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A Note on the Jacobian Conjecture in Two Variables

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Let $k[x, y]$ be the polynomial ring over a field k of characteristic zero. If f and g are polynomials in $k[x, y]$ then we denote by $[f, g]$ the jacobian of (f, g), that is, $[f, g]=(\partial f/\partial x)(\partial g/\partial y)-(\partial f/\partial y)(\partial g/\partial x)$.

The jacobian conjecture says (see [2]) that if $[f, g]$ is a non-zero constant then $k[x, y]=k[f, g]$. In this note we shall show that the jacobian conjecture is equivalent to the following

Proposition. *Let f and g be polynomials in $k[x, y]$. If $[f, g]$ is a non-zero constant then the ring $k[x, y]/(f)$ is isomorphic to $k[t]$, the ring of polynomials in one varaible over k.*

It is clear that if the jacobian conjecture is true then the proposition is true. For the proof of the converse we shall use the following two facts:

Theorem 1. *Let f and g be polynomials in $k[x, y]$ and assume that the ring $k[x, y]/(f)$ is isomorphic to $k[t]$. Then there exists a polynomial h in $k[x, y]$ such that $k[f, h]=k[x, y]$.*

Proof. See [1] and [3].

Theorem 2. *Let f and g be polynomials in $k[x, y]$. Assume that $[f, g]$ is a non-zero constant and assume that there exists a polynomial h in $k[x, y]$ such that $k[f, h]=k[x, y]$. Then $k[f, g]=k[x, y]$.*

Proof. Let α be a k-endomorphism of $k[x, y]$ such that $\alpha(x)=f$ and $\alpha(y)=h$. Since $k[f, h]=k[x, y]$, α is a k-automorphism of $k[x, y]$. Denote $\beta=\alpha^{-1}$. Then $[\beta(f), \beta(g)]$ is a non-zero constant and $\beta(f)=x$. Hence $k[\beta(f), \beta(g)]=k[x, y]$, because it is easy to verify that the jacobian conjecture is true in the case when one of degrees of the polynomials is equal to one. Therefore we have

$$k[f, g]=\beta^{-1}\beta k[f, g]=\beta^{-1}(k[\beta(f), \beta(g)]=\beta^{-1}(k[x, y])=k[x, y].$$

Now we may prove that if the proposition is valid then the jacobian conjecture is true. In fact, assume that $f, g\in k[x, g]$, $[f, g]\in k-\{0\}$ and assume that the proposition is valid. Then the ring $k[x, y]/(f)$ is isomorphic to $k[t]$ and so, by Theorem 1, $k[f, h]=k[x, y]$, for some $h\in k[x, y]$ and hence, by Theorem 2, $k[f, g]=k[x, y]$.

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