(S_h)^* H_h\sigma(S_h))$$
$$= (2-4\tau^2 a^2_{11}(x))\sin^2(h\xi/2)\{d(2-4\tau^2 a^2_{22}(x)\sin^2(h\xi/2)$$
$$(5.5) \qquad -\tau^2 a^2_{12}(x)\sin^2 h\xi + i\tau a_{12}(x) f_h\Delta_2 \sin h\xi - i\tau a_{12}(x)\bar{f}_h\bar{\Delta}_2\sin h\xi$$
$$-|f_h(1-\bar{\Delta}_1\Delta_2) - \bar{\Delta}_1 i\tau a_{12}(x)\sin h\xi|^2.$$

For the estimation of $\det(H_h - \sigma(S_h)^* H_h\sigma(S_h))$ from below the existence of the last term in the right hand side of (5. 5) raises a troublesome question. In order that $\sin^4(h\xi/2)$ may be a factor of the last term we put

$$(5.6) \qquad f_h = (a_{11}(x) - a_{22}(x))^{-1} a_{12}(x) \cos(h\xi/2).$$

Then we get

$$f_h = \bar{f}_h,$$
$$|f_h(1-\bar{\Delta}_1\Delta_2) - \bar{\Delta}_1 i\tau a_{12}(x) \sin h\xi|^2$$
$$\leqq (|f_h|(4+4\tau^2|a_{11}||a_{22}|) + 4\tau^2|a_{11}(x)a_{12}(x)|)^2 \sin^4(h\xi/2)$$

and

$$\det(H_h - \sigma(S_h)^* H_h\sigma(S_h))$$
$$(5.7) \qquad \geqq (2-4\tau^2 a^2_{11}(x))\ (d(2-4\tau^2 a^2_{22}(x)) - \tau^2 a^2_{12}(x) - 2\tau|a_{12}(x)|\,|f_h|)$$
$$-(|f_h|(4+4\tau^2|a_{11}|\,|a_{22}|) + 4\tau^2|a_{11}(x)a_{12}(x)|)^2\sin^4(h\xi/2),$$

where $|a_{ij}| = \sup|a_{ij}(x)|$.

Here we meet with two problems: the lst is the problem when f_h defined in (5. 6) is of class $\{S^0_{\lambda h}\}$ and the 2nd is the one when $\det(H_h - \sigma(S_h)^* H_h\sigma(S_h))$ is non-negative and $\det H_h = d - |f_h|^2$ is positive. The answer for the lst problem is affirmative if the following condition (5. 8) is satisfied;

(5.8) $(a_{11}(x)-a_{22}(x))^{-1}a_{12}(x)^{-1}\in B(\Omega)$
where $\Omega=\{x\in R^1|a_{11}(x)\neq a_{22}(x)\}$and this function is extensible to a function $b(x)$ of $B(R^1)$.

Then it is easily checked that $f_h=b(x)\cos(h\xi/2)\in\{S^0_{\lambda h}\}$. It should be noted that the condition (5. 8) does not exclude the equality $a_{11}(x)=a_{22}(x)$ if $a_{12}(x)=0$ for $x\in R^1-\Omega$.

Noting that the $C-F-L$ condition here is that $\tau^2\rho^2\leqq 1/2$ where $\rho=\max(|a_{11}|, |a_{22}|)$, we consider the 2nd problen in the following two cases (i), (ii).

(i) In the case $\tau^2\rho^2<1/2$.
From (5. 7) we get

$$\det(H_h-\sigma(S_h)^*H_h\sigma(S_h))$$

$$\geqq\alpha(d\alpha-\tau^2a^2_{12}(x)2\tau|a_{12}(x)||f_h|)-(M_1|f_h|^2+M_2|a_{12}(x)|^2)\times\sin^4(h\xi/2), \tag{5.9}$$

where $\alpha=1/2-\tau^2\rho^2(>0)$ and $M_j(j=1, 2)$ is positive constant depending on $|a_{ij}|$. Then by choosing d sufficiently large we get that $\det(H_h-\sigma(S_h)^*H_h\sigma(S_h))\geqq 0$ and $\det H_h=d-|f_h|^2>0$. Hence S_h is of Kreiss's class and L^2-stable by the theorem in the section 3.

(ii) In the case $\tau^2\rho^2=1/2$.

Put $\beta_j(x)=\rho^2-a^2_{jj}(x)$ $(j=1, 2)$. Then from (5. 7) we get

$$\det(H_h-\sigma(S_h)^*H_h\sigma(S_h))$$

$$\geqq 4\tau^2\beta_1(x)\ (d\beta_2(x)-\tau^2a^2_{12}(x)-2\tau|a_{12}(x)||f_h| \\ -M_3a^2_{12}(x)\ ((a_{11}(x)-a_{22}(x))^{-2}+1)\ \sin^4(h\xi/2) \tag{5.10}$$

for some constant M_3.
If there exist positive numbers γ, $\tilde{\gamma}$ such that

$$\gamma\beta(x)\geqq|a_{12}(x)| \tag{5.11.i}$$

and

$$\tilde{\gamma}\beta_1(x)\geqq|a_{12}(x)|((a_{11}(x)-a_{22}(x))^{-2}+1) \tag{5.11.ii}$$

for $x\varepsilon R^1$, we have also the L^2-stability of S_h by choosing d sufficiently large.

In the case when $a_{11}(x)$ is constant and $|a_{11}|\geqq|a_{22}(x)|$, we see that $\beta_1(x)=0$ and then the L^2-stability will not be expected unless $a_{12}(x)=0$. The condition (5. 11) states more; in the case when the off-diagonal element $a_{12}(x)$ vanishes at some order where $\beta_j(x)$ vanish the critical case (ii) produces the L^2-stability.

In the above we have assumed the condition (5. 8), however, this is seen to be natural by the following remark.

Remark 5. 1. It should be noted that the condition (5. 8) is a consequence of the uniform diagonalizability of $A(x)$. It is verified as follows.

From $AN=N\mathscr{D}(N\in B(R^1))$, $\mathscr{D}=\begin{pmatrix} a_{11}(x), & 0 \\ 0, & a_{22}(x) \end{pmatrix}$, we see

$$① \quad a_{11}n_{11}+a_{12}n_{21}=n_{11}a_{11},$$

$$② \quad a_{11}n_{12}+a_{12}n_{22}=n_{12}a_{22}$$

and

$$③ \quad a_{22}n_{21}=n_{21}a_{11}.$$

For $x\in\Omega=\{x\in R^1 \mid a_{11}(x)\neq a_{22}(x)\}$, we get from ③

$$n_{21}(x)=0.$$

Then $n_{22}(x)$ can not approach to zero in Ω because of the uniform diagonolizability of $A(x)$. From ② we get

$$a_{12}(x)\ (a_{11}(x)-a_{22}(x))^{-1}=n_{12}(x)n_{22}(x)^{-1}.$$

Therefore we see $a_{12}(x)\ (a_{11}(x)-a_{22}(x))^{-1}\in B(\Omega)$.
For $x\in R^1-\Omega$ we see from ② $a_{12}(x)n_{22}(x)=0$.
On the other hand we see from ① $a_{12}(x)n_{21}(x)=0$.
Since $n^2{}_{21}(x)+n^2{}_{22}(x)\neq 0$, we get $a_{12}(x)=0$ and $A(x)=a_{11}(x)I$. Then any matrix may be diagonalizer in $R^1-\Omega$. Choose a function $\tilde{n}_{22}(x)$ $(\in B(R^1))$ which does not approach to zero in $R^1-\Omega$ and coincides with $n_{22}(x)$ in Ω. Then $b(x)$ defined by $b(x)=n_{12}(x)\tilde{n}_{22}(x)^{-1}$ is an extension of $a_{12}(x)\ (a_{11}(x)-a_{22}(x))^{-1}$.

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