# *Global Isomorphisms of Lower Dimensional Lie Groups*

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Among lower dimesional Lie algebras, there exist some overlaps as in the following left side table (Helgason [1] 519-520). Now we give global explicit isomorphisms between linear Lie groups corresponding these Lie algebras. Our results are the following right side table which is a global answer of Helgason [1] 521-522.

| | |
|---|---|
| $\mathfrak{sp}(1) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$, | $Sp(1) \cong SU(2)$, $Sp(1)/\boldsymbol{Z}_2 \cong SO(3)$, |
| $\mathfrak{sp}(1, \boldsymbol{R}) \cong \mathfrak{sl}(1, \boldsymbol{R}) \cong \mathfrak{sp}(1, 1) \cong \mathfrak{so}(2, 1)$, | |
| $Sp(1, \boldsymbol{R}) \cong SL(2, \boldsymbol{R}) \cong SU(1, 1)$, | $Sp(1, \boldsymbol{R})/\boldsymbol{Z}_2 \cong O(2, 1)_0$, |
| $\mathfrak{sp}(1) \times \mathfrak{sp}(1) \cong \mathfrak{so}(4)$, | $(Sp(1) \times Sp(1))/\boldsymbol{Z}_2 \cong SO(4)$, |
| $\mathfrak{sl}(2, \boldsymbol{R}) \times \mathfrak{sl}(2, \boldsymbol{R}) \cong \mathfrak{so}(2, 2)$, | $(Sp(1, \boldsymbol{R}) \times Sp(1, \boldsymbol{R})/\boldsymbol{Z}_2 \cong O(2, 2)_0$, |
| $\mathfrak{sl}(2, \boldsymbol{R}) \times \mathfrak{su}(2) \cong \mathfrak{so}^*(4)$, | $(Sp(1, \boldsymbol{R}) \times Sp(1))/\boldsymbol{Z}_2 \cong SO^*(4)$, |
| $\mathfrak{sl}(2, \boldsymbol{C}) \cong \mathfrak{so}(3, 1)$, | $SL(2, \boldsymbol{C})/\boldsymbol{Z}_2 \cong O(3, 1)_0$, |
| $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, | $Sp(2)/\boldsymbol{Z}_2 \cong SO(5)$, |
| $\mathfrak{sp}(2, \boldsymbol{R}) \cong \mathfrak{so}(2, 3)$, | $Sp(2, \boldsymbol{R})/\boldsymbol{Z}_2 \cong O(2, 3)_0$, |
| $\mathfrak{sp}(1, 1) \cong \mathfrak{so}(4, 1)$, | $Sp(1, 1)/\boldsymbol{Z}_2 \cong O(4, 1)_0$, |
| $\mathfrak{su}(4) \cong \mathfrak{so}(6)$, | $SU(4)/\boldsymbol{Z}_2 \cong SO(6)$, |
| $\mathfrak{sl}(4, \boldsymbol{R}) \cong \mathfrak{so}(3, 3)$, | $SL(4, \boldsymbol{R})/\boldsymbol{Z}_2 \cong O(3, 3)_0$, |
| $\mathfrak{su}(2, 2) \cong \mathfrak{so}(2, 4)$, | $SU(2, 2)/\boldsymbol{Z}_2 \cong O(2, 4)_0$, |
| $\mathfrak{su}^*(4) \cong \mathfrak{so}(1, 5)$, | $SU^*(4)/\boldsymbol{Z}_2 \cong O(1, 5)_0$, |
| $\mathfrak{su}(1, 3) \cong \mathfrak{so}^*(6)$, | $SU(1, 3)/\boldsymbol{Z}_2 \cong SO^*(6)$, |
| $\mathfrak{so}^*(8) \cong \mathfrak{so}(6, 2)$, | ——————— |

where $\boldsymbol{Z}_2 = \{E, -E\}$ in any case.

Almost all of them might be known (e.g. [2]) except the the last several ones, but we will give their proves for all cases.

### 1. Notations and preliminaries

Let $\boldsymbol{R}$, $\boldsymbol{C} = \boldsymbol{R} \oplus \boldsymbol{R}\boldsymbol{i}$, $\boldsymbol{H} = \boldsymbol{R} \oplus \boldsymbol{R}\boldsymbol{i} \oplus \boldsymbol{R}\boldsymbol{j} \oplus \boldsymbol{R}\boldsymbol{k}$ be the fields of real, complex, quaternionic numbers, and $\boldsymbol{C}'$ ($\boldsymbol{i}'^2 = 1$), $\boldsymbol{H}' = \boldsymbol{C} \oplus \boldsymbol{C}\boldsymbol{j}'$ ($\boldsymbol{j}'^2 = 1$) be the algebras of split complex, split quaternionic numbers, respectively. Let $C = \boldsymbol{R}^c$, $\boldsymbol{C}^c$, $\boldsymbol{H}^c$ be the complexifications of $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$, respectively, and their complex conjugations are denoted by $\tau : \tau\,(x + yi) = x - yi$, $x, y \in K = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$.

We arrange the groups used in this paper, although they are familiar.

$SO(n) = \{A \in M(n, \boldsymbol{R}) \mid {}^tAA = E, \det A = 1\}$,

$O(m, n) = \{A \in M(m+n, \boldsymbol{R}) \mid {}^tAI_mA = I_m\}$, $O(m, n)_0$ is the identity connected componect of $O(m, n)$.

$SO^*(2n) = \{A \in M(2n, \boldsymbol{C}) \mid {}^tAA = E, J_nA = \bar{A}J_n, \det A = 1\}$,

$U(n) = \{A \in M(n, \boldsymbol{C}) \mid A^*A = E\}$, $SU(n) = \{A \in U(n) \mid \det A = 1\}$,

$SL(n, K) = \{A \in M(n, K) \mid \det A = 1\}$, $K = \boldsymbol{R}, \boldsymbol{C}$,

$SU(m, n) = \{A \in M(m+n, \boldsymbol{C}) \mid A^*I_mA = I_m, \det A = 1\}$,

$SU^*(2n) = \{A \in M(2n, \boldsymbol{C}) \mid J_nA = \bar{A}J_n, \det A = 1\}$,

$Sp(n) = \{A \in M(n, \boldsymbol{H}) \mid A^*A = E\}$,

$Sp(n, \boldsymbol{R}) = \{A \in M(2n, \boldsymbol{R}) \mid {}^tAJ_nA = J_n\}$,

$Sp(m, n) = \{A \in M(m+n, \boldsymbol{H}) \mid A^*I_mA = I_m\}$

where $I_m = \operatorname{diag}(\underbrace{-1, \cdots, -1}_{m}, 1, \cdots, 1)$, $J_n = \operatorname{diag}(\underbrace{J, \cdots, J}_{n})$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

LEMMA 1. *Define* $SU(n, \boldsymbol{C}') = \{A \in M(n, \boldsymbol{C}') \mid A^*A = E, \det A = 1\}$, $Sp(n, \boldsymbol{H}') = \{A \in M(n, \boldsymbol{H}') \mid A^*A = E\}$. *Then we have*

$$SU(n, \boldsymbol{C}') \cong SL(n, \boldsymbol{R}), \qquad Sp(n, \boldsymbol{H}') \cong Sp(n, \boldsymbol{R}).$$

PROOF. See Yokota [3], Propositions 0.2 and 0.4.

## 2. Explicit global isomorphisms

THEOREM 2. (1) $Sp(1) \cong SU(2)$, $Sp(1)/\boldsymbol{Z}_2 \cong SO(3)$.

(2) $Sp(1, \boldsymbol{R}) = SL(2, \boldsymbol{R}) \cong SU(1, 1)$, $Sp(1, \boldsymbol{R})/\boldsymbol{Z}_2 \cong O(2, 1)_0$.

PROOF. (1) The natural embedding $k : \boldsymbol{H} = \boldsymbol{C} \oplus \boldsymbol{C}\boldsymbol{j} \to M(2, \boldsymbol{C})$, $k(a+b\boldsymbol{j}) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ induces the isomorphism $k : Sp(1) \to SU(2)$. Next let $\boldsymbol{H}_0$ be all pure quaternionic numbers with the norm $N(x) = x\bar{x}$ and put $SO(3) = SO(\boldsymbol{H}_0)$. Then $f : Sp(1) \to SO(3)$, $f(p)x = px\bar{p}$, $x \in \boldsymbol{H}_0$ induces the isomorphism $Sp(1)/\boldsymbol{Z}_2 \cong SO(3)$.

(2) $Sp(1, \boldsymbol{R}) = SL(2, \boldsymbol{R})$ by the definition. The natural embedding $k' : \boldsymbol{H}' = \boldsymbol{C} \oplus \boldsymbol{C}\boldsymbol{j}' \to M(2, \boldsymbol{C})$, $k'(a+b\boldsymbol{j}') = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ induces the isomorphism $k' : Sp(1, \boldsymbol{H}') \to SU(1, 1)$ (note that $Sp(1, \boldsymbol{H}') \cong Sp(1, \boldsymbol{R})$ (Lemma 1)). Next let $\boldsymbol{H}'_0$ be all pure split quaternionic numbers with the norm $N(x) = x\bar{x}$ and put $O(2, 1) = O(\boldsymbol{H}'_0)$. Then $f : Sp(1, \boldsymbol{H}') \to O(2, 1)_0$, $f(p)x = px\bar{p}$, $x \in \boldsymbol{H}'_0$ induces the isomorphism $Sp(1, \boldsymbol{H}')/\boldsymbol{Z}_2 \cong O(2, 1)_0$.

THEOREM 3. (1) $(Sp(1) \times Sp(1))/\boldsymbol{Z}_2 \cong SO(4)$.

(2) $Sp(1, \boldsymbol{R}) \times Sp(1, \boldsymbol{R}))/\boldsymbol{Z}_2 \cong O(2, 2)_0$. (3) $(Sp(1, \boldsymbol{R}) \times Sp(1))/\boldsymbol{Z}_2 \cong SO^*(4)$.

PROOF. (1) Consider $\boldsymbol{H}$ with the norm $N(x) = x\bar{x}$ and put $SO(4) = SO(\boldsymbol{H})$. Then $f : Sp(1) \times Sp(1) \to SO(4)$, $f(p, q)x = px\bar{q}$, $x \in \boldsymbol{H}$ induces the required isomorphism.

(2) Use $\boldsymbol{H}'$ instead of $\boldsymbol{H}$ of (1) and put $O(2, 2) = O(\boldsymbol{H}')$. Then we have the required

isomorphism as similar to (1) (note that $Sp(1, \boldsymbol{H}') \cong Sp(1, \boldsymbol{R})$ (Lemma 1)).

(3) Let $\boldsymbol{H}^c$ be the complexification of $\boldsymbol{H}$ with the norm $N(x) = x\bar{x}$ and the Hermitian inner product $\langle x, y\rangle = (\tau x, \boldsymbol{i}y)$ (where $(x, y)$ is the usual inner product of $\boldsymbol{H}^c$ defined by $\frac{1}{2}(\bar{x}y + \bar{y}x)$) and put $SO^*(4) = \{\alpha \in \mathrm{Iso}_C(\boldsymbol{H}^c) \mid N(\alpha x) = N(x), \langle \alpha x, \alpha y\rangle = \langle x, y\rangle, \det\alpha = 1\}$. (This group $SO^*(4)$ is isomorphic to the ordinary group $SO^*(4)$. In fact, the matrix $A \in M(4, C)$ of $\alpha \in SO^*(4)$ with respect to the basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of $\boldsymbol{H}^c$ satisfies ${}^tAA = E$, $J_2A = (\tau A)J_2$, $\det A = 1$). Since $\{a + b\boldsymbol{i} + c\boldsymbol{ij} + d\boldsymbol{ik} \mid a, b, c, d \in \boldsymbol{R}\}$ is isomorphic to $\boldsymbol{H}'$, we identify these algebras. Then $f : Sp(1, \boldsymbol{H}') \times Sp(1) \to SO^*(4)$, $f(p, q)x = px\bar{q}$, $x \in \boldsymbol{H}^c$ induces the required isomorphism.

THEOREM 4. $SL(2, C)/\boldsymbol{Z}_2 \cong O(3, 1)_0$.

PROOF. Consider the 4-dimesional $\boldsymbol{R}$-vector space $\mathfrak{J}(2, \boldsymbol{C}) = \{X \in M(2, \boldsymbol{C}) \mid X^* = X\} = \{X = \begin{pmatrix} \xi & x \\ \bar{x} & \eta \end{pmatrix} \mid \xi, \eta \in \boldsymbol{R}, x \in \boldsymbol{C}\}$ with the norm $N(X) = \det X = \xi\eta - x\bar{x}$ and put $O(3, 1) = O(\mathfrak{J}(2, \boldsymbol{C}))$. Then $f : SL(2, \boldsymbol{C}) \to O(3, 1)_0$, $f(A)X = AXA^*$, $X \in \mathfrak{J}(2, \boldsymbol{C})$ induces the required isomorphism.

THEOREM 5. (1) $Sp(2)/\boldsymbol{Z}_2 \cong SO(5)$. (2) $Sp(2, \boldsymbol{R})/\boldsymbol{Z}_2 \cong O(2, 3)_0$.
(3) $Sp(1, 1)/\boldsymbol{Z}_2 \cong O(4, 1)_0$.

PROOF. (1) Consider the 5-dimensional $\boldsymbol{R}$-vector space $\mathfrak{J}(2, \boldsymbol{H})_0 = \{X \in M(2, \boldsymbol{H}) \mid X^* = X, \mathrm{tr}(X) = 0\} = \{X = \begin{pmatrix} \xi & x \\ \bar{x} & -\xi \end{pmatrix} \mid \xi \in \boldsymbol{R}, x \in \boldsymbol{H}\}$ with the norm $N(X) = -\det X = \frac{1}{2}\mathrm{tr}(X^2) = \xi^2 + x\bar{x}$ and put $SO(5) = SO(\mathfrak{J}(2, \boldsymbol{H}_0))$. Then $f : Sp(2) \to SO(5)$, $f(A)X = AXA^*$, $X \in \mathfrak{J}(2, \boldsymbol{H})_0$ induces the required isomorphism.

(2) Use $\mathfrak{J}(2, \boldsymbol{H}')_0$ instead of $\mathfrak{J}(2, \boldsymbol{H})_0$ of (1) and put $O(2, 3) = O(\mathfrak{J}(2, \boldsymbol{H}')_0)$. Then we have the required isomorphism as similar to (1) (note that $Sp(2, \boldsymbol{H}') \cong Sp(2, \boldsymbol{R})$(Lemma I)).

(3) Consider the 5-dimensional $\boldsymbol{R}$-vector space $\mathfrak{J}_1(2, \boldsymbol{H})_0 = \{X \in M(2, \boldsymbol{H}) \mid X = I_1XI_1, \mathrm{tr}(X) = 0\} = \{X = \begin{pmatrix} \xi & x \\ -\bar{x} & -\xi \end{pmatrix} \mid \xi \in \boldsymbol{R}, x \in \boldsymbol{H}\}$ with the norm $N(X) = -\det X = \frac{1}{2}\mathrm{tr}(X^2) = \xi^2 - x\bar{x}$ and put $O(4, 1) = O(\mathfrak{J}_1(2, \boldsymbol{H})_0)$. Then $f : Sp(1, 1) \to O(4, 1)_0$, $f(A)X = I_1AI_1XA^*$, $X \in \mathfrak{J}_1(2, \boldsymbol{H})_0$ induces the required isomorphism.

THEOREM 6. (1) $SU(4)/\boldsymbol{Z}_2 \cong SO(6)$. (2) $SL(4, \boldsymbol{R})/\boldsymbol{Z}_2 \cong O(3, 3)_0$.
(2) $SU(2, 2)/\boldsymbol{Z}_2 \cong O(2, 4)_0$. (4) $SU^*(4)/\boldsymbol{Z}_2 \cong O(1, 5)_0$.
(5) $SU(1, 3)/\boldsymbol{Z}_2 \cong SO^*(6)$.

PROOF. (1) Consider the 6-dimensional $\boldsymbol{R}$-vector space

$$\mathfrak{J}(\boldsymbol{C}) = \{X = \begin{pmatrix} 0 & \xi & -b & a \\ -\xi & 0 & -\bar{a} & -\bar{b} \\ b & \bar{a} & 0 & -\bar{\xi} \\ -a & \bar{b} & \bar{\xi} & 0 \end{pmatrix} \mid \xi, a, b \in \boldsymbol{C}\}$$

with the norm $N(X) = -\frac{1}{4}\mathrm{tr}(XX^*) = \xi\bar{\xi} + a\bar{a} + b\bar{b}$ and put $SO(6) = SO(\mathfrak{J}(\boldsymbol{C}))$.

Then $f: SU(4) \to SO(6)$, $f(A)X = AX{}^tA \in \mathfrak{J}(\boldsymbol{C})$ induces the required isomorphism.

(2) Use $\mathfrak{J}(\boldsymbol{C}')$ instead of $\mathfrak{J}(\boldsymbol{C})$ of (1) and put $O(3, 3) = O(\mathfrak{J}(\boldsymbol{C}'))$. Then we have the required isomorphism as similar to (1) (note that $SU(4, \boldsymbol{C}') \cong SL(4, \boldsymbol{R})$ (Lemma 1)).

(3) Consider the 6-dimensional $\boldsymbol{R}$-vector space

$$\mathfrak{J}_2(\boldsymbol{C}) = \{X = \begin{pmatrix} 0 & \xi & -b & a \\ -\xi & 0 & \bar{a} & \bar{b} \\ b & -\bar{a} & 0 & -\bar{\xi} \\ -a & -\bar{b} & \bar{\xi} & 0 \end{pmatrix} \mid \xi, a, b \in \boldsymbol{C}\}$$

with the norm $N(X) = -\dfrac{1}{4}\mathrm{tr}(I_2XI_2X^*) = -\xi\bar{\xi} + a\bar{a} + b\bar{b}$ and put $O(2, 4) = O(\mathfrak{J}_2(\boldsymbol{C}))$. Then $f: SU(2, 2) \to O(2, 4)_0$. $f(A)X = AX{}^tA$, $X \in \mathfrak{J}_2(\boldsymbol{C})$ induces the required isomorphism.

(4) Consider $\mathfrak{J}_1(\boldsymbol{C}) = \mathfrak{J}(\boldsymbol{C})$ with the norm $N(X) = \dfrac{1}{4}\mathrm{tr}(XJ_2XJ_2) = \dfrac{1}{2}(\xi^2 + \bar{\xi}^2) + a\bar{a} + b\bar{b}$ and put $O(1, 5) = O(\mathfrak{J}_1(\boldsymbol{C}))$. Then $f: SU^*(4) \to O(1, 5)_0$, $f(A)X = AX\bar{A}^{-1}$, $X \in \mathfrak{J}_1(\boldsymbol{C})$ induces the required isomorphism.

(5) Let $\mathfrak{J}(\boldsymbol{C})^C$ be the complexification of $\mathfrak{J}(\boldsymbol{C})$ with the norm $N(X) = -\dfrac{1}{4}\mathrm{tr}(XX^*)$ and the Hermitian inner product $\langle X, Y\rangle = \dfrac{1}{4}\mathrm{tr}((\tau X)I_1Y^*I_1)$ and put $SO^*(6) = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}(\boldsymbol{C})^C) \mid N(\alpha X) = N(X), \langle\alpha X, \alpha Y\rangle = \langle X, Y\rangle, \det\alpha = 1\}$. (This group $SO^*(6)$ is isomorphic to the ordinary group $SO^*(6)$. In fact, the matrix $A \in M(6, C)$ of $\alpha \in SO^*(6)$ with respect to the basis

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{J}(\boldsymbol{C})^C$ satisfies ${}^tAA = E$, $J_3A = (\tau A)J_3$, $\det A = 1$). Then $f: SU(1, 3) \to SO^*(6)$, $f(A)X = (\Gamma_1^{-1}A\Gamma_1)X{}^t(\Gamma_1^{-1}A\Gamma_1)$, $X \in \mathfrak{J}(\boldsymbol{C})^C$ (where $\Gamma_1 = \mathrm{diag}(i, 1, 1, 1) \in M(4, C)$) induces the required isomorphism.

### 3. Explicit local isomorphism between $\mathfrak{so}^*(8)$ and $\mathfrak{so}(6, 2)$

The centers of $SO^*(8)$ and $SO(6, 2)$ are both $\boldsymbol{Z}_2$, but the maximal compact subgroups of $SO^*(8)$ and $SO(6, 2)$ are $U(4)$ and $SO(6) \times SO(2)$, respectively. Hence there exists no epimorphism between $SO^*(8)$ and $SO(6, 2)$. So we will only give an isomorphism between their Lie algebras.

THEOREM 7. $\mathfrak{so}^*(8) \cong \mathfrak{so}(6, 2)$.

PROOF. The mapping $f_*: \mathfrak{so}^*(8) \to \mathfrak{so}(6, 2)$,

$$f_* \begin{pmatrix} 0 & -\lambda_1 & \alpha_1 & -\bar{\beta}_1 & \alpha_2 & -\bar{\beta}_2 & \alpha_3 & -\bar{\beta}_3 \\ \lambda_1 & 0 & \beta_1 & \bar{\alpha}_1 & \beta_2 & \bar{\alpha}_2 & \beta_3 & \bar{\alpha}_3 \\ -\alpha_1 & -\beta_1 & 0 & -\lambda_2 & \alpha_4 & -\bar{\beta}_4 & \alpha_5 & -\bar{\beta}_5 \\ \bar{\beta}_1 & -\bar{\alpha}_1 & \lambda_2 & 0 & \beta_4 & \bar{\alpha}_4 & \beta_5 & \bar{\alpha}_5 \\ -\alpha_2 & -\beta_2 & -\alpha_4 & -\beta_4 & 0 & -\lambda_3 & \alpha_6 & -\bar{\beta}_6 \\ \bar{\beta}_2 & -\bar{\alpha}_2 & \bar{\beta}_4 & -\bar{\alpha}_4 & \lambda_3 & 0 & \beta_6 & \bar{\alpha}_6 \\ -\alpha_3 & -\beta_3 & -\alpha_5 & -\beta_5 & -\alpha_6 & -\beta_6 & 0 & -\lambda_4 \\ \bar{\beta}_3 & -\bar{\alpha}_3 & \bar{\beta}_5 & -\bar{\alpha}_5 & \bar{\beta}_6 & -\bar{\alpha}_6 & \lambda_4 & 0 \end{pmatrix}$$

(where $\alpha_k = a_k + p_k i$, $\beta_k = b_k + q_k i$, $a_k$, $b_k$, $p_k$, $q_k$, and $\lambda_k \in \boldsymbol{R}$)

$$= \begin{pmatrix} 0 & -\lambda_{12} & a_{25} & \tilde{b}_{25} & \tilde{a}_{34} & b_{34} & \tilde{q}_{16} & -\tilde{p}_{16} \\ \lambda_{12} & 0 & b_{25} & -\tilde{a}_{25} & \tilde{b}_{34} & -a_{34} & -p_{16} & -q_{16} \\ -a_{25} & -b_{25} & 0 & -\lambda_{14} & -\tilde{a}_{16} & -\tilde{b}_{16} & \tilde{q}_{34} & -\tilde{p}_{34} \\ -\tilde{b}_{25} & \tilde{a}_{25} & \lambda_{14} & 0 & -b_{16} & a_{16} & -p_{34} & -q_{34} \\ -\tilde{a}_{34} & -\tilde{b}_{34} & \tilde{a}_{16} & b_{16} & 0 & -\lambda_{13} & -q_{25} & p_{25} \\ -b_{34} & a_{34} & \tilde{b}_{16} & -a_{16} & \lambda_{13} & 0 & \tilde{p}_{25} & \tilde{q}_{25} \\ \tilde{q}_{16} & -p_{16} & \tilde{q}_{34} & -p_{34} & -q_{25} & \tilde{p}_{25} & 0 & -\lambda/2 \\ -\tilde{p}_{16} & -q_{16} & -\tilde{p}_{34} & -q_{34} & p_{25} & \tilde{q}_{25} & \lambda/2 & 0 \end{pmatrix}$$

(where $a_{k\ell} = a_k + a_\ell$, $\tilde{a}_{k\ell} = a_k - a_\ell$ etc., and $\lambda_{k\ell} = (\lambda_k + \lambda_\ell) - \lambda/2$, $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$) is an isomorphism.

## References


[1] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, 1978.

[2] I. Yokota, Explicit isomorphism between $SU(4)$ and $Spin(6)$, J. Fac. Sci., Shinshu Univ., **14** (1979), 30-34.

[3] I. Yokota, Realizations of involutive automorphims $\sigma$ and $G^\sigma$ of exceptional linear Lie groups $G$, part I, $G = G_2$, $F_4$ and $E_6$, Tsukuba J. Math., **14** (1990), 185-223.