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A note on the Unit Groups of Burnside Rings as Burnside Ring Modules

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1. Introduction

Let G be a finite group and $Set_f{}^G$ the set of isomorphism classes of all finite G-sets. Then $Set_f{}^G$ is a semi-ring with addition and multiplication induced by the disjoint union and the cartesian product, respectively. The Burnside ring $A(G)$ is defined to be the Grothendieck ring of $Set_f{}^G$. Let $A(G)^*$ be the group of units in $A(G)$.

For a fixed $T \in Set_f{}^G$, we have an exponential map

$$(\quad)^T : Set_f{}^G \longrightarrow Set_f{}^G \quad (S \longmapsto S^T)$$

where $S^T = \{f : T \longrightarrow S \mid f;\text{ set theoretic map}\}$ with G-action

$$g.f : T \longrightarrow S \ (t \longmapsto gf(g^{-1}t)).$$

Exponential maps induce the following maps

$$(\quad)\uparrow(\quad) : Set_f{}^G \times Set_f{}^G \longrightarrow Set_f{}^G \ ((S,\ T) \longmapsto S\uparrow T = S^T)$$

and

$$(\quad)\uparrow(\quad) : A(G) \times A(G) \longrightarrow A(G) \ ((\alpha,\ \beta) \longmapsto \alpha\uparrow\beta)$$

(cf. [4] Section 3 and [1] Section 2). We have

$$(S_1 \times S_2)\uparrow T \cong (S_1\uparrow T) \times (S_2\uparrow T),\ S\uparrow(T_1 \times T_2) \cong (S\uparrow T_1)\uparrow T_2,$$

$$S\uparrow(T_1 + T_2) \cong (S\uparrow T_1) \times (S\uparrow T_2) \text{ and } S\uparrow\text{point} \cong S.$$

Therefore $A(G)^*$ is an $A(G)$-module (cf. [1]). Let $A(G)^{**}$ be the $A(G)$-submodule (subgroup) of $A(G)^*$ generated by -1 $(1 = [\text{point}])$.

Proposition 1. 1. ([1] Proposition 3. 4). $A(G)^{**} = \{1,\ -1\}$ *if and only if* G *is of odd order.*

In the paper [1], A. Dress says that

"One would like to conjecture that $A(G)^*$ is always generated by -1 as an $A(G)$-module, but unfortunately that is not true already for non abelian group of order 10. But still it seems interesting and promising to study the structure of $A(G)^*$ as an $A(G)$-module. Let me add one more elementary result in this direction."

In this paper, we shall prove the following result.

Main Theorem.

[I] *Let D_n be the dihedral group of order $2n$. Then we have $A(D_n)^{**}=A(D_n)^*$ if and only if $n=2$, 4, p^r or $2p^r$, where p is an odd prime such that $p\equiv 3 \bmod 4$.*

[II] *If G is an abelian group, then $A(G)^{**}=A(G)^*$.*

2. Preparation and well known retults

Throughout this paper we use the following notations:

(H) the conjugate classes of a subgroup H of G,
$C(G)$ the set $\{(H)\mid H$ is a subgroup of $G\}$,
G_x the isotropy subgroup at $x\in X$ ($X\in Set_f{}^G$),
$\langle Y\rangle$ the subgroup generated by a subset Y of a group,
$\sharp X$ the cardinal number of a set X,
N the trivial finite G-set with N elements,
n^* the set of positive divisors of an integer n,
n_e^* the set $\{i\in n^*\mid i$ is even$\}$,
n_o^* the set $\{i\in n^*\mid i$ is odd$\}$,
D_n the dihedral group $\langle b, a\mid b^2=a^n=1$ and $bab^{-1}=a^{-1}\rangle$.

For $(H)\in C(G)$ and $X\in Set_f{}^G$, we put

$$\mu_H(X)=\sharp\{x\in X\mid G_x=H\} \text{ and } \lambda_H(X)=\sharp\{e\in X/G\mid x\in e,\ (G_x)=(H)\}.$$

Theorem 2. 1 (cf. [3]). *If G is a finite abelian group, then*

$$A(G)^*=\langle -1,\ (1-[G/H])\mid (H)\in C(G) \text{ and } \sharp(G/H)=2\rangle.$$

Theorem 2. 2 (cf. [2] and [3]).

$$C(D_n)=\{(a^i),\ (b,\ a^i),\ (ba,\ a^j)\mid i\in n^*,\ j\in n_e^*\}.$$

In $A(D_n)$, we put

$$\alpha_i=D_n/\langle a^i\rangle,\ \beta_i=D_n/\langle b,a^i\rangle\ (i\in n^*),\ \gamma_j=D_n/\langle ba,\ a^j\rangle\ (j\in n_e^*),$$
$$\Delta_i=1+\alpha_i-2\beta_i\ (i\in n_o^*-\{1\}),\ \Theta_j=1+\alpha_{2j}-\beta_{2j}-\gamma_{2j}\ (j\in(n/2)^*-\{1\}),$$
$$\Delta_1=1-\alpha_1,\ \Theta_0=1-\beta_2 \text{ and } \Theta_1=1-\gamma_2.$$

For i, $j\in n^*$, we write m (resp. M) for the greatest common divisor (resp. the least common multiple) of i and j. Then we have the following (2.2.1) *and* (2.2.2).

$$\alpha_i\alpha_j=2m\alpha_M,\ \alpha_i\beta_j=\alpha_i\gamma_j=m\alpha_M,\ \gamma_i\gamma_j=2\gamma_M+((m/2)-1)\alpha_M,$$

(2.2.1)
$$\beta_i\beta_j=\begin{cases}\beta_M+((m-1)/2)\ \alpha_M & \textit{if } m \textit{ is odd}\\ 2\beta_M+((m/2)-1)\ \alpha_M & \textit{if } m \textit{ is even,}\end{cases}$$

$$\beta_i\gamma_j=\begin{cases}(m/2)\alpha_M & \textit{if } m \textit{ is even}\\ ((m-1)/2)\alpha_M+\gamma_M & \textit{if } m \textit{ is odd.}\end{cases}$$

$$A(D_n)^*=\begin{cases}\langle -1,\ \Delta_i \mid i\in n^*\rangle & \textit{if } n \textit{ is odd}\\ \langle -1,\ \Delta_i,\ \Theta_j,\ \Theta_0,\ \Theta_1 \mid i\in n_o{}^*,\ j\in(n/2)^*-\{1\}\rangle & \textit{if } n \textit{ is even,}\end{cases}$$

(2.2.2)
$$\sharp A(D_n)^*=\begin{cases}2^{\sharp n^*+1} & \textit{if } n \textit{ is odd}\\ 2^{\sharp n^*+2} & \textit{if } n \textit{ is even.}\end{cases}$$

Lemma 2.3. *Let X be a finite G-set. We have the following* (2.3.1)–(2.3.3):

(2.3.1) *If* $N\uparrow X=\sum_{(H)\in C(G)} f_{(H)}(N)\ [G/H]$, *then* $-1\uparrow X=\sum_{(H)\in C(G)} f_{(H)}(-1)\ [G/H]$.

(2.3.2) $X=\sum_{(H)\in C(G)} \lambda_H(X)\ [G/H]$.

(2.3.3) $A(G)^{**}=\langle -1\uparrow[G/H] \mid (H)\in C(G)\rangle$.

Proof. (2.3.1) follows from [4] (Section 3) and [1] (Section 2). (2.3.2)=[3] (2.2.1). Since $A(G)^*$ is an elementary abelian 2-group and an $A(G)$ module, (2.3.3) follows at once.

3. Proof of Main Theorem

By the simple caluculations, we have the following Lemmas 3.1 and 3.2.

Lemma 3.1. *For $m\in n^*$ and $l\in m^*$, we have the following* (3.1.1) *and* (3.1.2):

(3.1.1) $\sharp\{f\in(N\uparrow\beta_m) \mid (D_n)_f\supset\langle a^l\rangle\}=N^l$

$\sharp\{f\in(N\uparrow\beta_m) \mid (D_n)_f\supset\langle b\rangle\}=N^{[m/2]+1}$,

(*where $[k]$ denotes the largest integer that does not exceed k*),

$\sharp\{f\in(N\uparrow\beta_m) \mid (D_n)_f\supset\langle ba\rangle\}=N^{[(m+1)/2]}$,

$\sharp\{f\in(N\uparrow\beta_m) \mid (D_n)_f\supset\langle b,\ a^l\rangle\}=N^{[l/2]+1}$,

$\sharp\{f\in(N\uparrow\beta_m) \mid (D_n)_f\supset\langle ba,\ a^l\rangle\}=N^{[(l+1)/2]}$.

(3.1.2) *For $f,\ f'\in(N\uparrow\beta_m)$ $(f\neq f')$, if $(D_n)_f=(D_n)_{f'}=\langle b,\ a^l\rangle$ or $\langle ba, a^l\rangle$ and $f=f'$ in $(N\uparrow\beta_m)/D_n$, then l is even and $f=a^{(l/2)}f'$.*

Lemma 3.2. *For $m\in n^*$ and $l\in m^*$, we have the following* (3.2.1) *and* (3.2.2).

(3.2.1)
$$\sharp\{f\in(N\uparrow\alpha_m)\,|\,(D_n)_f\supset\langle a^l\rangle\}=N^{2l},$$
$$\sharp\{f\in(N\uparrow\alpha_m)\,|\,(D_n)_f\supset\langle b\rangle\}=N^{m},$$
$$\sharp\{f\in(N\uparrow\alpha_m)\,|\,(D_n)_f\supset\langle ba\rangle\}=N^{m},$$
$$\sharp\{f\in(N\uparrow\alpha_m)\,|\,(D_n)_f\supset\langle b,\ a^l\rangle\}=N^{l},$$
$$\sharp\{f\in(N\uparrow\alpha_m)\,|\,(D_n)_f\supset\langle ba,\ a^l\rangle\}=N^{l}.$$

(3.2.2) *For* $f,\ f'\in(N\uparrow\alpha_m)$ $(f\neq f')$, *if* $(D_n)_f=(D_n)_{f'}=\langle b,\ a^l\rangle$ *or* $\langle ba,\ a^l\rangle$ *and* $f=f'$ *in* $(N\uparrow\alpha_m)/D_n$, *then* l *is even and* $f=a^{(l/2)}f'$.

For $(N\uparrow\beta_m)\in A(D_n)$, $(N\uparrow\alpha_m)\in A(D_n)$ and $l\in m^*$, *we put*

$$\mu_{\langle b,a^l\rangle}(N\uparrow\beta_m)=\mu(l),\quad \mu_{\langle ba,a^l\rangle}(N\uparrow\beta_m)=\bar{\mu}(l),$$
$$\lambda_{\langle b,a^l\rangle}(N\uparrow\beta_m)=\lambda(l),\quad \lambda_{\langle ba,a^l\rangle}(N\uparrow\beta_m)=\bar{\lambda}(l),$$
$$\mu_{\langle b,a^l\rangle}(N\uparrow\alpha_m)=\mu^*(l),\quad \mu_{\langle ba,a^l\rangle}(N\uparrow\alpha_m)=\bar{\mu}^*(l),$$
$$\lambda_{\langle b,a^l\rangle}(N\uparrow\alpha_m)=\lambda^*(l),\quad \lambda_{\langle b,a^l\rangle}(N\uparrow\alpha_m)=\bar{\lambda}^*(l),$$
$$\lambda_{\langle a^l\rangle}(N\uparrow\beta_m)=\nu(l),\quad \lambda_{\langle a^l\rangle}(N\uparrow\alpha_m)=\nu^*(l),$$
$$\rho(l)=\begin{cases}1 & \text{if } l \text{ is odd}\\ 2 & \text{if } l \text{ is even.}\end{cases}$$

Lemma 3. 3. *We have the following identities*:

$$\mu(l)=\rho(l)\lambda(l),\ \ \bar{\mu}(l)=\rho(l)\bar{\lambda}(l),\ \ \mu^*(l)=\rho(l)\lambda^*(l),\ \ \bar{\mu}^*(l)=\rho(l)\bar{\lambda}^*(l),$$
$$\rho(l)\lambda^*(l)=N^l-\sum_{k\in l^*-\{l\}}\mu^*(k),\quad \rho(l)\bar{\lambda}^*(l)=N^l-\sum_{k\in l^*-\{l\}}\bar{\mu}^*(k),$$
$$\rho(l)\lambda(l)=N^{[l/2]+1}-\sum_{k\in l^*-\{l\}}\mu(k),\quad \rho(l)\bar{\lambda}(l)=N^{[(l+1)/2]}-\sum_{k\in l^*-\{l\}}\bar{\mu}(k),$$
$$2l\nu(l)=N^l-\Big(\sum_{k\in l^*}k\lambda(k)\Big)-\Big(\sum_{k\in l^*-\{l\}}2k\nu(k)\Big)-\Big(\sum_{k\in l_e^*}k\bar{\lambda}(k)\Big),$$
$$2l\nu^*(l)=N^{2l}-\Big(\sum_{k\in l^*}k\lambda^*(k)\Big)-\Big(\sum_{k\in l^*-\{l\}}2k\nu^*(k)\Big)-\Big(\sum_{k\in l_e^*}k\bar{\lambda}^*(k)\Big).$$

Proof. This follows from Lemmas 3.1 and 3.2.

Lemma 3. 4. *Let* p *be an odd prime and* $p^s\in m^*$. *We have the following identities*:

$$\lambda(p^s)=\bar{\lambda}(p^s)=N^{(p^s+1)/2}-N^{(p^{s-1}+1)/2}\ (s>0),$$
$$\lambda^*(p^s)=\bar{\lambda}^*(p^s)=N^{p^s}-N^{p^{s-1}}\ (s>0),$$

$\nu(p^s)=(1/2p^s)\ (N^{p^s}-N^{p^{s-1}}-p^s\lambda(p^s))\ (s>0),$

$\nu^*(p^s)=(1/2p^s)\ (N^{2p^s}-N^{2p^{s-1}}-p^s\lambda^*(p^s))\ (s>0),$

$\lambda(1)=\bar{\lambda}(1)=N,\ \lambda^*(1)=\bar{\lambda}^*(1)=N,\ \nu(1)=0$ *and* $\nu^*(1)=(N^2-N)/2.$

Proof. From Lemma 3. 3,

$$\lambda(p^s)=N^{(p^s+1)/2}-\sum_{i=0}^{s-1}\lambda(p^i)=N^{(p^s+1)/2}-\lambda(p^{s-1})-\sum_{i=0}^{s-2}\lambda(p^i)$$

$$=N^{(p^s+1)/2}-\left(N^{(p^{s-1}+1)/2}-\sum_{i=0}^{s-2}\lambda(p^i)\right)-\sum_{i=0}^{s-2}\lambda(p^i)$$

$$=N^{(p^s+1)/2}-N^{(p^{s-1}+1)/2},$$

and

$$2p^s\nu(p^s)=N^{p^s}-\sum_{i=0}^{s}p^i\lambda(p^i)-\sum_{i=0}^{s-1}2p^i\nu(p^i)$$

$$=N^{p^s}-\sum_{i=0}^{s}p^i\lambda(p^i)-\left(N^{p^{s-1}}-\sum_{i=0}^{s-1}p^i\lambda(p^i)-\sum_{i=0}^{s-2}2p^i\nu(p^i)\right)$$

$$-\sum_{i=0}^{s-2}2p^i\nu(p^i)=N^{p^s}-N^{p^{s-1}}-p^s\lambda(p^s).$$

The remaining part will be proved in the same way.

Lemma 3. 5. *For* $2^s\in m^*$ $(s>1)$, *we have the following identities*:

$\lambda(2^s)=(N^{2^{s-1}+1}-N^{2^{s-2}+1})/2,\ \lambda(2)=(N^2-N)/2,$

$\bar{\lambda}(2^s)=(N^{2^{s-1}}-N^{2^{s-2}})/2,\ \bar{\lambda}(2)=0,$

$\lambda^*(2^s)=(N^{2^s}-N^{2^{s-1}})/2,\ \lambda^*(2)=(N^2-N)/2,$

$\nu(2^s)=(N^{2^s}-N^{2^{s-1}}-2^s\lambda(2^s)-2^s\bar{\lambda}(2^s))/2^{s+1},\ \nu(2)=0,$

$\nu^*(2^s)=(N^{2^{s+1}}-N^{2^s}-2^s\lambda^*(2^s)-2^s\bar{\lambda}^*(2^s))/2^{s+1}$

(*this identity is true for* $s=1$).

Proof. From Lemma 3.3,

$$2\lambda(2^s)=N^{2^{s-1}+1}-\lambda(1)-\sum_{i=1}^{s-1}2\lambda(2^i)=N^{2^{s-1}+1}-\lambda(1)-2\lambda(2^{s-1})-\sum_{i=1}^{s-2}2\lambda(2^i)$$

$$=N^{2^{s-1}+1}-\lambda(1)-\left(N^{2^{s-2}+1}-\lambda(1)-\sum_{i=1}^{s-2}2\lambda(2^i)\right)-\sum_{i=2}^{s-2}2\lambda(2^i)$$

$$=N^{2^{s-1}+1}-N^{2^{s-2}+1},$$

and

$$2^{s+1}\nu(2^s)=N^{2^s}-\sum_{i=0}^{s}2^i\lambda(2^i)-\sum_{i=1}^{s}2^i\bar{\lambda}(2^i)-\sum_{i=0}^{s-1}2^{i+1}\nu(2^i)$$

$$=N^{2^s}-\sum_{i=0}^{s}2\lambda^i(2^i)-2^s\nu(2^{s-1})-\sum_{i=0}^{s-2}2^{i+1}\nu(2^i)-\sum_{i=1}^{s}2^i\bar{\lambda}(2^i)$$

$$=N^{2^s}-\sum_{i=0}^{s}2^i\lambda(2^i)-\sum_{i=1}^{s}2^i\bar{\lambda}(2^i)-\sum_{i=0}^{s-2}2^{i+1}\nu(2^i)$$

$$-\left(N^{2^{s-1}}-\sum_{i=0}^{s-1}2^i\lambda(2^i)-\sum_{i=1}^{s-1}2^i\bar{\lambda}(2^i)-\sum_{i=0}^{s-2}2^{i+1}\nu\,(2^i)\right)$$

$$=N^{2^s}-N^{2^{s-1}}-2^s\lambda(2^s)-2^s\bar{\lambda}(2^s).$$

The remaining part will be proved in the same way.

From Lemma 3.3, We have the following Lemma 3.6 by the same way as in Lemmas 3.4 and 3.5:

Lemma 3.6. *Let p and q be odd primes ($p\neq q$). We have the following identities*:

$$\lambda(pq)=\bar{\lambda}(pq)=N^{(pq+1)/2}-N^{(p+1)/2}-N^{(q+1)/2}+N,$$

$$\lambda^*(pq)=\bar{\lambda}^*(pq)=N^{pq}-N^p-N^q+N,$$

$$2pq\nu(pq)=N^{pq}-p\lambda(p)-q\lambda(q)-pq\lambda(pq)-\lambda(1)-2\nu(1)-2p\nu(p)-2q\nu(q),$$

$$2pq\nu^*(pq)=N^{2pq}-p\lambda^*(p)-q\lambda^*(q)-pq\lambda^*(pq)-\lambda^*(1)-2\nu^*(1)-2p\nu^*(p)-2q\nu^*(q),$$

$$2\lambda(2p^s)=N^{p^s+1}-N^{p^{s-1}+1}-\lambda(p^s),$$

$$2\bar{\lambda}(2p^s)=N^{p^s}-N^{p^{s-1}}-\bar{\lambda}(p^s),$$

$$2\lambda^*(2p^s)=N^{2p^s}-N^{2p^{s-1}}-\lambda^*(p^s),$$

$$2\bar{\lambda}^*(2p^s)=N^{2p^s}-N^{2p^{s-1}}-\bar{\lambda}^*(p^s),$$

$$4p^s\nu(2p^s)=N^{2p^s}-N^{2p^{s-1}}-2p^s\lambda(2p^s)-p^s\lambda(p^s)-2p^s\bar{\lambda}(2p^s)-2p^s\nu(p^s),$$

$$4p^s\nu^*(2p^s)=N^{4p^s}-N^{4p^{s-1}}-2p^s\lambda^*(2p^s)-p^s\lambda^*(p^s)-2p^s\bar{\lambda}^*(2p^s)-2p^s\nu^*(p^s),$$

$$2\lambda(4p^s)=N^{2p^s+1}-N^{2p^{s-1}+1}-\lambda(p^s)-2\lambda(2p^s),$$

$$2\bar{\lambda}(4p^s)=N^{2p^s}-N^{2p^{s-1}}-\bar{\lambda}(p^s)-2\bar{\lambda}(2p^s),$$

$$2\lambda^*(4p^s)=N^{4p^s}-N^{4p^{s-1}}-\lambda^*(p^s)-2\lambda^*(2p^s),$$

$$2\bar{\lambda}^*(4p^s)=N^{4p^s}-N^{4p^{s-1}}-\bar{\lambda}^*(p^s)-2\bar{\lambda}^*(2p^s),$$

$$8p^s\nu(4p^s)=N^{4p^s}-N^{4p^{s-1}}-4p^s\lambda(4p^s)-2p^s\lambda(2p^s)-p^s\lambda(p^s)-2p^s\bar{\lambda}(2p^s)$$

$$-4p^s\bar{\lambda}(4p^s)-2p^s\nu(p^s)-4p^s\nu(2p^s),$$

and

$$8p^s\nu^*(4p^s)=N^{4p^s}-N^{4p^{s-1}}-4p^s\lambda^*(4p^s)-2p^s\lambda^*(2p^s)-p^s\lambda^*(p^s)-2p^s\bar{\lambda}^*(2p^s)$$
$$-4p^s\bar{\lambda}^*(4p^s)-2p^s\nu^*(p^s)-4p^s\nu^*(2p^s).$$

Corollary 3.7. Let $\lambda(l)\,(-1)$, $\bar{\lambda}(l)\,(-1)$, $\lambda^*(l)\,(-1)$, $\bar{\lambda}^*(l)\,(-1)$, $\nu(l)\,(-1)$ and $\nu^*(l)\,(-1)$ be the rational integers obtained by substituting $N=-1$ into $\lambda\ (l)$, $\bar{\lambda}(l)$, $\lambda^*(l)$, $\bar{\lambda}^*(l)$, $\nu(l)$ and $\nu^*(l)$, respectively. Let s be an integer $(s>0)$. From Lemmas 3.4–3.6, we have the following table.

l	$\lambda(l)\,(-1)$	$\bar{\lambda}(l)\,(-1)$	$\lambda^*(l)(-1)$	$\bar{\lambda}(l)\,(-1)$	$\nu(l)\,(-1)$	$\nu^*(l)(-1)$
1	-1	-1	-1	-1	0	1
$p^s(p\equiv 3\ mod\ 4)$	$2(-1)^{s+1}$	$2(-1)^{s+1}$	0	0	$(-1)^s$	0
$p^s(p\equiv 1\ mod\ 4)$	0	0	0	0	0	0
2	1	0	1	1	0	-1
4	-1	1	0	0	0	0
$2^s(s>2)$	0	0	0	0	0	0
$2p^s(p\equiv 3\ mod\ 4)$	$(-1)^s$	$(-1)^s$	0	0	$(-1)^{s+1}$	0
$2p^s(p\equiv 1\ mod\ 4)$	0	0	0	0	0	0
$4p^s$	0	0	0	0	0	0
$pq\ \begin{pmatrix} p\equiv 3\ mod\ 4 \\ \text{and} \\ q\equiv 3\ mod\ 4 \end{pmatrix}$	-4	-4	0	0	2	0
$pq\ \begin{pmatrix} p\equiv 1\ \text{mod}\ 4 \\ \text{or} \\ q\equiv 1\ mod\ 4 \end{pmatrix}$	0	0	0	0	0	0

Remark 3.8. From (2.3.1) and (2.3.2), we have the following identities:

$$-1\uparrow\beta_m=\sum_{l\in m^*}(\nu(l)(-1)\alpha_l+\lambda(l)(-1)\beta_l)+\sum_{l\in m_e^*}\bar{\lambda}(l)(-1)\gamma_l,$$

and

$$-1\uparrow\alpha_m=\sum_{l\ni m^*}(\nu^*(l)(-1)\alpha_l+\lambda^*(l)(-1)\beta_l)+\sum_{l\in m_e^*}\bar{\lambda}^*(l)(-1)\gamma_l.$$

Since $D_n=\langle b,\ a\rangle=\langle ba,\ a\rangle$, $(b,\ a^{2i+1})=(ba,\ a^{2i+1})$ and $(b,\ a^{2i})\neq(ba,\ a^{2i})$, we have

$$-1\uparrow\gamma_m=\sum_{l\in m^*}\nu(l)(-1)\alpha_l+\sum_{l\in m_e^*}\lambda(l)(-1)\gamma_l+\sum_{l\in m_o^*}\lambda(l)(-1)\beta_l+\sum_{l\in m_e^*}\bar{\lambda}(l)(-1)\beta_l.$$

For $m,\ k\in n^*$ and $l\in m^*\cap k^*$, we have

$$\lambda_{\langle b,a^l\rangle}(N\uparrow\beta_m)=\lambda_{\langle b,a^l\rangle}(N\uparrow\beta_k),\quad \lambda_{\langle ba,a^l\rangle}(N\uparrow\beta_m)=\lambda_{\langle ba\cdot a^l\rangle}\ (N\uparrow\beta_k)$$

$$\lambda_{\langle b,\ a^l\rangle}(N\uparrow\alpha_m)=\lambda_{\langle b,\ a^l\rangle}(N\uparrow\alpha_k),\quad \lambda_{\langle ba,a^l\rangle}(N\uparrow\alpha_m)=\lambda_{\langle ba,\ a^l\rangle}\ (N\uparrow\alpha_k)$$

$$\lambda_{\langle a^l\rangle}(N\uparrow\beta_m)=\lambda_{\langle a^l\rangle}\ (N\uparrow\beta_k)\ \text{and}\ \lambda_{\langle a^l\rangle}(N\uparrow\alpha_m)=\lambda_{\langle a^l\rangle}(N\uparrow\alpha_k).$$

Therefore, for $m\in n^*$ and $k\in m^*$ we have the following identities:

$$-1\uparrow\beta_m=(-1\uparrow\beta_k)+\sum_{l\in m^*-k^*}(\nu(l)(-1)\alpha_l+\lambda(l)(-1)\beta_l)+\sum_{l\in m_e^*-k^*}\bar{\lambda}(l)(-1)\gamma_l,$$

$$-1\uparrow\alpha_m=(-1\uparrow\alpha_k)+\sum_{l\in m^*-k^*}(\nu^*(l)(-1)\alpha_l+\lambda^*(l)(-1)\beta_l)+\sum_{l\in m_e^*-k^*}\bar{\lambda}^*(l)\ (-1)\gamma_l,$$

and

$$-1\uparrow\gamma_m=(-1\uparrow\gamma_k)+\sum_{l\in m^*-k^*}\nu(l)(-1)\alpha_l+\sum_{l\in m_e^*-k^*}\lambda(l)(-1)\gamma_l+\sum_{l\in m_o^*-k^*}\lambda(l)(-1)\beta_l$$

$$+\sum_{l\in m_e^*-k^*}\bar{\lambda}(l)(-1)\beta_l.$$

Lemma 3.9. *Let r be an integer $(r>0)$. From Corollary 3.7 and Remark 3.8, we have the following identities*:

$$-1\uparrow\beta_{p^r}=\begin{cases}-1+\sum_{s=1}^{r}(-1)^s(\alpha_p s-2\beta_{p^s}) & if\ p\equiv 3\ mod\ 4\\ -1 & if\ p\equiv 1\ mod\ 4,\end{cases}$$

$$-1\uparrow\alpha_{p^r}=-1+\alpha_1,\quad -1\uparrow\alpha_1=-1+\alpha_1,\quad -1\uparrow\beta_1=-1,\quad -1\uparrow\beta_2=-1+\beta_2,$$

$$-1\uparrow\beta_{2^r}=-1+\beta_2-\beta_4+\gamma_4(r>1),\quad -1\uparrow\gamma_2=-1+\gamma_2,$$

$$-1\uparrow\gamma_{2^r}=-1+\gamma_2-\gamma_4+\beta_4(r>1),\quad -1\uparrow\alpha_{2^r}=-1+\beta_2+\gamma_2+\alpha_1-\alpha_2,$$

$$-1\uparrow\beta_{2p^r}=\begin{cases}-1+\beta_2+\sum_{s=1}^{r}(-1)^s(\alpha_{p^s}-2\beta_{p^s}-\alpha_{2p^s}+\beta_{2p^s}+\gamma_{2p^s}) & if\ p\equiv 3\ mod\ 4\\ -1+\beta_2 & if\ p\equiv 1\ mod\ 4,\end{cases}$$

$$-1\uparrow\gamma_{2p^r}=\begin{cases}-1+\gamma_2+\sum_{s=1}^{r}(-1)^s(\alpha_{p^s}-2\beta_{p^s}-\alpha_{2p^s}+\beta_{2p^s}+\gamma_{2p^s}) & if\ p\equiv 3\ mod\ 4\\ -1+r_2 & if\ p\equiv 1\ mod\ 4,\end{cases}$$

$$-1\uparrow\alpha_{2p^r}=-1+\alpha_1,\quad -1\uparrow\alpha_{4p^r}=-1+\alpha_1,\quad -1\uparrow\alpha_{pq}=-1+\alpha_1,$$

$$-1\uparrow\beta_{4p^r}$$

$$=\begin{cases}-1+\beta_2-\beta_4+\gamma_4+\sum_{s=1}^{r}(-1)^s(\alpha_{p^s}-2\beta_{p^s}-\alpha_{2p^s}+\beta_{2p^s}+\gamma_{2p^s}) & if\ p\equiv 3\ mod\ 4\\ -1+\beta_2-\beta_4+\gamma_4 & if\ p\equiv 1\ mod\ 4,\end{cases}$$

$$-1\uparrow\gamma_{4p^r}$$

$$=\begin{cases}-1+\gamma_2-\gamma_4+\beta_4+\sum_{s=1}^{r}(-1)^s(\alpha_{p^s}-2\beta_{p^s}-\alpha_{2p^s}+\beta_{2p^s}+\gamma_{2p^s}) & if\ p\equiv 3\ mod\ 4\\ -1+\gamma_2-r_4+\beta_4 & if\ p\equiv 1\ mod\ 4,\end{cases}$$

and

$$-1\uparrow\beta_{pq}=\begin{cases}-1-\alpha_p+2\beta_p-\alpha_q+2\beta_q+2\alpha_{pq}-4\beta_{pq} & \textit{if } p\equiv 3,\ q\equiv 3 \textit{ mod } 4\\ -1-\alpha_p+2\beta_p & \textit{if } p\equiv 3,\ q\equiv 1 \textit{ mod } 4\\ -1-\alpha_q+2\beta_q & \textit{if } p\equiv 1,\ q\equiv 3 \textit{ mod } 4\\ -1 & \textit{if } p\equiv 1,\ q\equiv 1 \textit{ mod } 4.\end{cases}$$

Lemma 3.10. *Let p and q be odd primes ($p\equiv 3$, $q\equiv 3$ mod 4). We have the following identities:*

$$-1\uparrow\beta_{p^r}=-\prod_{s=1}^{r}\Delta_{p^s},\qquad -1\uparrow\beta_{2p^r}=-(\prod_{s=1}^{r}\Delta_{p^s}\Theta_{p^s})\Theta_0,$$

$$-1\uparrow\gamma_{2p^r}=-(\prod_{s=1}^{r}\Delta_{p^s}\Theta_{p^s})\Theta_1,\qquad -1\uparrow\beta_{4p^r}=-(\prod_{s=1}^{r}\Delta_{p^s}\Theta_{p^s})\Theta_0\Theta_2,$$

$$-1\uparrow\gamma_{4p^r}=-(\prod_{s=1}^{r}\Delta_{p^s}\Theta_{p^s})\Theta_1\Theta_2,\qquad -1\uparrow\beta_{2^r}=-\Theta_0\Theta_2(r>1),$$

$$-1\uparrow\gamma_{2^r}=-\Theta_1\Theta_2(r>1),\qquad -1\uparrow\beta_2=-\Theta_0,\quad -1\uparrow\gamma_2=-\Theta_1,$$

$$-1\uparrow\alpha_{2^r}=-\Theta_0\Theta_1\Delta_1(r>0)\quad \textit{and}\quad -1\uparrow\beta_{pq}=-\Delta_p\Delta_q.$$

Proof. From (2.2.1) and Remark 3.8, the desired result follows by the induction on r.

Corollary 3.11. Let p be an odd prime ($p\equiv 3$ *mod* 4). Since $A(G)^*$ is an elementary abelian 2-group and an $A(G)$-module, we have the following identities:

$$-1\uparrow(\beta_{p^r}+\beta_{p^{r-1}})=\Delta_{p^r},\quad -1\uparrow(\beta_{2p^r}+\beta_{2p^{r-1}}+\beta_{p^r}+\beta_{p^{r-1}})=\Theta_{p^r},$$

$$-1\uparrow(1+\alpha_1)=\Delta_1,\quad -1\uparrow(1+\beta_2)=\Theta_0,\quad -1\uparrow(1+\gamma_2)=\Theta_1 \text{ and}$$

$$-1\uparrow(\beta_2+\beta_4)=-1\uparrow(\beta_2+\beta_{2^r})=\Theta_2.$$

Lemma 3.12. *For $m\in n^*$, we put*

$A(D_n)_m{}^{**}=\langle -1\uparrow\alpha_l,\ -1\uparrow\beta_l,\ -1\uparrow\gamma_{l'}\,|\, l\in m^*,\ l'\in m_e{}^*\rangle$. If $\Delta_m\in A(D_n)^{**}$, then $\Delta_m\in A(D_n)_m{}^{**}$. If $\Theta_m\in A(D_n)^*$ ($m>0$), then $\Theta_m\in A(D_n)_{2m}{}^{**}$.

Proof. If $\Delta_m\in A(D_n)^{**}$, then Δ_m has a following representation:

$$\Delta_m=(\prod_{l_1\in I_1}-1\uparrow\beta_{l_1})(\prod_{l_2\in I_2}-1\uparrow\alpha_{l_2})(\prod_{l_3\in I_3}-1\uparrow\gamma_{l_3}),$$

where I_1, $I_2\subset n^*$ and $I_3\subset n_e{}^*$. From (2.2.1) and Remark 3.8, we have the following identity:

$$\Delta_m=(\prod_{l_1\in I_1}-1\uparrow\beta_{(m,l_1)})(\prod_{l_2\in I_2}-1\uparrow\alpha_{(m,l_2)})(\prod_{l_3\in I_3}-1\uparrow\gamma_{(m,l_3)}),$$

where $(m,\ l)$ is the greatest common divisor of m and l. So, $\Delta_m\in A(D_n)_m{}^{**}$. Similarly

$\Theta_m \in A(D_n)_{2m}^{**}$ if $\Theta_m \in A(D_n)^{**}$.

Theorem 3.13. *Let p and q be odd primes ($p \neq q$). We have the following* (3.13.1)–(3.13.5).

(3.13.1) $A(D_n)^{**} \supset \langle -1, \Delta_{p^s}, \Theta_{p^s}, \Theta_0, \Theta_1, \Delta_1 \mid 1 \leq s \leq r \rangle$ *if* p^r, $2 \in n^*$ *and* $p \equiv 3$ *mod* 4.

(3.13.2) $\Delta_{p^s} \notin A(D_n)^{**}$ $(1 \leq s)$ *if* $p \equiv 1$ *mod* 4.

(3.13.3) $A(D_n)^{**} \cap \{\Theta_{2p}, \Theta_4, \Delta_{pq}\}$ *= empty set.*

(3.13.4) *If* $p \equiv 3$ *mod* 4, *then* $A(D_{p^r})^{**} = A(D_{p^r})^*$ *and* $A(D_{2p^r})^{**} = A(D_{2p^r})^*$.

(3.13.5) $A(D_4)^{**} = A(D_4)^*$.

Proof. From Lemmas 3.9–3.10, Corollary 3.11 and Lemma 3.12, we have

$$\Theta_{2p} \notin A(D_n)_{4p}^{**}, \quad \Theta_4 \notin A(D_n)_8^{**}, \quad \Delta_{pq} \notin A(D_n)_{pq}^{**} \text{ and}$$

$$\Delta_{p^s} \notin A(D_n)_{p^s}^{**} \text{ if } p \equiv 1 \text{ mod } 4.$$

Therefore, we have (3.13.2) and (3.13.3). The remaining part follows from Corollary 3.11 and (2.2.2).

Proof of Main Theorem. [I] follows from Theorem 3.13. Let G be an abelian group and H a subgroup of G with index 2. Then we have

$$N\uparrow[G/H] = N + ((N^2 - N)/2)[G/H]$$

$$(\text{so, } -1\uparrow(1+[G/H]) = 1 - [G/H]).$$

Therefore [II] follows form Theorem 2.1.

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