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Irreducible Decomposition of the U_{n+m}-module of (r,s)-forms on the Complex Grassmann Manifold $G_{n+m,n}$

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On the complex Grassmann manifold $SU_{n+m}/S(U_n\times U_m)$, irreducible decompositions of SU_{n+m}-modules of complex valued functions and 1-forms are calculated ([7], [5]). For the complex projective space $SU_{n+1}/S(U_n\times U_1)$, irreducible decompositons of the SU_{n+1}-module of (r, s)-forms are calculated ([2]). In [2] making use of the irreducible decomposition, spectra (eigenvalues of the Laplacian and their multiplicities) are also determined.

In this paper we represent the complex Grassmann manifold M as $U_{n+m}/U_n\times U_m$, and we try to calculate irreducible decompositions of the U_{n+m}-modules of (r, s)-forms. Our problem is reduced into the following two: What kind of an irreducible $U_n\times U_m$-module appears in the exterior product of the (complexified) isotropy representation ? How the U_{n+m}-module induced from the answer of above is decomposed into irreducible U_{n+m}-modules ? The main result of this paper is to show these two problems are solved by the use of the (L-R) rule (Littlewood-Richardson rule [4]). As an example we derive the results mentioned at the begining more accurately. At the end of this paper we compute the Hodge numbers making use of the answer of the first problem.

In appendix, we present a computer program for the calculation of the (L-R) rule due to Dr. H. Kamiya. I would like to thank Dr. H. Kamiya for the making of this program and permission of its insertion in this paper.

1. Notations and preliminaries.

Let G be U_{n+m} and K be $U_n\times U_m$. As in [2], $C^\infty(\Lambda^{r,s}M)$ denotes the U_{n+m}-module of smooth (r, s)-forms on $M=G/K$. We denote

$$\mathfrak{m}=\left\{\begin{pmatrix}0 & -{}^t\bar{A}\\ A & 0\end{pmatrix}\;\middle|\; A\in M(m, n, \mathbf{C})\right\},$$

$$J\begin{pmatrix}0 & -{}^t\bar{A}\\ A & 0\end{pmatrix}=\begin{pmatrix}0 & \sqrt{-1}\,{}^t\bar{A}\\ \sqrt{-1}A & 0\end{pmatrix},\quad \mathfrak{m}^\pm=\{X\in\mathfrak{m}^{\mathbf{C}}\mid J^{\mathbf{C}}X=\pm iX\}$$

where "i" is the imaginary unit for the complexification, and

$$\Lambda^{r,s}=\underbrace{\mathfrak{m}^-\wedge\cdots\wedge\mathfrak{m}^-}_{r}\wedge\underbrace{\mathfrak{m}^+\wedge\cdots\wedge\mathfrak{m}^+}_{s},$$

$$C^\infty(G,K,\Lambda^{r,s})=\{f\in C^\infty(G,\ \Lambda^{r,s})\,|\,f(gk)=k^{-1}\cdot f(g)\ \text{for any}\ g\in G,\ k\in K\}.$$

Here $\Lambda^{r,s}$ is a K-module, and we identify the induced G-module $C^\infty(G,\ K,\ \Lambda^{r,s})$ with $C^\infty(\Lambda^{r,s}M)$.

From now on, for the sake of later sections we give a brief review having connections with restricted Young diagrams and representations of a unitary group U_ℓ. We identify the follwing set with Young diagrams whose "depth" are not more than ℓ,

$$Y_\ell=\{\lambda\in \mathbf{Z}^\ell\mid \lambda_1\geq\cdots\geq\lambda_\ell\geq 0\}\ \ (\ell\in\mathbf{N}).$$

For a nonnegative integer p let $Y_\ell^{(p)}=\{\lambda\in Y_\ell\mid \lambda_1+\cdots+\lambda_\ell=p\}$, then $\lambda\in Y_\ell^{(p)}$ is identified with a diagram consisted of p "squares". The Young diagram λ ($p\neq 0$) whose squares are labeled with figures from 1 to p is called a tableau on λ, and we denote it by B. In particular B is called a "standard tableau" if the figures in its each row and column are in increasing order. For a given B a Young symmetrizer C_B and $\hat{C}_B$ are determined as as follows;

$$H_B=\sum_{\sigma\in\mathfrak{H}_B}\sigma,\ \ K_B=\sum_{\sigma\in\mathfrak{K}_B}(\mathrm{sgn}\ \sigma)\ \sigma,\ \ C_B=H_BK_B,\ \ \hat{C}_B=K_BH_B.$$

Here $\mathfrak{H}_B$ (resp. $\mathfrak{K}_B$) is the subgroup of the symmetric group $\mathfrak{S}_p$ which preserves the sets of figures in each row (resp. column). For $\hat{C}_B$ we can use the following

Lemma 1. 1 ([3]). (1) *If* $\lambda\neq\lambda'$ $(\lambda,\lambda'\in Y_\ell^{(p)})$ *and* B, B' *are tableaux on* λ, λ' *respectively, then we have* $\hat{C}_B\hat{C}_{B'}=0$.
(2) *There is a positive number* q *such that* $(\hat{C}_B)^2=q\ \hat{C}_B$.

Next $T=\mathbf{C}^\ell\otimes\cdots\otimes\mathbf{C}^\ell$ becomes a U_ℓ-module canonically, and a $\mathfrak{S}_p$-module by the action

$$\sigma(v_1\otimes\cdots\otimes v_p)=v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(p)}\ \text{for}\ \sigma\in\mathfrak{S}_p.$$

Since the two actions are commutative a representation $U_\ell\longrightarrow GL\ (\hat{C}_BT)$ is determined.

Remark. We choose a maximal torus as diag $(\varepsilon_1,\cdots,\varepsilon_\ell)$, $(\varepsilon_1=e^{2\pi\sqrt{-1}x_1},\cdots,\varepsilon_\ell=e^{2\pi\sqrt{-1}x_\ell})$. The charecter of this representation is given by the Schur function $\{\lambda\}$ whose variables are $\varepsilon_1,\cdots,\varepsilon_\ell$. On the other hand $\{\lambda\}$ is the character of the irreducible representation whose highest weight is $\lambda_1x_1+\cdots+\lambda_\ell x_\ell$. (We take a natural fundamental Weyl chamber of U_ℓ and "highest weight" means the weight which takes the maximal value on the closure of the fundamental Weyl chamber, see [1]).

By the above remark we have the following

Proposition 1. 2 (Weyl [3]). *T is decomposed into the irreducible U_ℓ-modules $\hat{C}_B T$, namely*

$$T = \bigoplus_{\lambda \in Y_\ell^{(p)}} \Big(\bigoplus_{B \in \mathfrak{B}(\lambda)} \hat{C}_B T \Big),$$

where $\mathfrak{B}(\lambda)$ denotes the set of all standard tableaux on λ.

Let ρ_λ be the irreducible representation, and ρ_0 the trivial representation ($0 = (\underbrace{0, \cdots, 0}_{\ell})$). For given λ, $\lambda' \in Y_\ell$ we can calculate the irreducible decomposition of $\rho_\lambda \otimes \rho_{\lambda'}$ by the following

Proposition 1. 3 ([6]). $\{\lambda\}\{\lambda'\} = \sum_{\lambda'' \in Y_\ell} g_{\lambda\lambda'\lambda''} \{\lambda''\}$.

Here $g_{\lambda\lambda'\lambda''}$ is the number of the Young diagram λ'' that can be built by adding to the Young diagram λ, λ'_1 squares with figures 1, $\cdots$, *λ'_ℓ squares with figures ℓ, subject to two conditions* : (a) *After the addition of each squares with identical figures, we must have Young diagram with no two identical figures in the same column.* (b) *If the total set of added figures is read from the right to the left and from the top to the bottom, we get following inequalities at all squares.*

(*the number of appeared* 1) $\geq \cdots \geq$ (*the number of appeared* ℓ).

The above algorithm is called the (L-R) rule ([4]). We denote this proposition by the equation

$$\lambda \otimes \lambda' = \bigoplus_{\lambda'' \in Y_\ell} g_{\lambda\lambda'\lambda''} \lambda''.$$

Then we close this section with following definitions.

$\lambda = (\lambda_1, \cdots, \lambda_{n+m})$, $\mu = (\mu_1, \cdots, \mu_n)$, $\nu = (\nu_1, \cdots, \nu_m)$: sets of integers which are in decreasing order,

$(\rho_\lambda, V_\lambda)$ (resp. (ρ_μ, V_μ), (ρ_ν, V_ν)) : the irreducible representation of G (resp. U_n, U_m) whose highest weight is $\lambda = \sum_{i=1}^{n+m} \lambda_i x_i$ (resp. $\mu = \sum_{i=1}^{n} \mu_i x_i$, $\nu = \sum_{i=1}^{m} \nu_i x_{n+i}$),

$* : \mathbf{Z}^\ell \ni h \longrightarrow *h = (-h_\ell, \cdots, -h_1) \in \mathbf{Z}^\ell$,

$\pm e : \mathbf{Z}^\ell \ni h \longrightarrow h \pm e = (h_1 \pm e, \cdots, h_\ell \pm e) \in \mathbf{Z}^\ell$ $(e \in \mathbf{Z})$,

$Y_{n,m}^{(r)} = \{\mu \in Y_n^{(r)} \mid \mu_1 \leq m\}$, $Y_{m,n}^{(r)} = \{\nu \in Y_m^{(r)} \mid \nu_1 \leq n\}$,

${}^t : Y_{n,m}^{(r)} \ni \mu \longrightarrow {}^t\mu \in Y_{m,n}^{(r)}$, $\mathfrak{B}(\mu) \ni B \longrightarrow {}^tB \in \mathfrak{B}({}^t\mu)$

where "$_t$" means the reflection in the leading diagonal.

2. Irreducible decomposition of the K-module $A^{r,s}$.

For the U_n-module V_1 and U_m-module V_2 let $V_1 \times V_2$ be the direct product of them (the representation of K whose representation module is $V_1 \otimes V_2$). As a fundametal Weyl chamber of K we choose the direct product of U_n's one and U_m's one. So a dominant integral form of K is expressed as $\mu+\nu$. Instead of the integral forms $\{x_{n+j} \mid j=1, \cdots, m\}$ we use $\{y_j = -x_{n+m-j+1} \mid j=1, \cdots, m\}$ in this section.

Lemma 2. 1. *For $\mu \in Y_{n,m}^{(r)}$ let ω_μ be $\mu+*{}^t\mu$. Then ω_μ is a weight of $\Lambda^{r,0}$ and its multiplicity is one.*

Proof. From the definition $\Lambda^{1,0}$ is equivalent with $\mathbf{C}^n \times \mathbf{C}^{m*}$. So the weights of $\Lambda^{1,0}$ are $\{x_i+y_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

x_1+y_1	x_1+y_2	$\cdots\cdots$	x_1+y_m
x_2+y_1	x_2+y_2	$\cdots\cdots$	x_2+y_m
$\vdots$	$\vdots$		$\vdots$
x_n+y_1	x_n+y_2	$\cdots\cdots$	x_n+y_m

Since these multiplicities are one, the weights of $\Lambda^{r,0}$ are obtained by adding different r squares chosen from the right figure. Let W be any shape of r squares which determines ω_μ, then

$\mu_i = \sharp$ {squares of W in the i-th row}.

So there is no square of W in the i-th row (${}^t\mu_1+1 \leq i \leq n$). On the other hand ${}^t\mu_1 = \sharp$ {squares of W in the first column}. Therefore in the first column of W there is no hole from the first row to the ${}^t\mu_1$-th row. So, as for the j-th column $(2 \leq j \leq m)$ there is no square of W in the i-th row (${}^t\mu_2+1 \leq i \leq {}^t\mu_1$). On the other hand ${}^t\mu_2 = \sharp$ {squares of W in the second column}. Therefore in the second column of W there is no hole of W from the first row to the ${}^t\mu_2$-th row. After the analogous steps we can see W must be the shape of μ. Since the multiplicity is equal to the number of W, we have Lemma 2. 1.

We set

$$T = \underbrace{\mathbf{C}^n \otimes \cdots \otimes \mathbf{C}^n}_{r} \ (U_n\text{-module}), \quad T^* = \underbrace{\mathbf{C}^{m*} \otimes \cdots \otimes \mathbf{C}^{m*}}_{r} \ (U_m\text{-module}),$$

$$\varphi : \underbrace{(\mathbf{C}^n \times \mathbf{C}^{m*}) \otimes \cdots \otimes (\mathbf{C}^n \times \mathbf{C}^{m*})}_{r} \longrightarrow T \times T^*$$

(canonical identification as K-module).

The action of $\mathfrak{S}_r$ induced by φ is as follows;

$$\sigma \cdot (v \otimes \xi) = (\sigma v) \otimes (\sigma \xi) \text{ for } \sigma \in \mathfrak{S}_r, \ v \in T \text{ and } \xi \in T^*.$$

Theorem 2. 2. *Let A_r be $\sum_{\sigma \in \mathfrak{S}_r} (\operatorname{sgn} \sigma)\sigma$, then we have the irreducible decomposition*

$$\varphi\{\Lambda^r(\mathbf{C}^n \otimes \mathbf{C}^{m*})\} = \bigoplus_{\mu \in Y_{n,m}^{(r)}} A_r \cdot (\hat{C}_B T \times \hat{C}^t{}_B T^*),$$

where B is a tableau on μ. Then $A_r \cdot (\hat{C}_B T \times \hat{C}^t{}_B T^*)$ is independent of the choice of B, and is equivalent with $V_\mu \times V_{*{}^t\mu}$.

Proof. Let ℓ be n in Proposition 1. 2, then we have the irreducible decomposition

$$T = \bigoplus_{\mu \in Y_n^{(r)}} \Big(\bigoplus_{B \in \mathfrak{B}(\mu)} \hat{C}_B T \Big).$$

We have the analogous irreducible decomposition for $\ell = m$ as U_m-module. Let us consider its contragredient representation, then we have

$$T^* = \bigoplus_{\nu \in Y_m^{(r)}} \Big(\bigoplus_{B' \in \mathfrak{B}(\nu)} \hat{C}_{B'} T^* \Big).$$

Considering the equation $\varphi\{\Lambda^r(\mathbf{C}^n \times \mathbf{C}^{m*})\} = A_r \cdot (T \times T^*)$, let us examine

$$A_r \cdot (\hat{C}_B T \times \hat{C}_{B'} T^*) \quad (B \in \mathfrak{B}(\mu),\ B' \in \mathfrak{B}(\nu)).$$

Since $A_r = (\mathrm{sgn}\ \sigma) A_r \sigma^{-1}$, $\mathfrak{K}_B = \mathfrak{H}^t{}_B$ and $\mathfrak{H}_B = \mathfrak{K}^t{}_B$, we have

$$\begin{aligned}
A_r \cdot (\hat{C}_B v \otimes \hat{C}_{B'} \xi) &= \sum_{\sigma \in \mathfrak{K}_B} A_r \cdot ((\mathrm{sgn}\ \sigma) \sigma H_B v \otimes \hat{C}_{B'} \xi) \\
&= \sum_{\sigma \in \mathfrak{H}^t{}_B} A_r \cdot (H_B v \otimes \sigma^{-1} \hat{C}_{B'} \xi) \\
&= A_r \cdot (H_B v \otimes H^t{}_B \hat{C}_{B'} \xi) \\
&= A_r \cdot (v \otimes K^t{}_B H^t{}_B \hat{C}_{B'} \xi).
\end{aligned}$$

Here $K^t{}_B H^t{}_B$ is $\hat{C}^t{}_B$. If $\nu \neq {}^t\mu$, by Lemma 1. 1 (1) we have

$$A_r \cdot (\hat{C}_B T \otimes \hat{C}_{B'} T^*) = \{0\}.$$

If $\nu = {}^t\mu$ and it is not $\{0\}$, we have

$$A_r \cdot (\hat{C}_B T \otimes \hat{C}_{B'} T^*) \simeq V_\mu \times V_{*}{}^t{}_\mu \ (\mu \in Y_{n,m}^{(r)}),$$

because the map $v \otimes \xi \longrightarrow A_r \cdot (v \otimes \xi)$ is a K-homomorphism. Hence

$$A_r \cdot (T \times T^*) = \bigoplus_{\mu \in Y_{n,m}^{(r)}} \Big\{ \sum_{\substack{B \in \mathfrak{B}(\mu) \\ B' \in \mathfrak{B}({}^t\mu)}} A_r \cdot (\hat{C}_B T \times \hat{C}_{B'} T^*) \Big\}. \tag{i}$$

Here we can see above $\{\ \ \}$ consists of just one non-zero term by Lemma 2. 1. To give a non-zero term let us examine the following

$$A_r \cdot (\hat{C}_{B_0} T \otimes \hat{C}^t{}_{B_0} T^*) \qquad \left(B_0 = \begin{array}{|l|}
\hline
1 \quad \cdots \quad \mu_1 \\
\hline
\mu_1+1 \quad \cdots \quad \mu_1+\mu_2 \\
\hline
\cdots \\
\hline
\underset{\substack{\uparrow \\ \mu_1+\cdots+\mu_{n-1}+1}}{} \quad \cdots \quad r \\
\hline
\end{array} \right).$$

By Lemma 1. 1 (2) we have

$$\begin{aligned}
A_r \cdot (\hat{C}_{B_0} T \times \hat{C}^t{}_{B_0} T^*) &= A_r \cdot (T \times \hat{C}^t{}_{B_0} \hat{C}^t{}_{B_0} T^*) \\
&= A_r \cdot (T \times \hat{C}^t{}_{B_0} T^*) \\
&= A_r \cdot (H_{B_0} T \times H^t{}_{B_0} T^*).
\end{aligned}$$

Using the natural basis $\{v_i \mid 1 \leq i \leq n\}$ (resp. $\{\xi_j \mid 1 \leq j \leq m\}$) of $\mathbf{C}^n$ (resp. $\mathbf{C}^{m*}$), we define

v_0 and ξ_0 as follows ;

$$v_0=\underbrace{v_1 \otimes v_1 \otimes \cdots \otimes v_1}_{\mu_1} \otimes \cdots\cdots \otimes \underbrace{v_n \otimes v_n \otimes \cdots \otimes v_n}_{\mu_n}$$

$$\xi_0=\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{\mu_1} \otimes \cdots\cdots \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{\mu_n}.$$

By these definitions, we have $H_{B_0}v_0=|\mathfrak{H}_{B_0}|v_0$, $H^t{}_{B_0}\xi_0=|\mathfrak{K}_{B_0}|\xi_0$ and

$$A_r\cdot(H_{B_0}v_0 \otimes H^t{}_{B_0}\xi_0)=|\mathfrak{H}_{B_0}||\mathfrak{K}_{B_0}|A_r\cdot(v_0 \otimes \xi_0),$$

which is not zero because

$$A_r\cdot(v_0 \otimes \xi_0)=\varphi\{(v_1 \otimes \xi_1)\wedge\cdots\wedge(v_1 \otimes \xi_{\mu_1})\wedge\cdots\cdots\wedge(v_n \otimes \xi_1)\wedge\cdots\wedge(v_n \otimes \xi_{\mu_n})\}.$$

Hence

$$A_r\cdot(\hat{C}_{B_0}T \times \hat{C}^t{}_{B_0}T^*) \simeq V_\mu \times V_*{}^t{}_\mu. \tag{ii}$$

Next we define $B=\tau B_0$ $(\tau \in \mathfrak{S}_r)$, then we have $\hat{C}_B=\tau\hat{C}_{B_0}\tau^{-1}$ and $\hat{C}^t{}_B=\tau\hat{C}^t{}_{B_0}\tau^{-1}$. Hence we have

$$A_r\cdot(\hat{C}_B T \times \hat{C}^t{}_B T^*)=A_r\cdot(\hat{C}_{B_0}T \otimes \hat{C}^t{}_{B_0}T^*). \tag{iii}$$

(i), (ii) and (iii) show Theorem 2. 2. Q.E.D.

Theorem 2. 3. *$\Lambda^{r,s}$ admits the following decomposiiton.*

$$\Lambda^{r,s} \simeq \bigoplus_{\substack{\mu \in Y^{(r)}_{n,m} \\ \nu \in Y^{(s)}_{m,n}}} \left(V_\mu \otimes V_*{}^t{}_\nu\right) \times \left(V_*{}^t{}_\mu \otimes V_\nu\right).$$

Proof. From the definition $\mathfrak{m}^+ \simeq (\mathfrak{m}^-)^*$. So $\Lambda^{0,s} \simeq (\Lambda^{s,0})^*$, and by Theorem 2. 2 $\Lambda^{0,s}$ is equivalent with $\bigoplus_{\nu \in Y^{(s)}_{m,n}} V_*{}^t{}_\nu \times V_\nu$. Considering $\mathfrak{m}^+ \cap \mathfrak{m}^-=\{0\}$, we have

$$\Lambda^{r,s}=\Lambda^{r,0}\wedge\Lambda^{0,s} \simeq \Big(\bigoplus_{\mu \in Y^{(r)}_{n,m}} V_\mu \times V_*{}^t{}_\mu\Big) \otimes \Big(\bigoplus_{\nu \in Y^{(s)}_{m,n}} V_*{}^t{}_\nu \times V_\nu\Big)$$

$$\simeq \bigoplus_{\substack{\mu \in Y^{(r)}_{n,m} \\ \nu \in Y^{(s)}_{m,n}}} \left(V_\mu \otimes V_*{}^t{}_\nu\right) \times \left(V_*{}^t{}_\mu \otimes V_\nu\right).$$

So we get Theorem 2. 3.

It is reduced to the (L-R) rule to give the irreducible decomposition of $\Lambda^{r,s}$ by the following

Theorem 2. 4. *$V_\mu \otimes V_*{}^t{}_\nu$ (resp. $V_*{}^t{}_\mu \otimes V_\nu$) is decomposed into irreducible U_n (resp. U_m)-modules by the (L-R) rule.*

Proof. Considering $V_*{}^t{}_\mu \otimes V_\nu \simeq V_\nu \otimes V_*{}^t{}_\mu$, the irreducible decomposition of $V_*{}^t{}_\mu \otimes V_\nu$ is analogous to that of $V_\mu \otimes V_*{}^t{}_\nu$. So we have only to state about $V_\mu \otimes V_*{}^t{}_\nu$.

Let k be the depth of ν, then

$$\rho_{\mu}\otimes\rho_{*^t\nu}=(\det)^{\mu_n-k}(\rho_{\mu-\mu_n}\otimes\rho_{*^t\nu+k}).$$

Since $\mu-\mu_n$ and $*^t\nu+k$ are elements of Y_n, we can decompose $\rho_{\mu-\mu_n}\otimes\rho_{*^t\nu+k}$ into irreducible representations by the (L−R) rule. Since the irreducibility is preserved under the tensor product with the 1-dimensional representation, we get Theorem 2. 4.

3. Irreducible decomposition of the G-module $C^\infty(G, K, \Lambda^{r,s})$.

Let us write the irreducible decomposition of $\Lambda^{r,s}$ as $\bigoplus_{W\subset\Lambda^{r,s}} W$, then $C^\infty(G, K, \Lambda^{r,s})$ admits the decomposition $\bigoplus_{W\subset\Lambda^{r,s}} C^\infty(G, K, W)$. Here $C^\infty(G, K, W)$ is the G-module induced from W. For its irreducible decomposition, we can use the following

Lemma 3. 1 (Frobenius [7]). *The following map is an isomorphism.*

$$\mathrm{Hom}_K(V_\lambda, W)\ni\varphi\longrightarrow\bar{\varphi}\in\mathrm{Hom}_G(V_\lambda, C^\infty(G, K, W)),$$

where $\bar{\varphi}(v)(g)=\varphi(\rho_\lambda(g^{-1})v)$ *for* $v\in V_\lambda$, $g\in G$.

We may replace W with $V_\mu\times V_\nu$. Then we define the number $m_{\lambda\mu\nu}=\dim\mathrm{Hom}_K(V_\lambda, V_\mu\times V_\nu)$. To determine $m_{\lambda\mu\nu}$ we can use the following

Proposition 3. 2 ([6]). *Define the Schur function* $\{\lambda\}$ *(resp.* $\{\mu\}$, $\{\nu\}$*) for* $\lambda\in Y_{n+m}$ *(resp.* $\mu\in Y_n$, $\nu\in Y_m$*) whose variables are* $\varepsilon_1, \cdots, \varepsilon_{n+m}$ *(resp.* $\varepsilon_1, \cdots, \varepsilon_n$, $\varepsilon_{n+1}, \cdots, \varepsilon_{n+m}$*). Then we have*

$$\{\lambda\}=\sum_{\substack{\mu\in Y_n\\ \nu\in Y_m}} g_{\bar{\mu}\bar{\nu}\lambda}\{\mu\}\{\nu\},$$

where $\bar{\mu}=(\mu_1, \cdots\mu_n, \underbrace{0, \cdots, 0}_{m})$, $\bar{\nu}=(\nu_1, \cdots, \nu_m, \underbrace{0, \cdots, 0}_{n})$ *and* $g_{\bar{\mu}\bar{\nu}\lambda}$ *is the number gained by the (L-R) rule.*

We can compute the irreducible decomposition of $C^\infty(G, K, V_\mu\times V_\nu)$ by the following

Theorem 3. 3. *Let us identify the irreducible components with these highest weights with multiplicities. Then we have*

$$C^\infty(G, K, V_\mu\times V_\nu)=\bigcup_{e\geq\mathrm{Max}(-\mu_n, -\nu_m)}\left\{\begin{matrix}\textit{Young diagrams appearing in}\\ (\overline{\mu+e})\otimes(\overline{\nu+e})\ \textit{whose}\ (n+m)-th\\ \textit{components are}\ 0\end{matrix}\right\}-e,$$

where the meaning of "——" is the same as in Proposition 3. 2, and $\{\ \}-e$ *means the set of dominant integral forms made by subtracting e from each Young diagram in* $\{\ \}$.

Proof. It is enough to show the equation

$$m_{\lambda\bar{\mu}\bar{\nu}}=\#\,\{\lambda\ \text{appearing in the right hand side}\}\ \text{for any}\ \lambda.$$

Only when $e=-\lambda_{n+m}$, we can find λ in the right hand side. So we have to divide the proof into two cases.

Case (1). $e=-\lambda_{n+m}\geq Max\ (-\mu_n, -\nu_m)$. As $\lambda+e\in Y_{n+m}$, we can use Proposition 3. 2.

$$\{\lambda+e\} = \sum_{\substack{\mu'\in Y_n \\ \nu'\in Y_m}} g_{\bar{\mu}'\bar{\nu}'\lambda+e}\{\mu'\}\ \{\nu'\}.$$

$\{\mu'\}$ and $\{\nu'\}$ are the irreducible characters of U_n and U_m determined by μ' and ν' respectively. So we get

$$m_{\lambda+e\mu'\nu'}=g_{\bar{\mu}'\bar{\nu}'\lambda+e} \text{ for any } \mu'\in Y_n,\ \nu'\in Y_m.$$

Moreover we have the following because $\left|\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\right|^e=|A|^e|B|^e$.

$$m_{\lambda+e\ \mu+e\ \nu+e}=m_{\lambda\mu\nu} \text{ for any dominant integral forms } \mu,\ \nu. \tag{iv}$$

Setting $\mu'=\mu+e$ and $\nu'=\nu+e$, we get $m_{\lambda\mu\nu}=g_{\overline{\mu+e}\ \overline{\nu+e}\ \lambda+e}$. As the $(n+m)$ – th component of $\lambda+e$ is 0,

$$g_{\overline{\mu+e}\ \overline{\nu+e}\ \lambda+e}=\sharp\ \{\lambda+e \text{ appearing in the } \{\ \ \}\}.$$

Therefore we get $m_{\lambda\mu\nu}=\sharp\ \{\lambda$ appearing in the right hand side$\}$.

Case (2). $-\lambda_{n+m} < Max\,(-\mu_n, -\nu_m)$. It is enough to show $m_{\lambda\mu\nu}=0$. As $\lambda-\lambda_{n+m}\in Y_{n+m}$, $\{\lambda-\lambda_{n+m}\}$ is a polynomial whose variables are $\varepsilon_1, \cdots, \varepsilon_{n+m}$. On the other hand either $\mu-\lambda_{n+m}$ or $\nu-\lambda_{n+m}$ is not a Young diagram owing to the assumption. So $(\varepsilon_1\cdots\varepsilon_n)^{-1}$ or $(\varepsilon_{n+1}\cdots\varepsilon_{n+m})^{-1}$ must appear in $\{\mu-\lambda_{n+m}\}\ \{\nu-\lambda_{n+m}\}$ as a common factor. By these properties of $\{\lambda-\lambda_{n+m}\}$ and $\{\mu-\lambda_{n+m}\}\ \{\nu-\lambda_{n+m}\}$, we can conclude $m_{\lambda-\lambda_{n+m}\ \mu-\lambda_{n+m}\ \nu-\lambda_{n+m}}=0$. By (iv), $m_{\lambda\mu\nu}=m_{\lambda-\lambda_{n+m}\ \mu-\lambda_{n+m}\ \nu-\lambda_{n+m}}$. Hence $m_{\lambda\mu\nu}$ is equal to 0. Q. E. D.

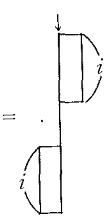
4. Examples.

In this section we decompose $C^\infty(G, K, A^{r,s})$ into irreducible G-modules making use of theorems in §2 and §3.

Remark. When we pull back the above irreducible decomposition by the natural identification $SU_{n+m}/S(U_n\times U_m)\simeq G/K$, we get the irreducible decomposition as SU_{n+m}-modules.

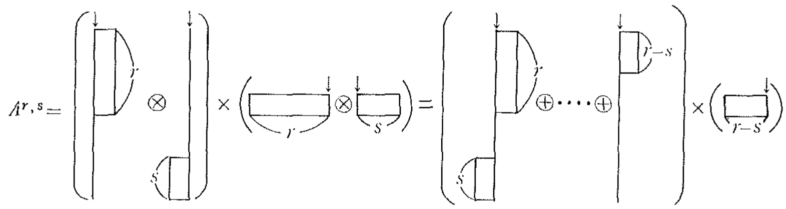
Assume $n\geqq m$ for convenience, and define the following fundamental dominant integral forms.

$$\Lambda_i=(x_1+\cdots+x_i-x_{n+m-i+1}-\cdots-x_{n+m})\ (1\leq i\leq m)$$

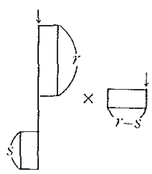
$\Lambda i=$ 

Even in the case dominant integral forms are not Young diagrams, let us express these as diagrams showing the origin by the mark " ↓ ".

Example 1. The case $m=1$. We may assume $s \leq r$, $r+s \leq n$ since $\Lambda^{s,r} \simeq (\Lambda^{r,s})^*$, $\Lambda^{r,s} \simeq \Lambda^{n-s,n-r}$. Then we can give the irreducible decomposition of $\Lambda^{r,s}$ by Theorem 2. 3 and 2. 4. This is shown in the following figure.



Therefore we have only to decompose the G-module induced from the K-module



into the irreduible G-modules. This is done by Theorem 3. 3 using the figure shown below. The answer is summarized in Table A.

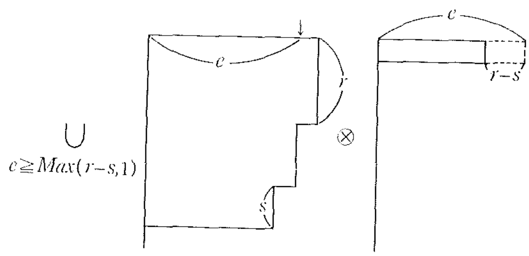


Table A

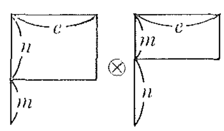
(r, s)	highest weights (These multiplicities are one.)	
$0=s=r$	$f\Lambda_1$	$(f \geq 0)$
$0=s<r<n$	$f\Lambda_1+(x_1+\cdots\cdots+x_r-rx_{n+1})$	$(f \geq 0)$
	$f\Lambda_1+(x_2+\cdots\cdots+x_{r+1}-rx_{n+1})$	$(f \geq 1)$
$0=s,\ r=n$	$f\Lambda_1+(x_1+\cdots\cdots+x_n-nx_{n+1})$	$(f \geq 0)$
$0<s \leq r<n$, $r+s=n$	$f\Lambda_1+(x_1+\cdots\cdots+x_r-x_{r+1}-\cdots\cdots-x_n-(r-s)x_{n+1})$	$(f \geq 0)$
	$f\Lambda_1+(x_2+\cdots\cdots+x_r-x_{r+2}-\cdots\cdots-x_n-(r-s)x_{n+1})$	$(f \geq 1)$
	$f\Lambda_1+(-x_1+x_2+\cdots\cdots+x_{r+1}-x_{r+2}-\cdots\cdots-x_n-(r-s)x_{n+1})$	$(f \geq 2)$
$0<s \leq r<n$, $r+s<n$	$f\Lambda_1+(x_1+\cdots\cdots+x_r-x_{n-s+1}-\cdots\cdots-x_n-(r-s)x_{n+1})$	$(f \geq 0)$
	$f\Lambda_1+(x_2+\cdots\cdots+x_{r+1}-x_{n-s+1}-\cdots\cdots-x_n-(r-s)x_{n+1})$	$(f \geq 1)$
	$f\Lambda_1+(x_2+\cdots\cdots+x_r-x_{n-s+2}-\cdots\cdots-x_n-(r-s)x_{n+1})$	$(f \geq 1)$
	$f\Lambda_1+(-x_1+x_2+\cdots\cdots+x_{r+1}-x_{n-s+2}-\cdots\cdots-x_n-(r-s)x_{n+1})$	$(f \geq 2)$

Example 2. The general case ($n \geq m \geq 1$). First we show the irreducible decomposition for $(r, s)=(0, 0)$, $(1, 0)$, $(0, 1)$ in TableB.

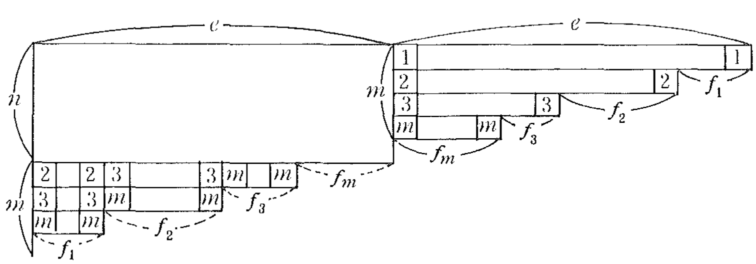
Table B

(r, s)	highest weights	multiplicities
(1) $(r, s)=(0, 0)$	$\sum_{i=1}^{m} f_i \Lambda_i \quad (f_i \geq 0)$	1
(2) $(r, s)=(1, 0)$	$\sum_{i=1}^{m} f_i \Lambda_i \quad (f_i \geq 0)$	$\sharp \{f_i \neq 0\}$
	$\sum_{i=1}^{m} f_i \Lambda_i + (x_j - x_k)$ $(f_i \geq 0,\ 1 \leq j \leq n,\ n+1 \leq k \leq n+m,\ *(x_j - x_k) \neq (x_j - x_k))$	1
(3) $(r, s)=(0, 1)$	$\sum_{i=1}^{m} f_i \Lambda_i \quad (f_i \geq 0)$	$\sharp \{f_i \neq 0\}$
	$\sum_{i=1}^{m} f_i \Lambda_i + (-x_j + x_k)$ $(f_i \geq 0, 1 \leq j \leq n,\ n+1 \leq k \leq n+m,\ *(-x_j + x_k) \neq (-x_j + x_k))$	1

(1) Let us put $\mu=0$, $\nu=0$ in Theorem 3. 3, then we can see the following set is equal to the set $\{\sum_{i=1}^{m} f_i \Lambda_i \mid f_i \geq 0\}$.

$\bigcup_{e \geq 0}$ {Young diagrams in  whose $(n+m)$-th components are 0} $-e$.

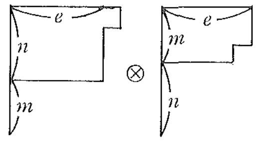
The elements in { } correspond to the ones in $\{(f_1, \cdots, f_m) \in \mathbf{Z}^m \mid \sum_{i=1}^{m} f_i = e,\ f_i \geq 0\}$ bijectively as is shown in the following diagram.



(Since the numbers of each figures are equa', the figure 1 does not appear in the $(n+1)$-th row to get the diagrams whose $(n+m)$-th row are 0. So we can write the right hand side as above. Then the arrangement of the left hand side is determined as above.)

(2) Let us examine the following set as in (1).

$$\bigcup_{e\geq 1}\left(\begin{array}{l}\text{Young diagrams in} \\ \\ \text{whose } (n+m)\text{-th components are } 0\end{array}\right)-e.$$



In the above { }, we can classify the diagrams into the following $m+1$ (m) cases.

Case 1. There is not 1 in the $(e+1)$-th column.

Case 2. There is a 1, but not 2, in the $(e+1)$-th column.

....

Case m. There is a $1, \cdots$, a m-1, but not m, in the $(e+1)$-th column. Unless $n=m$, the following case must be considered.

Case $m+1$. There is a $1, \cdots$, and a m in the $(e+1)$-th column. For each case there are two cases, namely,

Case j_A. There is not 1 in the $(n+1)$-th row.

Case j_B. There is a 1 in the $(n+1)$-th row.

We can show the following equations as in (1), (The elements in the right hand sides are assumed to be dominant integral forms).

$$\bigcup_{e\geq 1}\{\text{case } j_A\}-e=\{\sum_{i=1}^{m} f_i\Lambda_i+(x_j-x_k)\mid f_i\geq 0,\ n+1\leq k\leq n+m-1\}.$$

$$\bigcup_{e\geq 1}\{\text{case } j_B\}-e=\{\sum_{i=1}^{m} f_i\Lambda_i+(x_j-x_{n+m})\mid f_i\geq 0\}.$$

We get (2) by adding these two equations from $j=1$ to $j=m+1(m)$.

(3) Let us think the image of (2) under the map $*$, then we get (3).

5. Computation of the Hodge numbers.

By Theorem 2. 3 we can get the Hodge nurmbers $h^{r,s}(M)$. For this purpose we need to prepare some results on the spectra of M. Let $\mathfrak{g}$ be the Lie algebra of G, and $< >$ the inner product of $\mathfrak{g}$ defined by $<X, Y>=-2tr.XY$. For this $< >$ we define the Cassimir operator C. Then we have the following lemma in the same way as [2] (Since G is unimodular and has no boundary we get the fact : If dg is biinvariant measure of G, $\int_G X(f)\,dg$ is equal to 0 for $f\in C^\infty(G)$ and $X\in\mathfrak{g}$).

Lemma 5. 1. *Let us identify $C^\infty(\Lambda^{r,s}M)$ with $C^\infty(G, K, \Lambda^{r,s})$, then*

$$\Delta\alpha=-C\alpha \text{ for any } \alpha\in C^\infty(\Lambda^{r,s}M).$$

The eigenvalues of C are given by the following

Lemma 5. 2 ([8]). *Let $\delta=\frac{1}{2}\sum$ (positive roots of G), and we identify the irreducible submodule of $C^\infty(G, K, \Lambda^{r,s})$ with V_λ. Then,*

$$Cf = -4\pi^2 < \lambda,\ \lambda+2\delta > f \text{ for any } f \in V_\lambda.$$

By the aid of these lemmas we can derive the following fact.

Proposition 5. 3. $h^{r,s}(M) = \begin{cases} 0 & (r \neq s) \\ \sharp Y^{(r)}_{n,m} & (r = s). \end{cases}$

Proof. Let V_λ be an irreducible component corresponding to the harmonic forms in $C^\infty(\Lambda^{r,s}M)$. Then we get $\lambda=0$ because of $<\lambda,\ \delta> \geq 0$.
So $h^{r,s}(M) = \dim \mathrm{Hom}_K(\mathrm{C},\ \Lambda^{r,s})$. By Theorem 2. 3 we have,

$$\begin{aligned} \mathrm{Hom}_K(\mathrm{C},\ \Lambda^{r,s}) &\simeq \bigoplus_{\substack{\mu \in Y^{(r)}_{n,m} \\ \nu \in Y^{(s)}_{m,n}}} \mathrm{Hom}_{U_n}(\mathrm{C},\ V_\mu \otimes V_*{}^{t}{}_\nu) \otimes \mathrm{Hom}_{U_m}(\mathrm{C},\ V_*{}^{t}{}_\mu \otimes V_\nu) \\ &\simeq \bigoplus_{\substack{\mu \in Y^{(r)}_{n,m} \\ \nu \in Y^{(s)}_{m,n}}} \mathrm{Hom}_{U_n}(V^t{}_\nu,\ V_\mu) \otimes \mathrm{Hom}_{U_m}(V^t{}_\mu,\ V_\nu). \end{aligned}$$

Hence we have the above fact.

Appendix

We show a program in N88-BASIC for the calculation of the (L-R) rule by Dr. Hisao Kamiya.

```
10  REM save "YNG"
20  KEY 10, "LIST 30-50"+CHR$ (13)
30  REM S 1 2 3 4 5 6 7 8 9 === SET DATA===
40  DATA 5,3,2,2,0,0
50  DATA 5,2,1,0,0,0
60  REM ====================================
70  DEFINT A-Z : DIM A(10, 10), W(10, 10), Z(1000, 10)
80  RESTORE 40
90  LPRINT CHR$ (27)+"L010"+"INITIAL DATA"
100  READ SX : FOR I=1 TO SX : READ X(I) : LPRINT X(I);:NEXT : LPRINT
110  READ SY : FOR I=1 TO SY : READ Y(I) : LPRINT Y(I);:NEXT : LPRINT
120  FOR I=1 TO SX
130  A(0, I)=X(I)
140  NEXT : A(0, 0)=999
150  L=1 : J=0
160  REM ============== NEXT L ==============
170  FOR I=1 TO SX : W(L, I)=0 : A(L, I)=A(L-1, I) : NEXT
180  W (L, L)=Y(L) : A(L, L)=W(L, L)+A(L-1, L)
190  IF A(L-1, L-1)<A(L, L) THEN GOTO 210 ELSE 320
210  K=1
220  IF W(L, K)=0 THEN 300
230  WW=W(L, K)-1 : W(L, K)=0 : W(L, L)=WW : W(L, K+1)=W(L, K+1)+1
```

```
240 FOR K=0 TO SX
250 A(L, K)=A(L-1,K)+W(L, K)
260 NEXT
270 FOR KK=L TO SX : IF A(L-1, KK-1)<A(L,KK) THEN 210
280 NEXT : GOTO 320
290 REM ===================================
300 K=K+1 : IF K>=SX THEN 310 ELSE 220
310 L=L-1 : IF L=0 THEN STOP ELSE 210
320 IF L=1 THEN 360
330 REM ==== LATTICE PERMUTATION CHECK=====
340 T=W(L, 1) : FOR K=2 TO SX+1 : IF T>0 THEN 210
350 T=T-W(L-1, K-1)+W(L, K) : NEXT
360 IF L=SY THEN 380
370 L=L+1 : GOTO 170
380 CLS 1 : PRINT J+1 ; "-th answer of this type"
390 FOR II=1 TO SX : PRINT : FOR IJ=1 TO X(II) : PRINT " 0" ;
395 NEXT : FOR IJ=1 TO SY
400 FOR IK=1 TO W(IJ, II) : PRINT USING"##"; IJ ;: NEXT : NEXT : NEXT :
  PRINT
410 FOR II=1 TO SX : LPRINT : FOR IJ=1 TO X(II) : LPRINT" 0" ;
415 NEXT : FOR IJ=1 TO SY
420 FOR IK=1 TO W(IJ, II) : LPRINT USING"##" ; IJ ;: NEXT : NEXT : NEXT :
  LPRINT
430 IK=1 : FOR II=1 TO SX : Z(J, II)=A(SY, II) : NEXT
440 FOR II=1 TO J-1 : FOR IJ=1 TO SX : IF Z(II, IJ)< >Z(J, IJ) THEN 460
450 NEXT : IK=IK+1
460 NEXT II : PRINT"This is" ; IK ; "-th answer of this type."
470 LPRINT"This is" ; IK ; "-th answer of this type. " : GOSUB 500
480 J=J+1 : PRINT : GOTO 210
490 REM ========== DATA check ============
500 PRINT : FOR L=1 TO SY : FOR KL=0 TO 10 : PRINT A(L, KL);: NEXT :
  PRINT : NEXT
510 PRINT : FOR L=1 TO SY : FOR KL=0 TO 10
515 PRINT W(L, KL);: NEXT : PRINT : NEXT
520 T=W(2, 1) : FOR K=2 TO SX
530 T=T-W(1, K-1)+W(2,K) : PRINT T;:NEXT : RETURN
540 FOR I=1 TO 200 : NEXT : INPUT X$ : RETURN
550 END
```

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