J. FAC. SCI., SHINSHU UNIVERSITY Vol. 18, No. 2, 1983

On Rings Satisfying the Polynomial Identity $(x+x^2)^2=0$

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(Received 8th June, 1983)

Throughout, R will represent a ring with center C. Let N be the set of nilpotents in R, and E the set of idempotents in R. If E is contained in C, R is said to be normal. It is well known that R is normal if and only if $[E,N]=0$ (resp. $[E,E]=0$).

Let n be a positive integer, and consider the following property:

$(*)_n$ $(x+x^2+\cdots+x^n)^{(n)}(=(1+x+x^2+\cdots+x^n)^n-1)=0$ for all $x\in R$.

If $2R=0$ then $(*)_2$ becomes $(x+x^2)^2(=x^2+x^4)=0$. In [2], R is called a generalized Boolean-like ring if $2R=0$ and $(x+x^2)(y+y^2)=0$ for all x, $y\in R$. According to [2, Theorem 2], a ring with $2R=0$ is a generalized Boolean-like ring if and only if N is an ideal with $N^2=0$ and R/N is a Boolean ring.

The present objective is to generalize [2, Theorems 3 and 4] as follows:

Theorem 1. *Suppose that* $2R=0$ *and* R *satisfies the polynomial identity* $(x+x^2)^2=0$. *Then the following are equivalent:*

1) *R is commutative.*
2) *E is an additive subsemigroup of R.*
3) *E is a subring of R.*
4) $[E,N]=0$.
5) *N is central.*
6) *Every element of R can be uniquely written as the sum of an element in E and an element in N.*

In preparation for proving the theorem, we state four lemmas. First, we quote [1, Lemma 3].

Lemma 1. *If R satisfies $(*)_{2k}$ and $2^\alpha R=0$, then N is an ideal and R/N is a Boolean ring.*

Corollary 1 (cf. [2, Theorem 2]). *Suppose that* $2R=0$. *If R satisfies the polynomial identity* $(x+x^2)^2=0$, *then N is a commutative nil ideal of bounded index 2 and R/N is a Boolean ring (and conversely).*

Lemma 2. *Suppose that* $2^\alpha R=0$. *If E is an additive subsemigroup of R, then R*

is normal and E is a Boolean ring.

Proof. We claim first that E is a group. In fact, if $e\in E$ then $2e\in E\cap N=0$, and therefore $-e=e\in E$. Moreover, for any $x\in R$ we have $e+ex(1-e)\in E$, and therefore $ex(1-e)\in E\cap N=0$, namely $ex=exe$. Similarly, $xe=exe$. Hence, R is normal and E is a Boolean ring.

Lemma 3. *Suppose that N is an ideal and R/N is a Boolean ring. Then the following are equivalent*:

1) *R is commutative.*
2) *R is normal and N is commutative.*
3) *$[E, N]=0$ and N is commutative.*
4) *N is central.*

Proof. As is well known, every idempotent of R/N can be lifted to an idempotent of R. Thus, every element of R is the sum of an element in E and an element in N, and the equivalence of 1) – 4) is almost clear.

Lemma 4. *Suppose that N is commutative and every element of R can be uniquely written as the sum of an element in E and an element in N. Then R is commutative.*

Proof. Given $e\in E$ and $x\in R$, we have $e+ex(1-e)\in E$ and $e+(1-e)xe\in E$. Hence, by the uniqueness, $ex(1-e)=0=(1-e)xe$, namely $ex=exe=xe$. This proves that R is normal, and therefore R is commutative

We are now ready to complete the proof of our theorem.

Proof of Theorem 1. According to Corollary 1, N is a commutative ideal and R/N is a Boolean ring. Obviously, 1), 4) and 5) are equivalent by Lemma 3, and 1), 2) and 3) are so by Lemmas 2 and 3. Finally, if R is commutative, then E is a Boolean ring and $R=E\oplus N$ (grouptheoretic direct sum). Hence, 1) and 6) are equivalent by Lemma 4.

References

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