J. FAC. SCI., SHINSHU UNIVERSITY, Vol. 21, No. 2, 1986

Conjugate Sets of Gaussian Random Fields on a Hilbert Space

By Kazuyuki Inoue

Department of Mathematics, Faculty of Science
Shinshu University
(Received Sept. 25, 1986)

1. Introduction

This is the detailed exposition of [2]. In the theory of Gaussian random fields (G.r.f.'s), the notion of conjugate sets has been introduced by P. Lévy and employed by the author to study the conditional independence of G.r.f.'s with parameter space $\boldsymbol{R}^d$ ([1]−[5]). The aim of this paper is to give a description of G.r.f.'s with projective invariance by using the conjugate sets associated with them. In particular, the parameter space is taken to be the real Hilbert space ℓ^2 defined by $\ell^2=\{\boldsymbol{x}=(x_n)_{n\geqslant 1};\ \sum_{n=1}^{\infty} x_n{}^2<\infty,\ \ x_n\in \boldsymbol{R}\ (n\geqslant 1)\}$, in which every space $\boldsymbol{R}^d$ $(d\geqslant 1)$ is embedded. Let $\boldsymbol{X}=\{X(\boldsymbol{x});\ \boldsymbol{x}\in \ell^2\}$ be a mean zero G.r.f. on ℓ^2 with homogeneous and isotropic increments such that the variance of $X(\boldsymbol{x})-X(\boldsymbol{y})$ is given by $r(|\boldsymbol{x}-\boldsymbol{y}|)$, where the *structure function* $r(t)$ is assumed to be continuous and satisfy the normalizing condition $r(1)=1$ ([8]). We may identify two G.r.f.'s on ℓ^2 with common structure function $r(t)$, because such G.r.f.'s have the same probabilistic structure related to conditional dependence. From this point of view, we often use the notation $(\boldsymbol{X}, r(t))$ instead of $\boldsymbol{X}$ ([3]). We note that there exists a one-to-one correspondence between the class of these G.r.f.'s $(\boldsymbol{X}, r(t))$ on ℓ^2 and the class $\mathscr{S}$ of all the functions $r(t)$ on $[0, \infty)$ expressed as follows:

$$r(t)=ct^2+\int_0^{\infty}(1-e^{-t^2u})u^{-1}d\gamma(u) \qquad (t\geqslant 0),$$

where c is a non-negative constant and γ denotes a measure on $(0, \infty)$ such that $\int_0^{\infty}(1+u)^{-1}d\gamma(u)<\infty$ and $r(1)=1$ ([7]). An important subclass of $\mathscr{S}$ is given by

$$\mathscr{L}=\{r(t)=t^{\alpha};\ 0<\alpha\leqslant 2\},$$

which corresponds to the class of G.r.f.'s with projective invariance in the sense of [6].

We now proceed to the definition of the conjugate sets associated with $(\boldsymbol{X}, r(t))$. Given $E\subset \ell^2$ $(E\neq\phi)$, we denote by $R_r(\boldsymbol{x}, \boldsymbol{y}|E)$ the conditional covariance function of $(X, r(t))$ relative to E, i.e.,

$$R_r(x, y|E)=E[(X(x)-\mu_r(x|E))(X(y)-\mu_r(y|E))] \qquad (x, y\in \ell^2),$$

where $\mu_r(x|E)$ stands for the conditional expectation of $X(x)$ under the conditioning by $\{X(z);\ z\in E\}$ (see Section 2). Then, for every $x\in \ell^2$, the *maximal conjugate set* (or shortly *conjugate set*) $\mathscr{T}_X(x|E)$ of x relative to E is defined as follows:

$$\mathscr{T}_X(x|E)=\{y\in \ell^2;\ R_r(x, y|E)=0\}.$$

Since $(X, r(t))$ is a Gaussian system, the set $\mathscr{T}_X(x|E)$ proves to be the locus of $y\in \ell^2$ for which $X(x)$ and $X(y)$ are independent conditioned by $\{X(z);\ z\in E\}$. In this paper, we assume that E is finite and contains at least two points:

$$(1.1) \qquad E=\{a_k\}_{1\leqslant k\leqslant n} \quad \text{and} \quad n=\sharp E\geqslant 2,$$

where $\sharp E$ denotes the cardinal number of E. Then $\mu_r(x|E)$ can be expressed in the form

$$\mu_r(x|E)=\sum_{k=1}^{n} X(a_k)\gamma_r{}^k(x|E) \qquad (x\in \ell^2),$$

where $\gamma_r{}^k(x|E)$ $(1\leqslant k\leqslant n)$ stand for certain real numbers satisfying the equation $\sum_{k=1}^{n}\gamma_r{}^k(x|E)=1$. A mapping Φ_E: $\ell^2\to R^n$ is defined by $\Phi_E(x)=(|x-a_1|, \cdots, |x-a_n|)$ $(x\in \ell^2)$. By an inversion on ℓ^2 with center $z\in \ell^2$ and radius $t>0$, we mean the following transformation T on ℓ^2:

$$Tx=t^2|x-z|^{-2}(x-z)+z \quad (x\neq z) \quad \text{and} \quad Tz=z.$$

We denote by $\mathscr{I}(\ell^2)$ the set of all inversions on ℓ^2. We are now in a position to state our main result.

THEOREM 1. *Let $(X, r(t))$ be a G.r.f. on ℓ^2 rigged with $\{a, E\}$, where $r(t)\in\mathscr{S}$, $a\notin E$ and E is given by* (1.1). *Suppose that $\{a, E, r(t)\}$ satisfies the following conditions:*

(1.2) *$\Phi_E(\mathscr{T}_X(a|E))$ contains an interior point; and*

(1.3) *$\gamma_r{}^i(Ua|UE)\gamma_r{}^j(Ua|UE)\neq 0$ for some $U\in\mathscr{I}(\ell^2)$ with center $z_U\in E$ and some i, j $(i\neq j)$.*

Then it holds that $r(t)\in\mathscr{L}$ if and only if

(1.4) *$\mathscr{T}_X(Ta|TE)=T\mathscr{T}_X(a|E)$ for any $T\in\mathscr{I}(\ell^2)$ with center z_U.*

We note that Theorem 1 remains true even if ℓ^2 is replaced by R^d. Formerly we obtained an analogous result with respect to similar transformations on R^d (see Theorem 1 of [3]). The proof of Theorem 1 is given in Section 2 based on a certain functional equation. Finally, in Section 3, we shall give a simplified version of this theorem by specializing the set E.

2. Proof of the main theorem

Let $(X, r(t))$ be a G.r.f. on ℓ^2 and E be a non-empty subset of ℓ^2. First we note that the conditional expectation $\mu_r(\boldsymbol{x}|E)$ is defined in the sense of [5]. In other words, we set

$$\mu_r(\boldsymbol{x}|E)=X(z_0)+E[X(\boldsymbol{x})-X(z_0)|X(z)-X(z_0);\ z\in E] \quad (\boldsymbol{x}\in\ell^2,\ z_0\in E).$$

Here the right side does not depend on the choice of $z_0\in E$. In the special case $E=\{z\}$, we have $\mu_r(\boldsymbol{x}|z)=X(z)$ and so the conditional covariance function $R_r(\boldsymbol{x}, \boldsymbol{y}|z)$ is given by

$$R_r(\boldsymbol{x}, \boldsymbol{y}|z)=\{r(|\boldsymbol{x}-z|)+r(|\boldsymbol{y}-z|)-r(|\boldsymbol{x}-\boldsymbol{y}|)\}/2 \quad (\boldsymbol{x}, \boldsymbol{y}\in\ell^2).$$

In general, if E is given by (1.1), we have the following expression: For any $\boldsymbol{x}$, $\boldsymbol{y}\in\ell^2$,

$$2R_r(\boldsymbol{x}, \boldsymbol{y}|E)=r(|\boldsymbol{x}-\boldsymbol{a}_1|)+\Lambda_r(\boldsymbol{x}, \boldsymbol{y}|E)-r(|\boldsymbol{x}-\boldsymbol{y}|)-\Lambda_r(\boldsymbol{x}, \boldsymbol{a}_1|E), \tag{2.1}$$

where we set $\Lambda_r(\boldsymbol{x}, \boldsymbol{y}|E)=\sum_{k=1}^{n} r(|\boldsymbol{y}-\boldsymbol{a}_k|)\gamma_r{}^k(\boldsymbol{x}|E)$. As for the class $\mathscr{S}$, we know some interesting properties ([3]). Among them, we note that each function $r(t)\in\mathscr{S}$, as well as its inverse function $r^{-1}(x)$, is strictly increasing and analytic.

Proof of Theorem 1. The *"only if"* part immediately follows from the projective invariance of $(X, r(t))$ in the sense of [6]. Therefore it suffices to prove the *"if"* part. Without loss of generality, we may assume that $\gamma_r{}^i(T_s\boldsymbol{a}|T_sE)\gamma_r{}^j(T_s\boldsymbol{a}|T_sE)\neq 0$ for some $s>0$ and some i, j $(i\neq j)$, where T_t denotes the inversion on ℓ^2 with center $\boldsymbol{a}_1$ and radius t $(t>0)$. Because of the assumption (1.2), there exist open intervals I_k $(1\leqslant k\leqslant n)$ contained in $(0, \infty)$ such that $\prod_{k=1}^{n} I_k\subset\Phi_E(\mathscr{T}_X(\boldsymbol{a}|E))$. It follows that, for each $\boldsymbol{u}=(u_1, \cdots, u_n)\in\prod_{k=1}^{n} I_k$, there exists $\boldsymbol{y}[\boldsymbol{u}]\in\mathscr{T}_X(\boldsymbol{a}|E)$ such that $\Phi_E(\boldsymbol{y}[\boldsymbol{u}])=\boldsymbol{u}$ or equivalently $|\boldsymbol{y}[\boldsymbol{u}]-\boldsymbol{a}_k|=u_k$ $(1\leqslant k\leqslant n)$. Then we see by (1.4) that

$$T_t\boldsymbol{y}[\boldsymbol{u}]\in\mathscr{T}_X(T_t\boldsymbol{a}|T_tE) \quad \text{for any } (t, \boldsymbol{u})\in(0, \infty)\times\prod_{k=1}^{n} I_k.$$

In other words, by using the expression (2.1), we have the following: For any $(t, \boldsymbol{u})\in(0, \infty)\times\prod_{k=1}^{n} I_k$,

$$\begin{aligned} r(|T_t\boldsymbol{a}-T_t\boldsymbol{y}[\boldsymbol{u}]|)=&\sum_{k=1}^{n} r(|T_t\boldsymbol{y}[\boldsymbol{u}]-T_t\boldsymbol{a}_k|)\gamma_r{}^k(T_t\boldsymbol{a}|T_tE) \\ &+r(|T_t\boldsymbol{a}-T_t\boldsymbol{a}_1|)-\Lambda_r(T_t\boldsymbol{a}, T_t\boldsymbol{a}_1|T_tE). \end{aligned} \tag{2.2}$$

For the sake of convenience, we set $l_k=|\boldsymbol{a}_k-\boldsymbol{a}_1|$ $(2\leqslant k\leqslant n)$ and further introduce the following functions:

$$\begin{cases} p_k(t)=\gamma_r{}^k(T_{\sqrt{t}}\boldsymbol{a}|T_{\sqrt{t}}E) \quad (t>0,\ 1\leqq k\leqq n), \\ g(t)=r(|T_{\sqrt{t}}\boldsymbol{a}-T_{\sqrt{t}}\boldsymbol{a}_1|)-\Lambda_r(T_{\sqrt{t}}\boldsymbol{a}, T_{\sqrt{t}}\boldsymbol{a}_1|T_{\sqrt{t}}E) \quad (t>0) \text{ and} \\ h(u_1, \cdots, u_n)=r^{-1}\left(r\left(\frac{1}{u_1}\right)p_1(1)+\sum_{k=2}^{n} r\left(\frac{u_k}{l_k u_1}\right)p_k(1)+g(1)\right) \quad (\boldsymbol{u}\in \prod_{k=1}^{n} I_k). \end{cases}$$

Then it follows from (2.2) that, for any $(t, \boldsymbol{u})\in(0, \infty)\times \prod_{k=1}^{n} I_k$,

$$r(th(u_1, \cdots, u_n))=r\left(\frac{t}{u_1}\right) p_1(t)+ \sum_{k=2}^{n} r\left(\frac{tu_k}{l_k u_1}\right) p_k(t)+g(t). \tag{2.3}$$

Now we set

$$D_r(\boldsymbol{a}, E)= \{(v_1, \cdots, v_n);\ \frac{1}{v_1}\in I_1 \text{ and } \frac{l_k v_k}{v_1}\in I_k\ (2\leqq k\leqq n)\} \text{ and}$$

$$\tilde{h}(v_1, \cdots, v_n)=h\left(\frac{1}{v_1}, \frac{l_2 v_2}{v_1}, \cdots, \frac{l_n v_n}{v_1}\right) \qquad (\boldsymbol{v}\in D_r(\boldsymbol{a}, E)).$$

Then the equation (2.3) is replaced by the following: For any $(t, \boldsymbol{v})\in(0, \infty)\times D_r(\boldsymbol{a}, E)$,

$$r(t\tilde{h}(v_1, \cdots, v_n))= \sum_{k=1}^{n} r(tv_k)p_k(t)+g(t). \tag{2.4}$$

It should be noted that $p_i(s^2)p_j(s^2)\neq 0$ and $\tilde{h}(v_1, \cdots, v_n)>0$ on the domain $D_r(\boldsymbol{a}, E)$ of $\boldsymbol{R}^n$. By applying Lemma 7 of [3] to the equation (2.4), we see that $r(t)$ can be expressed in the form

$$r(t)=C_1 t^\alpha+C_2 \qquad \text{or} \qquad r(t)=\beta\log t+C_3 \qquad (t>0),$$

where α, β and C_k $(1\leqq k\leqq 3)$ are real constants $(\alpha C_1\neq 0,\ \beta\neq 0)$. Therefore we obtain the desired expression $r(t)=t^\alpha$ $(0<\alpha\leqq 2)$ by using the conditions $r(0)=0$, $r(1)=1$ and the concavity of $r(\sqrt{t})$ ([3]). The proof is thus completed.

3. A simplified result

In this section we shall give an interesting version of Theorem 1 by specializing the set E. Let $\{e_n\}_{n\geqq 1}$ be the canonical orthonormal basis of l^2. We mean by E_n $(n\geqq 2)$ the subsets of l^2 defined as follows:

$$E_n= \{\boldsymbol{a}_{nk}\}_{1\leqq k\leqq n} \text{ and } \boldsymbol{a}_{nk}=e_{k+1}-\frac{1}{n}\sum_{j=1}^{n} e_{j+1} \quad (1\leqq k\leqq n).$$

For each $t>0$, T_{nt} denotes an inversion on l^2 with center $\boldsymbol{a}_{n1}$ and radius t, and further S_t denotes a similar transformation on l^2 given by $S_t\boldsymbol{x}=t\boldsymbol{x}$. Then we have the following theorem.

THEOREM 2. *Let $(X, r(t))$ be a G.r.f. on l^2 with $r(t)\in\mathscr{S}$. Then the following three conditions are equivalent:*

(i) $r(t) \in \mathscr{L}$;

(ii) $\mathscr{T}_X(T_{nt}e_1|T_{nt}E_n)=T_{nt}\mathscr{T}_X(e_1|E_n)$ *for any* $n \geqslant 2$ *and any* $t>0$;

(iii) $\mathscr{T}_X(S_te_1|S_tE_n)=S_t\mathscr{T}_X(e_1|E_n)$ *for any* $n \geqslant 2$ *and any* $t>0$.

Proof. In order to apply Theorem 1 we shall prove that $\{e_1, E_n, r(t)\}$ satisfies the conditions (1.2) and (1.3) for sufficiently large n. First we note that $|e_1-\boldsymbol{a}_{nk}|=\sqrt{(2n-1)/n}$, $|\boldsymbol{a}_{nk}|=\sqrt{(n-1)/n}$ $(1 \leqslant k \leqslant n)$ and $|\boldsymbol{a}_{nj}-\boldsymbol{a}_{nk}|=\sqrt{2}$ $(1 \leqslant j<k \leqslant n)$. Then we see by (2.1) that

$$\begin{aligned}&\lim_{n\to\infty} 2R_r(e_1, -e_1|E_n)\\ &=\lim_{n\to\infty} \{2r(|e_1-\boldsymbol{a}_{n1}|)-r(|2e_1|)-\frac{1}{n}\sum_{k=1}^{n} r(|\boldsymbol{a}_{n1}-\boldsymbol{a}_{nk}|)\}\\ &=r(\sqrt{2})-r(2)<0.\end{aligned}$$

Therefore we have $R_r(e_1, -e_1|E_n)<0$ for sufficiently large n. In addition, the points of E_n are independent and the point e_1 lies in the orthogonal complement of E_n. Thus we see by Proposition 1 of [3] that $\{e_1, E_n, r(t)\}$ satisfies the condition (1.2) for sufficiently large n. On the other hand, we can easily show that there exists a sequence $\{g_n(t)\}_{n \geqslant 2}$ of functions on $(0, \infty)$ such that $\lim_{n\to\infty} g_n(t)=1$ $(t>0)$ and

$$\gamma_r{}^k(T_{nt}e_1|T_{nt}E_n)=g_n(t)/n \qquad (t>0,\ 2 \leqslant k \leqslant n,\ n \geqslant 2),$$

which guarantees the condition (1.3) for sufficiently large n. Consequently by applying Theorem 1, we have the equivalence of (i) and (ii). Moreover, we easily obtain the equivalence of (i) and (iii) by using Theorem 1 of [3].

References

[1] K. Inoue: Conjugate sets of self-similar Gaussian random fields, Lecture Notes in Math., **1021**, Springer, 1983, 252-256.

[2] K. Inoue: Conjugate sets of Gaussian random fields on a Hilbert space, Stochastic Process. Appl., **21** (1985), 16-17.

[3] K. Inoue: Conjugate sets of Gaussian random fields, Osaka J. Math., **23** (1986), 363–388.

[4] K. Inoue and A. Noda: Independence of the increments of Gaussian random fields, Nagoya Math. J., **85** (1982), 251-268.

[5] P. Lévy: *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1965.

[6] A. Noda: Gaussian random fields with projective invariance, Nagoya Math. J., **59** (1975), 65-76.

[7] I. J. Schoenberg: Metric spaces and completely monotone functions, Annals of Math., **39** (1938), 811-841.

[8] A. M. Yaglom: Some classes of random fields in n-dimensional space related to stationary random processes, Theory Prob. Appl., **2** (1957), 273-320 (English translation).