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A class of time-dependent solutions of the SU (2) gauge theory with spherical symmetry

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Abstract

A method proposed by Ikeda, Maekawa and the present author to look for static solutions of the *SU* (2) gauge theory is applied to get nonstatic solutions with spherical symmetry, and a class of time-dependent solutions are obtained in a systematic way for a special type of field strength. The solutions are not self-dual and contain those obtained previously by several authors.

Introduction. Recent developments in high energy physics teach us that the gauge principle plays a fundamental role in the laws of nature. In order to understand the structure of the so-called elementary particles and various interactions among them, it is important to investigate physical contents of the non-Abelian gauge theory. Especially, a deep knowledge on classical configurations of the field will give us an important clue to its quantum mechanical behaviour.

In a previous paper [1], we proposed a method to classify static and spherically symmetric solutions of the Minkowski *SU*(2) gauge theory according to the types of field strength in a systematic way. The method can naturally be applied also to the nonstatic case. In this note we present a class of time-dependent solutions which are obtained in the course of investigations along this line of reasoning. Our solutions are not self-dual and contain those obtained previously by several authors.

General formulation. The field strength $F^a{}_{\mu\nu}$ and the potential $B^a{}_\mu$ in Yang-Mills theory are related by

$$D^\nu F^a{}_{\mu\nu} \equiv \partial^\nu F^a{}_{\mu\nu} + \varepsilon_{abc} B^{b\nu} F^c{}_{\mu\nu} = 0, \tag{1}$$

$$F^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \varepsilon_{abc} B^b{}_\mu B^c{}_\nu. \tag{2}$$

The most general form of a spherically symmetric field is given by [1], [2], [3]

$$B_0{}^a = \frac{x^a}{r} B,$$
$$B^a{}_i = \varepsilon_{aij} \frac{x^j}{r^2}(1+A) + \left(\delta_{ai} - \frac{x_a x_i}{r^2}\right) C/r + \frac{x_a x_i}{r^2} D. \tag{3}$$

$$F^a{}_{0i} = \varepsilon_{aij} \frac{x^j}{r^2} f_1 + \left(\delta_{ai} - \frac{x_a x_i}{r^2}\right) f_2/r + \frac{x_a x_i}{r^2} f_3,$$
$$F^a{}_{ij} = \varepsilon_{ijk} \left[\varepsilon_{akl} \frac{x^l}{r^2} g_1 + \left(\delta_{ak} - \frac{x_a x_k}{r^2}\right) g_2/r + \frac{x_a x_k}{r^2} g_3\right]. \tag{4}$$

Here A, B, C, D, f_i and g_i ($i=1, 2, 3$) are functions of r and t.

Substituting (3) and (4) into (1), we can rewrite the field equation (1) in the following form:

$$\begin{aligned} &\partial_r(r^2 f_3) = 2(Cf_1 - Af_2), \\ &\partial_0(r^2 f_3) = -2(Cg_2 + Ag_1), \\ &\partial_0 f_1 + \partial_r g_2 = Bf_2 - Dg_1 - Ag_3, \\ &\partial_0 f_2 - \partial_r g_1 = -Bf_1 - Dg_2 - Cg_3. \end{aligned} \tag{5}$$

Also we see from (2) that f's and g's can be written in terms of A, B, C, D

$$\begin{aligned} &f_1 = \partial_0 A - BC, \quad g_1 = \partial_r C + AD, \\ &f_2 = \partial_0 C + AB, \quad g_2 = -\partial_r A + CD, \\ &f_3 = \partial_0 D - \partial_r B, \quad g_3 = (A^2 + C^2 - 1)/r^2. \end{aligned} \tag{6}$$

Special type of solutions. We look for a solution whose field strength F belongs to the type $f_1 g_2 g_3 \neq 0$, $f_2 = f_3 = g_1 = 0$. It is easy to find from (5) and (6) that the only non-vanishing component of the potential in this case is $A(r,t)$ and that the field strengths are given by

$$f_1 = \partial_0 A, \quad g_2 = -\partial_r A, \quad g_3 = (A^2-1)/r^2. \tag{7}$$

Here $A(r,t)$ is a solution of the equation

$$r^2(\partial^2{}_0 A - \partial^2{}_r A) = A(1-A^2). \tag{8}$$

We consider that the solution $A(r,t)$ depends on r and t through a combination $\rho = \rho(r,t)$. Then, Eq. (8) reads

$$r^2 \left\{(\ddot{\rho} - \rho'') \frac{dA}{d\rho} + (\dot{\rho}^2 - \rho'^2) \frac{d^2 A}{d\rho^2}\right\} = A(1-A^2), \tag{9}$$

where a dot and a prime denote differentiation with respect to t and r, respectively.

Now we require that ρ is a function of r and t such that Eq. (9) can be expressed in terms of a single variable ρ. In other words, $r^2(\ddot{\rho}-\rho'')$ and $r^2(\dot{\rho}^2-\rho'^2)$ must be some functions of ρ. We tried to look for this possibility by assuming the form $\rho=F(r,t)/G(r,t)$, where F and G are polynomials of r and t. Two cases will be discussed below.

I $\rho\ (r,t)=r/(t+a)$, where a is a constant.

In this case we obtain

$$r^2(\ddot{\rho}-\rho'')=2\rho^3,\quad r^2(\dot{\rho}^2-\rho'^2)=\rho^4-\rho^2.$$

Then Eq. (9) can be written

$$2\rho^3\frac{dA}{d\rho}+(\rho^4-\rho^2)\frac{d^2A}{d\rho^2}=A(1-A^2), \tag{10}$$

which is the equation derived by Arodz [4]. It is also a special case of the equation derived by Babu Joseph and Baby [5] (for a vanishing Higgs field). Using a variable $\tau=\frac{1}{\rho}-1$, we get

$$\frac{d}{d\tau}\{(2+\tau)\tau\frac{dA}{d\tau}\}=A(A^2-1). \tag{11}$$

The regular solution of Eq. (11) has been analysed in Ref. [4]. We shall discuss the solution of Eq. (10) in the next paragraph.

II $\rho=r/(t^2-r^2+at+b)$, where a and b are constants.

In this case we get after some calculation the following equation for A:

$$-8\alpha\rho^3\frac{dA}{d\rho}+(-\rho^2-4\alpha\rho^4)\frac{d^2A}{d\rho^2}=A(1-A^2), \tag{12}$$

where $\alpha=b-a^2/4$.

(i) If $\alpha=0$, Eq. (12) is reduced to

$$-\rho^2\frac{d^2A}{d\rho^2}=A(1-A^2). \tag{13}$$

We notice that Eq. (13) has the same form as the one considered by Wu and Yang [6]. We remark that the Wu-Yang monopole is a constant solution $A=0$, while a nonstatic solution in which we are interested here must be a non-constant solution, whose asymptotic behaviour has been analysed numerically in Ref. [6].

(ii) For a nonvanishing value of α, we can rewrite Eq. (12) by rescaling the variable ρ.

For $\alpha>0$, we put $\xi=2\alpha^{1/2}\rho$, and obtain

$$-2\xi^3\frac{dA}{d\xi}-(\xi^2+\xi^4)\frac{d^2A}{d\xi^2}=A(1-A^2). \tag{14}$$

Using a variable $\xi=\tan\theta$, we get

$$-\sin^2\theta\frac{d^2A}{d\theta^2}=A(1-A^2), \tag{15}$$

which has a solution

$$A=\pm\cos\theta=\pm(1+\xi^2)^{-1/2}. \tag{16}$$

For $\alpha<0$, we put $\xi=2|\alpha|^{1/2}\rho$, and obtain

$$2\xi^3\frac{dA}{d\xi}+(-\xi^2+\xi^4)\frac{d^2A}{d\xi^2}=A(1-\mathrm{A}^2), \tag{17}$$

which has the same form as Eq. (10). Using a variable $\xi=\tanh\theta$, we get

$$-\sinh^2\theta\frac{d^2A}{d\theta^2}=A(1-A^2), \tag{18}$$

which has a solution

$$A=\pm\cosh\theta=\pm(1-\xi^2)^{-1/2} \tag{19}$$

Thus the solution of Eq. (10) is given by

$$A=\pm t(t^2-r^2)^{-1/2}. \tag{20}$$

Taking the definition of ξ into account, we see that the two solutions (16) and (19) of Eq. (12) can be put into the following single form:

$$A=\pm(t^2-r^2+\alpha)\ [(t^2-r^2+\alpha)^2+4\alpha r^2]^{-1/2}. \tag{21}$$

Results and discussions. We now get a class of time-dependent solutions of the SU (2) gauge theory for a special type of field strength. The results are summarized below. (Hereafter we take the constant a in the definintion of ρ to be zero.)

$$B^a{}_0=0,\ \ B^a{}_i=\varepsilon_{aij}\frac{x^j}{r^2}(1+A(r,t)),$$

$$E^a{}_i = F^a{}_{0i} = \varepsilon_{aij}\frac{x^j}{r^2} f_1, \tag{22}$$

$$H^a{}_i = \frac{1}{2}\varepsilon_{ijk}F^{ajk} = (\delta_{ai} - \frac{x_a x_i}{r^2})\, g_2/r + \frac{x_a x_i}{r^2} g_3.$$

Solution 1. This solution is singular on the light cone.

$$\begin{aligned} A &= \pm t(t^2-r^2)^{-1/2}, \\ f_1 &= \mp r^2(t^2-r^2)^{-3/2}, \\ g_2 &= \mp rt(t^2-r^2)^{-3/2}, \\ g_3 &= (t^2-r^2)^{-1}. \end{aligned} \tag{23}$$

Solution 2. The potential A in this case is a non-constant solution of the equation

$$-\rho^2\frac{d^2A}{d\rho^2} = A(1-A^2),$$

where

$$\rho = r/(t^2-r^2).$$

It has the following asymptotic behaviour [6]:

$\rho \to \infty$: $A = \pm(1+c/\rho) + O(1/\rho^2)$, where c is a constant,

$\rho \to 0$: $A \to 0$ as oscillatory functions of ρ with minima and maxima $= O(\rho^{1/2})$. Notice that for a fixed t (say $t=0$), $\rho \to \infty$ and $\rho \to 0$ correspond to $r \to 0$ and $r \to \infty$ respectively.

Solution 3. For a non-vanishing value of α, we have

$$\begin{aligned} A &= \pm(t^2-r^2+\alpha)\,[(t^2-r^2+\alpha)^2+4\alpha r^2]^{-1/2}, \\ f_1 &= \pm 8\alpha r^2 t\,[(t^2-r^2+\alpha)^2+4\alpha r^2]^{-3/2}, \\ g_2 &= \pm 4\alpha r(t^2+r^2+\alpha)[(t^2-r^2+\alpha)^2+4\alpha r^2]^{-3/2}, \\ g_3 &= 4\alpha\,[(t^2-r^2+\alpha)^2+4\alpha r^2]^{-1}. \end{aligned} \tag{24}$$

The solution with $\alpha > 0$ is regular, while the one with $\alpha < 0$ is singular. For $\alpha = 1$, the solution is reduced to

$$A = \pm(1+t^2-r^2)\,[(1+t^2-r^2)^2+4r^2]^{-1/2}, \tag{25}$$

which is a solution given by Actor [7].

Our solutions have both "electric" and "magnetic" components. Notice, however, that the radial component of the field is purely magnetic and is given by g_3. In our formulation, the solutions are classified according to the type of field strength. Under a gauge transformation $U=\exp\left[\frac{i}{2}\frac{x_a}{r}\sigma^a\theta(r,t)\right]$ which preserves the spherical symmetry of the field, components of F transform simply as

$$\begin{pmatrix} f'_1 \\ f'_2 \\ f'_3 \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{26}$$

and the same formula for g's. Notice that f_3 and g_3 remain invariant, thus the relation of various solutions with respect to a gauge transformation is manifest in our approach.

Our method can be applied to get solutions for other types of field strength. Investigations covering all cases are now in progress, and results will be published elsewhere.

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