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On Blow-up Sets for the Parabolic Equation $\partial_t\beta(u)=\Delta u+f(u)$ *in a Ball*

Dedicated to Professor Mutsuhide Matsumura on his sixtieth birthday

by

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Abstract: Radially symmetric solutions of nonlinear degenerate parabolic equations are considered in a ball, under some blow-up conditions on $\beta(\xi)$, $f(\xi)$ and the initial date. Blow-up sets of solutions are classified by the increasing order of the heat source $f(\xi)$ as $\xi \to \infty$.

Key words: blow-up, blow-up set, nonlinear degenerate parabolic equation

1. Introduction and Results

The present paper is a continuation of the previous one [7] of the first two authors, and deals with some blow-up probloms for the parabolic intial-boundary value problem

(1.1) $\partial_t\beta(u)=\Delta u+f(u)$ in $B(R)\times(0, T)$,

(1.2) $Bu(x, t)=0$ on $\partial B(R)\times(0, T)$,

(1.3) $u(x, 0)=u_0(r)$, $r = |x|$, in $B(R)$.

Here $B(R) = \{x\in\mathbf{R}^N;\ |x|<R\}$, $0 < T \leq \infty$, $\partial_t=\frac{\partial}{\partial t}$, Δ is the N-dimensional Laplacian and (1.2) stands for the Dirichlet, Neumann or Robin condition:

$$Bu(x, t) = \begin{cases} u(x, t) & \text{(Dirichlet) or} \\ \partial_r u(x, t) & \text{(Neumann) or} \\ (\partial_r+\sigma)u(x, t),\ \sigma > 0. & \text{(Robin).} \end{cases}$$

Throughout this paper we assume the following conditions.

(A1) $\beta(\xi), f(\xi) \in C^\infty((0, \infty)) \cap C([0, \infty))$; $\beta(\xi) > 0$, $\beta'(\xi) > 0$, $\beta''(\xi) \leq 0$ and

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$f(\xi) > 0$ for $\xi > 0$, where $' = \frac{d}{d\xi}$; $\lim_{\xi\to\infty} \beta(\xi) = \infty$; $f\circ\beta^{-1}$ is is locally Lipschitz continuous in $[\beta(0), \infty)$.

(A2) $u_0(r) \in C(B(R))$ and ≥ 0; $u_0(0) > 0$.

These conditions guarantee the unique existence of local weak solutions (see e.g., Oleinik et al [11]), and as a consequence of the uniqueness we have $u = u(r, t)$. If u does not exist globally in time, its existence time $T < \infty$ is defined as

$$T = \sup\{\tau > 0\,;\ u(r, t) \text{ is bounded in } [0, R]\times[0, \tau]\}. \tag{1.4}$$

In this case we say that u is a *blow-up solution* and T is the *blow-up time*.

In order to state a blow-up condition, let $(s(r), \lambda)$ be the principal eigensolution of $-\Delta$ in $B(R)$ with boundary condition (1.2) (s is normalized: $s > 0$ and $\int_{B(R)} s(r)\,dx = 1$), and let

$$J(t) = \int_{B(R)} \beta(u(r, t))\, s(r) dx \text{ for } t \in [0, T). \tag{1.5}$$

(A3) There exist a continuous funcion $g(\xi)$ of $\xi \geq 0$ and a $\xi_0 \geq 0$ such that

$$g(\xi) \leq f(\xi) - \lambda\xi \text{ in } \xi \geq 0\,; \tag{1.6}$$

$$\Gamma \equiv g\circ\beta^{-1} \text{ is convex in } (\beta(0), \infty)\,; \tag{1.7}$$

$$g(\xi) > 0 \text{ and } \int_\xi^\infty \frac{\beta'(\eta)}{g(\eta)}\, d\eta < \infty \text{ if } \xi > \xi_0\,; \tag{1.8}$$

$$J(0) > \beta(\xi_0). \tag{1.9}$$

We have proved in [7] (cf., also Itaya [8] and Galaktionov [6]) that under (A3) every nontrivial solution blows up in finite time:

$$J(t) \to \infty \text{ as } t \uparrow T. \tag{1.10}$$

The blow-up set of u is defined as

$$S_u = \{x \in B(R)\,; \text{ there exists a sequence } (x_i, t_i) \in B(R)\times(0, T) \text{ such that } x_i \to x,\ t_i \uparrow T \text{ and } u(x_i, t_i) \to \infty \text{ as } i \to \infty\}. \tag{1.11}$$

Clearly S_u is closed and is nonempty under (A3). Moreover, in the present case (since $u = u(r, t)$), we have $\{x\,;\ r = |x_0|\} \subset S_u$ if $x_0 \in S_u$.

To characterize S_u more precisely we require

(A4) $u_0(r) \in C(B(R))$ in the Dirichlet case [or $\in C^1(B(R))$ in the Neumann and Robin case] and is nonincreasing in $r \in (0, R)$; $u_0(0) > 0$ and $Bu_0(R) = 0$.

(A5) $u_0(r) \in C^2(B(R))$ and $\Delta u_0(r) + f(u_0(r)) \geq 0$ in $(0, R)$.

For each $\rho > 0$ we denote by Δ_D in $B(\rho)$ the Laplacian with zero Dirichlet boundary condition on $\partial B(\rho)$. For $\mu > 0$ let $R_\mu > 0$ be chosen so that the principal eigenvalue of $-\Delta_D$ in $B(R_\mu)$ is given by μ. On the other hand, for $\rho > 0$ let $\lambda_\rho > 0$ be the principal eigenvalue of $-\Delta_D$ in B(ρ). Our main results are:

Theorem 1. (1) *Assume* (A3), (A4) *and*

$$f(\xi) = \lambda\xi + o(\xi) \text{ as } \xi \to \infty. \tag{1.12}$$

Then we have

$$S_u = \bar{B}(R). \tag{1.13}$$

Moreover, u blows up locally uniformly in $B(R)$:

$$\lim_{t \uparrow T} \inf_{0<r<\rho} u(r, t) = \infty \text{ for any } 0 < \rho < R. \tag{1.14}$$

(2) *Assume* (A3), (A4) *and*

$$f(\xi) \leq \gamma\xi + C \text{ in } \xi \geq 0 \text{ for some } \gamma > \lambda \text{ and } C > 0. \tag{1.15}$$

Then we have

$$S_u \supset \bar{B}(R_\gamma), \tag{1.16}$$

and u blows up locally uniformly in $B(R_\gamma)$.

(3) *Assume* (A3), (A4), (A5) *and the Friedman-McLeod conditions on* $f(\xi)$ (*see* [4]). *Then we have*

$$S_u = \{0\}. \tag{1.17}$$

In assertion (3) condition (A5) is required to ensure the inequality $\partial_t u(r, t) \geq 0$. With this inequality one can easily follow the method given in [7] for one dimensional problem (see also Chen [3] or Mochizuki-Suzuki [10]) to obtain (1.17). So, in this paper we omit the proof of (3).

Assertions (1) is also proved in [7] for the one dimensional Neumann problem. The proof is based on an energy method, and as a result we obtain the exact blow-up rate of solutions. Thus, in [7] the uniform blow-up property automatically follows. It seems difficult to extend the former proof directly to higher dimensional problems. So, in this paper we give up to obtain the exact blow-up rate, and use a nonblow-up result and a monotonicity of solutions. To ensure the monotonicity we require (A4). Cf., Friedman-McLeod [5], where is studied the Dirichlet problem for the equation

$$\partial_t u = u^2\{\Delta u + u\} \text{ in a bounded domain in } \mathbf{R}^N. \tag{1.18}$$

Our nonblow-up result is given to the inhomogeneous Dirichlet problem (1.1), (1.3) and

$$(1.19) \qquad u(R, t) = b(t) \qquad \text{on } (0, T).$$

(A6) $f(\xi) \leq \gamma\xi + C$ in $\xi \geq 0$ for some $0 \leq \gamma < \lambda$ and $C > 0$.

(A7) $b(t) \in C([0, \infty))$ *and* $u_0(r) \in C([0, R])$; $0 \leq b(t) \leq M$ *in* $[0, \infty)$ *and* $u_0(r) \geq 0$ *in* $[0, R]$; $b(0) = u_0(R)$.

Theorem 2. *Assume* (A6) *and* (A7). *Then the initial-boundary value problem* (1.1), (1.3) *and* (1.19) *has a unique global solution* u, *which is uniformly bounded in* $B(R) \times (0, \infty)$.

This extends the corresponding result of [7] for $N = 1$. The main reason that we had to restrict ourselves to the case $N = 1$ is also in the use of energy estimates. In this paper we shall compare the solution $u(x, t)$ directly with a steady state supersolution of (1.1).

Finally, we note that the Cauchy problem for (1.1) has been studied in Suzuki [12] for $N = 1$ and in [10] for $N \geq 2$, where asymptotic behaviors of the free boundary are discussed near the blow-up time.

2. Proof of Theorems

We begin with the definition of the weak solutions of (1.1).

Definition 2.1. By a solution of (1.1) and (1.2) [or (1.19)] we mean a function $u = u(x, t)$ such that

(i) $u(x, t) \in C(\bar{B}(R) \times [0, T))$ and ≥ 0 in $B(R) \times (0, T)$.

(ii) For $0 < \tau < T$ and nonnegative $\varphi(x, t) \in C^2(\bar{B}(R) \times [0, T))$ which satisfies the condition $B\varphi(x, t)$ [or $\varphi(x, t)$] $= 0$ on $\partial B(R) \times [0, T)$,

$$(2.1) \qquad \int_{B(R)} \beta(u(x, \tau))\varphi(x, \tau)dx - \int_{B(R)} \beta(u(x, 0))\varphi(x, 0)dx$$
$$= \int_0^\tau \int_{B(R)} \{\beta(u)\varphi_t + u\Delta\varphi + f(u)\varphi\} dxdt$$
$$\left[\text{or} = \int_0^\tau \int_{B(R)} \{\beta(u)\varphi_t + u\Delta\varphi + f(u)\varphi\} dxdt - \int_0^\tau \int_{\partial B(R)} u\partial_r\varphi dSdt\right].$$

A supersolution of (1.1) and (1.2) [or (1.19)] is defined by (i) and (ii) with equality (2.1) replaced by $\geq$. A subsolution of (1.1) and (1.2) [or (1.19)] is similarly defined with equation (2.1) replaced by $\leq$.

Lemma 2.2 (*Comparison principle*). (i) *Let* u [*or* v] *be a supersolution* [*or subsolution*] *of* (1.1) *and* (1.19). *If* $u \geq v$ *on the parabolic boundary of* $\{B(R) \times \{0\}\} \cup \{\partial B(R) \times (0, T)\}$, *then* $u \geq v$ *in the whole* $\bar{B}(R) \times [0, T)$.

(ii) *Let* u [*or* v] *be a supersolution* [*or subsolution*] *of* (1.1) *and the*

Neumann or Robin condition (1 . 2). *If* $u \geq v$ *on the initial domain* $B(R)\times\{0\}$, *then* $u \geq v$ *in the whole* $\bar{\Omega}\times[0, T]$.

Proof. See Aronson et al [1] and Bertsch et al [2]. □

Next, let u be a solution of (1 . 1), (1 . 2) [or (1 .19)] and (1 . 3).

Lemma 2. 3. *If* $u(\bar{r}, \bar{t}) > 0$ *for some* $(\bar{r}, \bar{t}) \in (0, R)\times(0, T)$, *then* u *is smooth in a neighborhood of* $(\bar{r}, \bar{t})$ *and becomes a classical solution there.*

Proof. Note that $\beta(\xi)$, $f(\xi) \in C^\infty((0, \infty))$ and $\beta(\xi) > 0$ for $\xi > 0$. Then the proposition follows from the usual parabolic regularization method (see e.g. Ladyzenskaja et al [9]). □

Let $\omega = \{x\,;\, R_1 < r < R_2\}$ for $0 < R_1 < R_2 \leq R$, and let $(s_\omega(r), \lambda_\omega)$ be be the principal eigensolution of $-\Delta_D$ in ω, where s_ω is normalized as $\sup_{x\in\omega} s_\omega(r) = 1$. For some $\bar{t} \in (0, T)$ and $a > 0$ let $\eta = \eta(t)$ be the solution to

$$\eta' = -\lambda_\omega \frac{\eta}{\beta'(\eta)} \text{ in } t > \bar{t} \text{ with } \eta(\bar{t}) = a. \tag{2.2}$$

Integrating this, we have

$$\eta(t) = W^{-1}\{W(a) - \lambda_\omega(t - \bar{t})\}, \text{ where } W(s) = \int_1^s \frac{\beta'(\xi)}{\xi}\, d\xi. \tag{2.3}$$

Since $\beta'(\xi) > 0$ and $\beta''(\xi) \leq 0$ in $\xi > 0$, $W(s)$ is increasing in $s > 0$ and $W(s) \to -\infty$ as $s \downarrow 0$. Thus, we have $\eta(t) > 0$ in $t \geq \bar{t}$.

Lemma 2. 4. *Assume* $u(r, \bar{t}) \geq a > 0$ *in* ω. *Then we have*

$$u(r, t) \geq \eta(t)s_\omega(r) \text{ in } \omega\times(\bar{t}, T). \tag{2.4}$$

Proof. Cf. [7] or [12]. We put $v(r, t) = \eta(t)s_\omega(r)$. Since $0 < s_\omega(r) \leq 1$, we then have $\beta'(\eta) \leq \beta'(v)$ and hence

$$\partial_t\beta(v) = -\frac{\beta'(v)}{\beta'(\eta)}\eta\lambda_\omega s_\omega \leq \Delta v + f(v) \text{ in } \omega\times(\bar{t}, T).$$

This shows that v is a subsolution of (1 . 1) and (1 .19) in $\omega\times(\bar{t}, T)$. Since

$$v(r, \bar{t}) \leq a \leq u(r, \bar{t}) \quad \text{on } \omega \text{ and}$$
$$v(r, t) = 0 \leq u(r, t) \quad \text{on } \partial\omega\times(\bar{t}, T),$$

Lemma 2. 2 (i) shows (2. 4). □

Lemma 2.5. *Assume* (A4). *Then the corresponding solution* u *is nonincreasing in* r *for fixed any* $0 < t < T$. *Moreover,*

$$\partial_r u(r, t) < 0 \text{ in the domain in } (0, R)\times(0, T) \text{ where } u > 0. \tag{2.5}$$

Proof. First consider the Dirichlet problem. In this case a reflection principle (cf.

[4] or [10]) will play an important role. For $0 < \alpha < R$ put $\Omega_\alpha^+ = B(R) \cap \{x = (x_1, x'); x_1 > \alpha\}$ and $\Omega_\alpha^- = \{(x_1, x'); (2\alpha - x_1, x') \in \Omega_\alpha^+\}$. We compare two solutions $u(x, t)$ and $v(x, t) = u(2\alpha - x_1, x', t)$ of (1.1) in $\Omega_\alpha^- \times (0, T)$. Noting $u \geq 0$ in $B(R) \times (0, T)$ and $u = 0$ on $\partial B(R) \times (0, T)$, we have $u \geq v$ on $\partial\Omega_\alpha^- \times (0, T)$. On the other hand, (A4) implies that $u \geq v$ on $\Omega_\alpha^- \times \{0\}$. Thus, it follows from Lemma 2.2 (i) that

$$u(x, t) \geq v(x, t) \text{ in the whole } \Omega_\alpha^- \times (0, T). \tag{2.6}$$

Morever, if $u(\alpha, x', t) > 0$, then the inequality

$$\partial_{x_1} u(\alpha, x', t) < 0 \tag{2.7}$$

follows from a strong maximum principle. (2.6) and (2.7) show the assertions of the lemma.

Next consider the Neumann or Robin problem. In this case we have only to consider the problem in the domain $B(R) \times (\tau, T)$, where $\tau = \sup\{t ; u(R, t) = 0\}$. For our problem is reduced to the above Dirichlet problem in $B(R) \times (0, \tau)$. We note that

$$u(r, t) > 0 \text{ in } [0, R] \times (\tau, T). \tag{2.8}$$

In fact, if $u(\bar{r}, \bar{t}) = 0$ for some $(\bar{r}, \bar{t}) \in (0, R) \times (\tau, t)$, then $u(\bar{r}, t) = 0$ in $(0, \bar{t})$ by Lemma 2.4. Hence, we can apply a uniqueness theorem in $\{\bar{r} < |x| < R\} \times (\tau, t)$ to obtain $u(r, t) = 0$ in $[\bar{r}, R] \times (0, \bar{t})$. This is a contradiction since we have $u(R, t) > 0$ in (τ, T),

Now, (2.8) and Lemmas 2.3 show that $u(r, t)$ is smooth in the whole $[0, R] \times (\tau, T)$. Putting $w = \partial_r u$, we then have the initial-boundary value problem (cf. [3])

$$\begin{aligned}
&\beta'(u)\partial_t w = \partial_r^2 w + \frac{N-1}{r}\partial_r w - \frac{N-1}{r^2} w \\
&\qquad + \{-\beta''(u)\partial_r u + f'(u)\} w, \quad (r, t) \in (0, R) \times (\tau, T), \\
&w(0, t) = w(R, t) + \tilde{\sigma} u(R, t) = 0, \qquad t \in (\tau, T), \\
&w(r, 0) = \partial_r u(r, \tau) \leq 0, \qquad r \in (0, R),
\end{aligned}$$

where $\tilde{\sigma} = 0$ in the Neumann problem and $\tilde{\sigma} = \sigma$ in the Robin problem. Since $\tilde{\sigma} u(R, t) \geq 0$, a maximum principle shows that $w(r, t) = \partial_r u(r, t) < 0$ in $(0, R) \times (\tau, T)$ and the lemma is proved. □

Proof of Theorem 2. Since $\gamma < \alpha$ in (A6), we have $R_\gamma > R$. Let $s_\gamma(r) > 0$ be the principal eigenfuncthon of $-\Delta_D$ in $B(R_\gamma)$ corresponding to γ. Put

$$v_h(r) = h s_\gamma(r) - \frac{C}{\gamma} \text{ for } h > 0.$$

Since $s_\gamma(r) \geq c > 0$ in $B(R)$, noting (A7), we have

(2.9) $\quad v_h(r) \geq u_0(r)$ in $B(R)$ and $V_h(R) \geq b(t)$ on $[0, \infty)$

if we choose $h > 0$ sufficiently large. Moreover, it follows from (A6) that

(2.10) $\quad -\Delta v_h = -h\Delta s_\gamma = \gamma v_h + C \geq f(v_h).$

Hence, v_h is a supersolution of (1.1) and (1.19) in $B(R)$. With (2.9) and (2.10) we can apply Lemma 2.2 (i) to obtain the a-priori estimate

(2.11) $\quad v_h(r) \geq u(r, t)$ for any $r \in [0, R)$ and $t \geq 0$,

which ensures simultaneously the global existence and the boundedness of solutions. □

Proof of Theorem 1. (1) [or (2)] Let $T > 0$ be the blow-up time of u. Contrary to (1.13) [or (1.16)] suppose that $u(\rho, t) \leq M$ in $(0, T)$ for some $0 < \rho < R$ [or R_γ]. Then since λ [or γ] $< \lambda_\rho$, we see that u in $B(\rho)\times(0, T)$ satisfies the conditions of Theorem 2. Thus, u stays bounded in $\bar{B}(\rho)\times[0, T)$. On the other hand, $u(r, t) \leq M$ in $(\rho, R]\times[0, T)$ by Lemma 2.5. Thus, $\sup_{0<\gamma<R} u(r, t)$ remains bounded as $t \uparrow T$. This is a contradiction, and we conclude (1.13) [or (1.16)].

Next, we shall show (1.14). The corresponding result in (2) is similarly proved. Put $0 < \rho < R' < R'' < R$. Since $\{x\, ;\, r=R''\} \subset S_u$, there exist sequences $\{r_m\}$ and $\{t_m\}$ such that

(2.12) $\quad r_m \to R'', \ t_m \uparrow T$ and $a_m = u(r_m, t_m) \to \infty$ as $m \to \infty$.

Without loss of generality, we can assume $\{r_m\} \subset (R', R)$. Since $u(r, t_m) \geq a_m$ in $B(R')$ by Lemma 2.5, we can apply Lemma 2.4 with $\omega = B(R')$, $\bar{t} = t_m$ and

(2.13) $\quad \eta(t) = \eta_m(t) = W^{-1}\{W(a_m) - \lambda_\omega(t-t_m)\}.$

We then have

(2.14) $\quad u(r, t) \geq \eta_m(t)s_\omega(r)$ in $\omega\times(t_m, T)$.

As is shown above, $\eta_m(t) > 0$ and is decreasing in $t > t_m$. Thus, $\eta_m(t) \geq \eta_m(T)$. On the hand, it follows from (2.12) and (2.13) that $\eta_m(T) \to \infty$ as $m \to \infty$. Hence, (2.14) shows (1.14). □

References

[1] D.G. Aronson, M.G. Crandall and L.A. Peletier, Stabilization of solutions of a degenerate nonlinear diffusion problem, Nonlinear Anal. **6** (1982), 1001-1022.

[2] M. Bertsch, R. Kersner and L.A. Peletier, Positivity versus localization in degenerate diffusion equations, Nonlinear Analy. **9** (1985), 987-1008.

[3] Y.-G. Chen, Blow-up solutions of a semilinear parabolic equation with the Neumann and Robin boundary conditions, Hokkaido Univ. Preprint series in Math. # 65, 1989.

[4] A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. **34** (1985), 425-447.

[5] A. Friedman and B. McLeod, Blow-up of solutions of nonlinear degenerate parabolic equations, Arch. Rat. Mech. Analy. **96** (1986), 55-80.

[6] V.A. Galaktionov, Boundary-value problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta}$, Differents. Urab. **21** (1985), 836-842.

[7] T. Imai and K. Mochizuki, On blow-up of solutions for quasilinear degenerate parabolic equations, Publ. RIMS, Kyoto Univ. to appear.

[8] N. Itaya, A note on the blowup-nonblowup problems in nonlinear parabolic equations, Proc. Japan Acad. **55** (1979), 241-244.

[9] O.A. Ladyzenskaja, V.A. Solonokov and N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs, **23** AMS Providence, R.I. 1968.

[10] K. Mochizuki and R. Suzuki, Blow-up sets and asymptotic behavior of interfaces for quasilinear degenerate parabolic equations in R^N, preprint 1990.

[11] O.A. Oleinik, A.S. Kalashnikov and Chzou Yui-Lin, The Cauchy problem and boundary problems for equations of the type of nonlinear filtration, Izv. Akad. Nauk SSSR Ser. Math. **22** (1958), 667-704.

[14] R. Suzuki, On blow-up sets and asymptotic behavior of interfaces of one dimensional quasilinear degenerate parabolic equations, Pub. RIMS, Kyoto Univ. to appear.