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# On determining sets for $N(B_n)$ and $H^p(B_n)$

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## 1. Introduction

Let $n\geqq 2$ be an integer. Let $N(B_n)$ denote the Nevanlinna class on the unit open ball $B_n$ of the complex $n$-dimensional Euclidean space $\mathbf{C}^n$. Let $H^p(B_n)$, $0<p\leqq\infty$, denote the Hardy spaces on $B_n$. $H^\infty(B_n)$ is the space of all bounded holomorphic functions in $B_n$.

In [2], R. O. Kujala proposed three problems. One of them is on the complete characterization of the zero sets of functions in $N(B_n)$, by the Blaschke condition. This problem was solved affirmatively by G. M. Henkin [1] and H. Skoda [8], independently (See below Theorem A in §2.). Secondly Kujala asked whether a certain necessary condition for the zero sets in $H^\infty(B_n)$ (which is easily obtained through the Jensen Formula) is also sufficient. This problem has recently solved negatively in [3]. In the present paper we shall study the Kujala's last problem. He asked ([2], p. 260):

Can determining sets (or divisors) for $H^\infty(B_n)$ or $N(B_n)$ be reasonably characterized?

We shall consider when zero sets (of holomorphic functions) in $B_n$ are determining sets for $H^p(B_n)$, $0<p\leqq\infty$, or $N(B_n)$. By using the Henkin-Skoda theorem (Theorem A in §2), we can characterize completely the determining sets for $N(B_n)$ (Theorem 1 in §3). The characterization of the determining sets for $H^p(B_n)$, $0<p\leqq\infty$, is much more complicated. We shall only show the existence of various determining sets and non-determining sets for $H^p(B_n)$, $0<p\leqq\infty$ (See Theorem 2～Theorem 5 in §4 and §5.).

## 2. Preliminaries

Let $H(B_n)$ denote the space of all holomorphic functions in $B_n$. Put

$$H(B_n)^*=\{f\in H(B_n);\ f\not\equiv 0 \text{ in } B_n\},$$

$$N(B_n)^*=N(B_n)\cap H(B_n)^*,$$

$$H^p(B_n)^*=H^p(B_n)\cap H(B_n)^* \quad (0<p\leqq\infty).$$

We note that

$$H^{\infty}(B_n)\subset H^q(B_n)\subset H^p(B_n)\subset N(B_n)\subset H(B_n) \quad \text{if} \quad 0<p<q<\infty.$$

Let $f\in H(B_n)^*$. In a neighborhood of each point $a\in B_n$, $f$ can be expanded in a series of homogeneous polynomials:

$$f(z)=\sum_{k=0}^{\infty}P_k(z-a).$$

The integer

$$\nu_f(a)=\mathrm{Min}\,\{k\geqq 0;\ P_k\not\equiv 0\}$$

is called the zero multiplicity of $f$ at $a$. The integer-valued function $\nu_f$ defined in $B_n$ is called the zero-divisor of $f$.

Let $\mu$ be a non-negative integer-valued function defined in $B_n$. Then $\mu$ is called a *positive divisor* on $B_n$ if and only if it is locally the zero-divisor of some holomorphic function, that is, for each point $a\in B_n$ there exist a connected neighborhood $V$ of $a$ and a holomorphic function $f$ in $V$ such that $f\not\equiv 0$ and $\mu=\nu_f$ in $V$.

We denote by $\mathfrak{D}^+(B_n)$ the set of all positive divisors on $B_n$. Then we have the divisor map $\nu$ from $H(B_n)^*$ into $\mathfrak{D}^+(B_n)$ defined by letting $\nu(f)$ for $f$ in $H(B_n)^*$ be $\nu_f$.

Let $\mu$ be a positive divisor on the unit open disc $B_1$ in the complex plane. Define

$$n_\mu(r)=\sum_{\lambda\in rB_1}\mu(\lambda)$$

for $0<r\leqq 1$, and

$$N_\mu(r, s)=\int_s^r \{n_\mu(t)/t\}\,dt$$

for $0<s<r\leqq 1$, where $rB_1=\{\lambda\in\mathbf{C};\ |\lambda|<r\}$.

Let $n\geqq 2$ be an integer. Let $\mu\in\mathfrak{D}^+(B_n)$. Take a point $\zeta\in\partial B_n$, where $\partial B_n$ is the boundary of $B_n$. Define

$$\mu[\zeta](\lambda)=\mu(\lambda\zeta) \quad \text{for } \lambda\in B_1.$$

Put $E=\{\zeta\in\partial B_n;\ \mu[\zeta]\in\mathfrak{D}^+(B_1)\}$. Then $\sigma(\partial B_n\backslash E)=0$, where $\sigma$ is the rotation-invariant positive Borel measure on $\partial B_n$ for which $\sigma(\partial B_n)=1$. (See e.g. Stoll [9], p. 13.) We write

$$N_\mu(r,\ s;\ \zeta)=N_{\mu[\zeta]}(r,\ s)$$

if $\zeta\in E$, and we define

$$N_\mu(r,\ s)=\int_{\partial B_n} N_\mu(r,\ s;\ \zeta)d\sigma(\zeta)$$

for $0<s<r\leqq 1$.

Let $f\in H(B_n)^*$. We denote by $Z(f)$ the zero set of $f$:

$$Z(f)=\{z\in B_n;\ f(z)=0\}.$$

Then $Z(f)=\{z\in B_n;\ \nu_f(z)>0\}$.

We shall say that a positive divisor $\mu\in\mathfrak{D}^+(B_n)$ (resp. a zero set $Z(f)$ of some $f\in H(B_n)^*$) satisfies the *Blaschke condition* if and only if

$$N_\mu(1,\ s)<\infty \quad (\text{resp. } N_{\nu_f}(1,\ s)<\infty)$$

for some $s\in(0,\ 1)$. (See Kujala[2], p. 252 and Stoll[9], p. 41.)

**Theorem A** (Henkin[1]; Skoda[8]). *For* $\mu\in\mathfrak{D}^+(B_n)$, *the following two conditions are equivalent*:

(a) $\mu\in\nu(N(B_n)^*)$.

(b) $\mu$ *satisfies the Blaschke condition.*

Let $X$ be a subspace of $H(B_n)$. A zero set $M$ in $B_n$ (i.e. $M=Z(g)$ for some $g\in H(B_n)^*$) is said to be a *determining set* for $X$ if the assumptions $f\in X$, $M\subset Z(f)$ force $f\equiv 0$. Here the symbol $M\subset Z(f)$ means the inclusion relation with multiplicity; i.e. for two zero sets $M(=Z(g))$ and $Z(f)$, we write $M\subset Z(f)$ if and only if $\nu_g\leqq\nu_f$ in $B_n$.

We recall some results about the Hardy spaces $H^p(B_n)$ and the Bergman spaces $A^p(B_n)$. Assume $0<p<\infty$. For $f\in H(B_n)$, we define the $H^p$-norm and the $A^p$-norm as follows:

$$||f||_{H^p}=\sup_{0<r<1}\{\int_{\partial B_n}|f(r\zeta)|^p d\sigma(\zeta)\}^{1/p},$$

$$||f||_{A^p}=\{\int_{B_n}|f(z)|^p d\lambda(z)\}^{1/p}.$$

Here $\lambda$ is the Lebesgue measure on $\mathbf{C}^n$ normalized so that $\lambda(B_n)=1$. Then

$$H^p(B_n)=\{f\in H(B_n);\ ||f||_{H^p}<\infty\},$$

$$A^p(B_n)=\{f\in H(B_n);\ ||f||_{A^p}<\infty\}.$$

We note that $H^p(B_n)\subset A^p(B_n)$ $(0<p<\infty)$.

Suppose $n\geqq 2$. Let $f$ and $g$ be functions defined in $B_n$ and $B_{n-1}$, respectively, and define a restriction operator $\rho$ and an extension operator $E$ by

$$(\rho f)\ (z')=f(z',\ 0) \qquad (z'\in B_{n-1}),$$

$$(Eg)\ (z',\ z_n)=g(z') \qquad ((z',\ z_n)\in B_n).$$

We note that $\rho E$ is the identity operator.

The following two theorems will be used in §5 :

**Theorem B** (Rudin [6], p. 127). *Assume* $n \geqq 2$, $0 < p < \infty$.

(a) *The extension* $E$ *is a linear isometry of* $A^p(B_{n-1})$ *into* $H^p(B_n)$.

(b) *The restriction* $\rho$ *is a linear norm-decreasing map of* $H^p(B_n)$ *onto* $A^p(B_{n-1})$.

**Theorem C** (Rudin [6], p. 128). *Assume* $n \geqq 1$, $0 < p < \infty$. *If* $f \in H^p(B_n)$, *then*

$$|f(z)| \leqq 2^{n/p} \|f\|_{H^p} (1-|z|)^{-n/p} \qquad (z \in B_n).$$

## 3. Determining sets for $N(B_n)$

**Theorem 1.** *Suppose* $n \geqq 1$. *Let* $M$ *be a zero set in* $B_n$. *Then* $M$ *is a determining set for* $N(B_n)$ *if and only if* $M$ *does not satisfy the Blaschke condition.*

**Proof.** Suppose that $M$ satisfies the Blaschke condition. Then, by Theorem A, there exists a $g \in N(B_n)^*$ such that $Z(g) = M$. Hence $M$ cannot be a determining set for $N(B_n)$.

Conversly, suppose that $M$ is not a determining set for $N(B_n)$. Then there exists an $h \in N(B_n)^*$ such that $Z(h) \supset M$. By Theorem A, $Z(h)$ satisfies the Blaschke condition, so does $M$.

Since $N(B_n) \supset H^p(B_n)$ $(0 < p \leqq \infty)$, we obtain

**Corollary.** *Suppose* $n \geqq 1$. *Let* $M$ *be a zero set in* $B_n$. *If* $M$ *does not satisfy the Blaschke condition, then* $M$ *is a determining set for* $H^p(B_n)$ $(0 < p \leqq \infty)$.

## 4. Determining sets for $H^p(B_n)$, $0 < p \leqq \infty$

In this section we need the results of Rudin [5]. Let $\phi : (-\infty, \infty) \to [0, \infty)$ be a nondecreasing convex function, not identically 0, and let $H\phi(B_n)$ be the class of all $f \in H(B_n)$ whose growth is restricted by the requirement

$$\sup_{0<r<1} \int_{\partial B_n} \phi(\log|f(r\zeta)|) d\sigma(\zeta) < \infty.$$

If $\phi(t) = \max(0, t)$, then $H\phi(B_n) = N(B_n)$. If $0 < p < \infty$ and $\phi(t) = e^{pt}$, then $H\phi(B_n) = H^p(B_n)$.

**Theorem D** (Rudin [5], p. 58). *Fix* $n \geqq 2$. *Assume that* $\phi$ *and* $\psi$ *are non-constant, nonnegative, nondecreasing convex functions defined on* $(-\infty, \infty)$, *and that*

$$\lim_{t \to \infty} \psi(t)/\phi(t) = \infty.$$

*Then there exists an* $f \in H\phi(B_n)$ *with the following property:*

*If* $b \in H^\infty(B_n)$, $g \in H(B_n)^*$, *and* $h = (f+b)g$, *then some constant multiple of* $h$ *fails to be in* $H\psi(B_n)$.

Rudin remarked that the following theorem is obtained as a special case of the above Theorem D (See [5], p. 59.).

**Theorem E.** *Let $n\geqq 2$ be an integer and $0<p<\infty$. Then there exists an $f\in H^p(B_n)$ such that $Z(f)$ is a determining set for $\bigcup_{q>p} H^q(B_n)$ $(\supset H^\infty(B_n))$.*

Now we show that Theorem D furnishes two similar results to Theorem E.

**Theorem 2.** (cf. Rudin [4], pp. 60–62, Theorem 4. 1. 1) *Let $n\geqq 2$ be an integer. Then there exists an $f\in \bigcap_{0<p<\infty} H^p(B_n)$ such that $Z(f)$ is a determining set for $H^\infty(B_n)$.*

**Proof.** Put

$$\phi(t)=\begin{cases}\exp(t^2) & \text{if } t\geqq 0\\ 1 & \text{if } t<0,\end{cases}$$

$$\psi(t)=\begin{cases}\exp(t^3) & \text{if } t\geqq 0\\ 1 & \text{if } t<0.\end{cases}$$

Then $\phi$ and $\psi$ satisfy the assumptions in Theorem D. Hence there is an $f\in H_\phi(B_n)$ such that $f$ has the property described in the conclusion of Theorem D. We note that

$$H^\infty(B_n)\subset H_\psi(B_n)\subset H_\phi(B_n)\subset \bigcap_{0<p<\infty} H^p(B_n).$$

Since $f\in H_\phi(B_n)$, $f\in \bigcap_{0<p<\infty} H^p(B_n)$.

Suppose $h\in H^\infty(B_n)$ and $Z(h)\supset Z(f)$ i.e. $\nu_h\geqq \nu_f$ in $B_n$. Then $h=gf$ for some $g\in H(B_n)$. If $g\not\equiv 0$ in $B_n$, there is a constant $c\in \mathbf{C}$ such that $ch \notin H_\psi(B_n)$. Since $H^\infty(B_n)\subset H_\psi(B_n)$ we have $ch\notin H^\infty(B_n)$. This contradicts the assumption $h\in H^\infty(B_n)$. Therefore $g\equiv 0$ in $B_n$, and so, $h\equiv 0$ in $B_n$. Thus $Z(f)$ is a determining set for $H^\infty(B_n)$. Q.E.D.

**Theorem 3.** *Let $n\geqq 2$ be an integer. Then there exists an $f\in N(B_n)$ such that $Z(f)$ is a determining set for $\bigcup_{0<p<\infty} H^p(B_n)$.*

**Proof.** Put

$$\phi(t)=\max(0,\ t)\qquad (-\infty<t<\infty),$$

$$\psi(t)=\begin{cases}\exp(\sqrt{t}) & (t\geqq 1)\\ e & (t<1).\end{cases}$$

Then $\phi$ and $\psi$ satisfy the assumptions in Theorem D. Hence we can find an $f\in H_\phi(B_n)=N(B_n)$ with property described in the conclusion of Theorem D. We note that

$$\bigcup_{0<p<\infty} H^p(B_n)\subset H_\psi(B_n)\subset N(B_n).$$

The same reasoning as in the proof of Theorem 2 shows that $Z(f)$ is a determining set for $\bigcup_{0<p<\infty} H^p(B_n)$. Q.E.D.

## 5 Non-determining sets for $H^p(B_n)$, $0<p\leqq\infty$

In this section, in addition to the theorems (Theorem B and Theorem C) described in §2, we shall use the following J.H. Shapiro's result:

**Theorem F.** (See Shapiro [7], p. 245, Corollary 2.2 and p. 246, Corollary 2.5.) *Let $n\geqq 1$ be an integer and $0<p<\infty$. Then there exists an $f\in A^p(B_n)^*$ such that $\nu_f \notin \nu(\bigcup_{q>p} A^q(B_n)^*)$.*

**Theorem 4.** (cf. Rudin [6], §7.3.6) *Let $n\geqq 2$ be an integer and $0<p<\infty$. Then there exists an $f\in H^p(B_n)$ which satisfies the following two conditions:*

(a) *$Z(f)$ is not a determining set for $H^\infty(B_n)$.*

(b) *$\nu_f \notin \nu(\bigcup_{q>p} (H^q(B_n)^*)$.*

**Proof.** By Theorem F, there is a $g\in A^p(B_{n-1})$ such that $\nu_g \notin \nu(\bigcup_{q>p} A^q(B_{n-1})^*)$. Define $f=Eg$ in $B_n$, where $E$ is the extension operator defined in §2. Then it follows from Theorem B that $f\in H^p(B_n)$. If $\nu_f \in \nu(\bigcup_{q>p} H^q(B_n)^*)$, then there is an $h\in \bigcup_{q>p} H^q(B_n)^*$ with $\nu_h=\nu_f$. Therefore $h=fk$ for some $k\in H(B_n)$ with $Z(k)=\phi$. Put $h'=\rho h$ and $k'=\rho k$, where $\rho$ is the restriction operator defined in §2. Then

$$h'(z')=g(z')k'(z') \qquad \text{for } z'\in B_{n-1}.$$

Since $Z(k')=\phi$, we have $\nu_g=\nu_{h'}$. By theorem B we have $h'\in \bigcup_{q>p} A^q(B_{n-1})$, since $h\in \bigcup_{q>p} H^q(B_n)$. Thus $\nu_g\in \nu(\bigcup_{q>p} A^q(B_{n-1})^*)$. This contradicts the choice of the function $g$. Hence $\nu_f \notin \nu(\bigcup_{q>p} H^q(B_n)^*)$.

We turn to the proof of (a). Since $f\in H^p(B_n)$, Theorem C gives

$$|f(z)|\leqq 2^{n/p}\|f\|_{H^p}(1-|z|)^{-n/p} \qquad (z\in B_n).$$

Hence

$$|g(z')|\leqq 2^{n/p}\|f\|_{H^p}(1-|z'|)^{-n/p} \qquad (z'\in B_{n-1}). \tag{1}$$

Choose a positive integer $m$ with $n/p<m$. Define

$$F(z)=f(z)z_n^{2m}=g(z')z_n^{2m} \qquad (z=(z',z_n)\in B_n). \tag{2}$$

Then $F\in H(B_n)$. By (1) and (2),

$$|F(z)|\leqq 2^{n/p}\|f\|_{H^p}(1-|z'|)^{-n/p}|z_n|^{2m} \tag{3}$$

for $z=(z', z_n)\in B_n$. Since $|z_n|^2<1-|z'|^2<2(1-|z'|)$, we have

$$(1-|z'|)^{-n/p}<2^{n/p}|z_n|^{-2n/p}. \tag{4}$$

It follows from (3) and (4) that

$$|F(z)|\leq 2^{2n/p}||f||_{H^p}|z_n|^{2m-2n/p}\leq 2^{2n/p}||f||_{H^p}<\infty$$

for $z=(z', z_n)\in B_n$. Thus $F\in H^\infty(B_n)$. On the other hand, $Z(F)\supset Z(f)$ and $F\not\equiv 0$ in $B_n$, by virtue of (2). Hence $Z(f)$ is not a determining set for $H^\infty(B_n)$. Q. E. D.

**Theorem 5.** *Let $n\geq 3$ be an integer. Then there exists an $f\in\bigcap_{0<p<\infty}H^p(B_n)$ which satisfies the following two conditions :*

(a) *$Z(f)$ is not a determining set for $H^\infty(B_n)$.*

(b) $\nu_f\notin\nu(H^\infty(B_n)^*)$.

**Proof.** Since $n-1\geq 2$, Theorem 2 establishes the existence of a function $g\in\bigcap_{0<p<\infty}H^p(B_{n-1})$ such that $Z(g)$ is a determining set for $H^\infty(B_{n-1})$. Since $H^p(B_{n-1})\subset A^p(B_{n-1})$ $(0<p<\infty)$, we have $g\in\bigcap_{0<p<\infty}A^p(B_{n-1})$. Put $f=Eg$. Then Theorem B implies

$$f\in\bigcap_{0<p<\infty}H^p(B_n). \tag{5}$$

Choose a positive number $p$ with $n<p$. Define

$$F(z)=f(z)z_n^2=g(z')z_n^2 \tag{6}$$

for $z=(z', z_n)\in B_n$. By (4), (5), (6) and Theorem C, we have

$$|F(z)|\leq 2^{2n/p}||f||_{H^p}|z_n|^{2-2n/p}\leq 2^2||f||_{H^p}<\infty$$

for $z=(z', z_n)\in B_n$. Hence $F\in H^\infty(B_n)$. It follows from (6) that $Z(F)\supset Z(f)$ and $F\not\equiv 0$ in $B_n$. This proves (a)

The repetition of the argument used to prove that $\nu_f\notin\nu(\bigcup_{q>p}H^q(B_n)^*)$ in Theorem 4 gives now (b). Q. E. D.

## References


[1] G. M. Henkin, Solutions with estimates of H. Lewy and Poincaré–Lelong equations. Construction of functions of the Nevanlinna class with prescribed zeros in strictly pseudoconvex domains, Dokl. Akad. Nauk SSSR, **224** (1975), 2–13. Soviet Math. Dokl., **16** (1975), 1310–1314.

[2] R. O. Kujala, Generalized Blaschke conditions on the unit ball in $\mathbf{C}^p$, *Value Distribution Theory*, Part A, Marcel Dekker, New York, (1974), 250–261.

[3] Y. Matsugu, On a problem of Kujala, Mem. Fac. Sci. Kyushu Univ. Ser. A, **36** (1982), 145–150.

[4] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.

[5] W. Rudin. Zeros of holomorphic functions in balls, Indag. Math., **38** (1976), 57–65.

[6] W. Rudin, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

[7] J. H. Shapiro, Zeros of functions in weighted Bergman spaces, Michigan Math. J., **24** (1977), 243–256.

[8] H. Skoda, Valeurs au bord pour les solutions de l'opérateur d'', et caractérisation des zéros des fonctions de la classe de Nevanlinna, Bull. Soc. Math. France, **104** (1976), 225–299.

[9] W. Stoll, *Holomorphic Functions of Finite Order in Several Complex Variables*, Regional Conf. Series in Math., Amer. Math. Soc., Providence, R. I., 1974.