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On the zero sets of functions in the Bergman spaces and the Hardy spaces

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1. Introduction

Let $n\geqq 1$ be an integer. Let $H(B_n)$ denote the space of all holomorphic functions in the open unit ball B_n of the complex n-dimensional Euclidean space $\mathbf{C}^n$. Let Γ denote the class of all functions defined on $(-\infty, \infty)$ which are nonconstant, nonnegative, nondecreasing and convex. For each $\phi\in\Gamma$, we define

$$A^{\phi}(B_n)=\{f\in H(B_n);\int_{B_n}\phi(\log|f|)d\lambda<\infty\},$$

$$H^{\phi}(B_n)=\{f\in H(B_n);\ \sup_{0<r<1}\int_{\partial B_n}\phi(\log|f(r\zeta)|)d\sigma(\zeta)<\infty\}.$$

Here λ is the usual Lebesgue measure on $\mathbf{C}^n=\mathbf{R}^{2n}$, ∂B_n is the boundary of B_n and σ is the rotation invariant positive Borel measure on ∂B_n for which $\sigma(\partial B_n)=1$. If $\phi(t)=e^{pt}$, $0<p<\infty$, then $A^{\phi}(B_n)$ are the *Bergman spaces* $A^p(B_n)$ and $H^{\phi}(B_n)$ are the *Hardy spaces* $H^p(B_n)$. If $\phi(t)=\max(0, t)$, then $H^{\phi}(B_n)$ is the *Nevanlinna space* $N(B_n)$ and $A^{\phi}(B_n)$ is denoted by $A^0(B_n)$ throughout this paper. $H^{\infty}(B_n)$ stands for the space of all bounded holomorphic functions in B_n.

The open unit disc in the complex plane $\mathbf{C}$ will be denoted by U in place of B_1. It is well known that all of the spaces $H^p(U)$ $(0<p\leqq\infty)$ and the space $N(U)$ admit the same zero sets which are completely characterized by the Blaschke condition. (See e. g. [1], §2. 2.) When $n\geqq 2$, the situation is considerably more complicated. It was proved by W. Rudin [5] that for two different values of $p>0$ the zero sets of functions in the corresponding $H^p(B_n)$ differ.

Regarding the Bergman spaces $A^p(U)$, an analogous result was proved by C. Horowitz [2]: If $0<p<q<\infty$, then the zero sets of functions in $A^p(U)$ and those of functions in $A^q(U)$ are different. J. H. Shapiro [6] extended this theorem to the weighted Bergman spaces and to the case of several variables.

The purpose of this paper is to amplify the above results of Rudin, Horowitz and Shapiro. The summary of our results will be stated at the end of §2.

2. Preliminaries

By definition, it holds that

$$H^p(B_n)\subset A^p(B_n)$$

for any $p\in(0, \infty)$, and that

$$H^\infty(B_n)\subset H^q(B_n)\subset H^p(B_n)\subset N(B_n),$$

$$H^\infty(B_n)\subset A^q(B_n)\subset A^p(B_n)\subset A^0(B_n)$$

if $0<p<q<\infty$. For each $p\in(0, \infty)$, we define

$$H^{p-}(B_n)=\bigcup_{p<q<\infty} H^q(B_n),\quad H^{p+}(B_n)=\bigcap_{0<q<p} H^q(B_n),$$

$$A^{p-}(B_n)=\bigcup_{p<q<\infty} A^q(B_n),\quad A^{p+}(B_n)=\bigcap_{0<q<p} A^q(B_n).$$

Then

$$H^{p-}(B_n)\subset H^p(B_n)\subset H^{p+}(B_n),$$

$$A^{p-}(B_n)\subset A^p(B_n)\subset A^{p+}(B_n).$$

Let f be a holomorphic function in a connected open subset Ω of $\mathbf{C}^n$. Suppose $f\not\equiv 0$ in Ω. Take a point $a\in\Omega$. Then a series

$$f(z)=\sum_{k=m}^\infty P_k(z-a)$$

converges in some neighborhood of a and represents f in this neighborhood. Here P_k is a homogeneous polynomial of degree k and $P_m\not\equiv 0$. The polynomials P_k depend on f and a only. The integer

$$\nu_f(a)=m\geqq 0$$

is called the *zero multiplicity* of f at a. The integer-valued function ν_f defined in Ω is called the *zero-divisor* of f.

Let μ be a nonnegative integer-valued function in Ω. Then μ is called a *positive divisor* on Ω if and only if it is locally the zero-divisor of some holomorphic function, that is, for each point $a\in\Omega$ there exist a connected neighborhood V of a and a holomorphic function f in V such that $f\not\equiv 0$ and $\mu=\nu_f$ in V.

We denote by $\mathfrak{D}^+(B_n)$ the set of all positive divisors on B_n. Then we have the divisor map ν from $H(B_n)^*$ into $\mathfrak{D}^+(B_n)$ defined by letting $\nu(f)$ for f in $H(B_n)^*$ be ν_f. Here, for any subspace X of $H(B_n)$ we write

$$X^*=\{f\in X;\ f\not\equiv 0 \text{ in } B_n\}.$$

We recall that $\mu\in\mathfrak{D}^+(U)$ satisfies the *Blaschke condition* if and only if

$$\sum_{z\in U}\mu(z)(1-|z|)<\infty.$$

The set of positive divisors on U which satisfy the Blaschke condition will be denoted by $\mathfrak{D}_0$. The following theorem is classical:

Theorem A (See e.g. [1], §2.2.). *For any* $p\in(0,\infty)$,

$$\nu(H^\infty(U)^*)=\nu(\bigcap_{0<q<\infty}H^q(U)^*)=\nu(H^{p-}(U)^*)=\nu(H^p(U)^*)$$

$$=\nu(H^{p+}(U)^*)=\nu(\bigcap_{0<q<\infty}H^q(U)^*)=\nu(N(U)^*)$$

$$=\mathfrak{D}_0.$$

The main result of W. Rudin [5] is the following:

Theorem B ([5], p. 58). *Fix* $n\geqq 2$. *Suppose* ϕ, $\psi\in\Gamma$ *and*

$$\lim_{t\to\infty}\psi(t)/\phi(t)=\infty.$$

Then there exists an $f\in H^\phi(B_n)$ *with the following property:*

If $b\in H^\infty(B_n)$, $g\in H(B_n)^*$, *and*

$$h=(f+b)g,$$

then some constant multiple of h *fails to be in* $H^\psi(B_n)$.

Applying Theorem B to the case

$$\phi(t)=e^{pt},\quad \psi(t)=(2+p^2t^2)e^{pt},\quad 0<p<\infty,$$

Rudin showed the following:

Theorem C ([5], p. 59). *For any integer* $n\geqq 2$ *and any* $p\in(0,\infty)$,

$$\nu(H^{p-}(B_n)^*)\subsetneqq\nu(H^p(B_n)^*).$$

To describe the results of C. Horowitz and J. H. Shapiro, we define the "weighted" Bergman spaces $A_\mu{}^\phi$. From now on, μ will denote a finite, positive, rotation invariant Borel measure on U which gives positive mass to each annulus $r<|z|<1$. For each $\phi\in\Gamma$, we define

$$A_\mu{}^\phi=\{f\in H(U);\int_U\phi(\log|f|)d\mu<\infty\}.$$

The main result of J. H. Shapiro [6] is the following:

Theorem D ([6], Theorem 2.1.). *Assume that* ϕ *and* ψ *are strictly positive, convex, increasing, unbounded functions defined on* $(-\infty,\infty)$, *and that*

$$\sup_{-\infty<t<\infty}\phi(t+1)/\phi(t)<\infty,\qquad \sup_{-\infty<t<\infty}\psi(t+1)/\psi(t)<\infty,$$

$$\lim_{t\to-\infty}\phi(t)=0, \qquad \lim_{t\to-\infty}\psi(t)=0, \qquad \lim_{t\to\infty}\psi(t)/\phi(t)=\infty.$$

Then there is an $f\in A_\mu{}^\phi$ *such that for any positive integer* m, *any* $b\in H^\infty(U)$ *and any* $g\in H(U)^*$,

$$(f^m+b)g\notin A_\mu{}^{\psi_m},$$

where $\psi_m(t)=\psi(t/m)$.

C. Horowitz [2] considered the case $d\mu(z)=(1-|z|)^\alpha dxdy$, $\alpha>-1$. Shapiro noticed that with $\phi(t)=e^{pt}$ and $\psi(t)=(2+p^2t^2)e^{pt}$ Theorem D gives the Horowitz's result :

Theorem E ([2], Theorem 4.6 and Theorem 6.11; [6], Corollary 2.2 and Corollary 2.5). *For any integer* $n\geqq1$ *and any* $p\in(0, \infty)$,

$$\nu(A^{p-}(B_n)^*)\subsetneqq\nu(A^p(B_n)^*).$$

In §3, we shall prove some generalizations of Theorem D. In §4, making use of them and Theorem B, we shall describe the zero sets of functions in the spaces $A^p(B_n)$ and $H^p(B_n)$. The summary of our results is the following :

Theorem.

(a) $\nu(\bigcap_{0<q<\infty} A^q(U)^*)\subsetneqq\mathfrak{D}_0$, *so that,* $\bigcap_{0<q<\infty} A^q(U)\subsetneqq\bigcap_{0<q<\infty} H^q(U)$.

(b) *For any integer* $n\geqq2$ *and any* $p\in(0, \infty)$,

$$\nu(H^\infty(B_n)^*)\subsetneqq\nu(\bigcap_{0<q<\infty} H^q(B_n)^*)\subsetneqq\nu(H^{p-}(B_n)^*)\subsetneqq\nu(H^p(B_n)^*)$$

$$\subsetneqq\nu(H^{p+}(B_n)^*)\subsetneqq\nu(\bigcup_{0<q<\infty} H^q(B_n)^*)\subsetneqq\nu(N(B_n)^*).$$

(c) *For any integer* $n\geqq1$ *and any* $p\in(0, \infty)$,

$$\nu(H^\infty(B_n)^*)\subsetneqq\nu(\bigcap_{0<q<\infty} A^q(B_n)^*)\subsetneqq\nu(A^{p-}(B_n)^*)\subsetneqq\nu(A^p(B_n)^*)$$

$$\subsetneqq\nu(A^{p+}(B_n)^*)\subsetneqq\nu(\bigcup_{0<q<\infty} A^q(B_n)^*)\subsetneqq\nu(A^0(B_n)^*).$$

3. Generalizations of the Shapiro's theorem

Theorem 1. *Suppose* ϕ, $\psi\in\Gamma$ *and*

$$\lim_{t\to\infty}\psi(t)/\phi(t+1)=\infty.$$

Then there exists an $f\in A_\mu{}^\phi$ *such that for any positive integer* m, *any* $b\in H^\infty(U)$ *and any* $g\in H(U)^*$,

$$c(f^m+b)g\notin A_\mu{}^{\psi_m}$$

for some constant c, *where* $\phi_m(t)=\phi(t/m)$.

Proof. Our proof follows the same lines as [6], §3, pp. 248–251. (cf. [3], §3) Without loss of generality, we may assume that $\phi(t)=0$ for $t\leqq 0$. For $t\geqq 0$, we define

$$\Phi(t)=\phi(\log t),\quad \Phi_0(t)=\phi(\log t+1),\quad \Psi_m(t)=\phi_m(\log t).$$

Then Φ_0 is a continuous nondecreasing nonnegative function on $[0, \infty)$, and

$$\Phi_0(0)=0,\quad \lim_{t\to\infty}\Phi_0(t)=\infty,\quad \lim_{t\to\infty}\Psi(t)/\Phi_0(t)=\infty.$$

Using a W. Rudin's lemma ([5], pp. 59–60), we can show the following lemma (cf. [6], p. 248, Lemma; [3], §3, Lemma 2):

Lemma. *There exist sequences* $\{t_k\}$ *and* $\{a_k\}$ *of positive numbers increasing to* ∞, *and* $\{n_k\}$ *of positive integers increasing to* ∞, *and* $\{r_k\}$ *and* $\{\rho_k\}$ *with*

$$0<r_1<\rho_1<r_2<\rho_2<\cdots\to 1$$

such that if $u_k(z)=a_k z^{n_k}$ *and* $R_k=\{r_k<|z|\leqq\rho_k\}$, *then for* $k\geqq 2$ *the following conditions hold :*

(a) $t_k\geqq 4\sum_{j=1}^{k-1}a_j$ *and* $\Psi(t)/\Phi_0(t)>k$ *for* $t\geqq t_k$;

(b) $\int_U \Phi_0(|u_k|)d\mu=k^{-2}$;

(1) (c) $\int_{R_k}\Phi_0(|u_k|)d\mu>(2k^2)^{-1}$;

(d) $|u_k(z)|\geqq t_k$ if $|z|\geqq r_k$;

(e) $|u_k(z)|\leqq|u_{k-1}(z)|/5$ *if* $r_1\leqq|z|\leqq\rho_{k-1}$.

We now define

$$f(z)=\sum_{k=1}^{\infty}u_k(z)\qquad (z\in U).$$

The series converges uniformly on compact subsets of U, by (1$-$e). Hence $f\in H(U)$. By (1$-$a), (1$-$d) and (1$-$e), we have

(2) $|f|\leqq 5|u_k|/4+5|u_{k+1}|/4$ on $\{r_k\leqq|z|\leqq\rho_{k+1}\}$,

(3) $|f|\geqq|u_k|/2$ on R_k.

Using (2), we have

$$\int_{\{r_k<|z|\leqq r_{k+1}\}}\Phi(|f|)d\mu\leqq\int_{\{r_k<|z|\leqq r_{k+1}\}}\Phi(5|u_k|/4+5|u_{k+1}|/4)d\mu$$

$$\leqq\int_{\{r_k<|z|\leqq r_{k+1}\}}\{\Phi(5|u_k|/2)+\Phi(5|u_{k+1}|/2)\}\,d\mu$$

$$\leqq\int_U \Phi_0(|u_k|)d\mu+\int_U \Phi_0(|u_{k+1}|)d\mu.$$

It follows from (1—b) that

$$\int_U \Phi(|f|)d\mu=\Big(\int_{\{|z|\leqq r_1\}}+\sum_{k=1}^{\infty}\int_{\{r_k<|z|\leqq r_{k+1}\}}\Big)\Phi(|f|)d\mu$$

$$\leqq\int_{\{|z|\leqq r_1\}}\Phi(|f|)d\mu+\sum_{k=1}^{\infty}\Big\{k^{-2}+(k+1)^{-2}\Big\}<\infty.$$

Thus $f\in A_\mu^{\phi}$.

Fix a positive integer m. Suppose that $b\in H^\infty(U)$, $g\in H(U)^*$ and $h=(f^m+b)g$. Put

$$\beta=\sup_{z\in U}|b(z)|,\quad \delta=(2\pi)^{-1}\int_{-\pi}^{\pi}\log|g(r_1e^{i\theta})|d\theta$$

Since $\log|g|$ is subharmonic in U,

(4) $$-\infty<\delta\leqq(2\pi)^{-1}\int_{-\pi}^{\pi}\log|g(re^{i\theta})|d\theta<\infty\qquad (r_1\leqq r<1).$$

Choose a positive number c so that

(5) $$\log c+\delta-m\log 4>0.$$

We shall see that ch is not in $A_\mu^{\phi_m}$.

Since $t_k\to\infty$, there exists a positive integer $K\geqq 2$ such that

$$(t_k/4)^m>\beta\qquad \text{if } k\geqq K.$$

It follows from (1—d) and (3) that

(6) $$|f^m+b|\geqq(|u_k|/4)^m\qquad \text{on } R_k$$

for $k\geqq K$. Fix $k\geqq K$ and $r\in(r_k,\ \rho_k]$. By Jensen's convexity theorem, (4), (6) and (5), we have

$$(2\pi)^{-1}\int_{-\pi}^{\pi}\Psi_m(|ch(re^{i\theta})|)d\theta\geqq\psi((2\pi)^{-1}\int_{-\pi}^{\pi}\log|u_k(re^{i\theta})|d\theta)$$

$$=\Psi(|u_k(r)|).$$

Hence

(7) $$\int_{R_k}\Psi_m(|ch|)d\mu\geqq\int_{R_k}\Psi(|u_k|)d\mu\qquad \text{for } k\geqq K.$$

By (1—d), (1—a) and (1—c),

$$\int_{R_k}\Psi(|u_k|)d\mu>(2k)^{-1}\qquad \text{for } k\geqq 2.$$

It follows from (7) that

$$\int_U \Psi_m(|ch|)d\mu \geqq \sum_{k=K}^{\infty} \int_{R_k} \Psi_m(|ch|)d\mu \geqq \sum_{k=K}^{\infty} (2k)^{-1} = \infty.$$

Hence $ch \notin A_\mu^{\psi_m}$. This completes the proof.

Remark. If ψ satisfies the growth condition

$$\limsup_{t\to\infty} \psi(t+1)/\psi(t) < \infty,$$

then $A_\mu^{\psi_m}$ is closed under scalar multiplication. (cf. [5], p. 58) In that case, the conclusion $ch \notin A_\mu^{\psi_m}$ is simply that $h \notin A_\mu^{\psi_m}$. Moreover, if ϕ also satisfies the growth condition

$$\limsup_{t\to\infty} \phi(t+1)/\phi(t) < \infty,$$

then the condition $\lim_{t\to\infty} \psi(t)/\phi(t) = \infty$ implies that $\lim_{t\to\infty} \psi(t)/\phi(t+1) = \infty$. Theorem D is therefore a special case of Theorem 1.

Using Theorem 1, we obtain its analogue in the case of several complex variables :

Theorem 2. *Let $n \geqq 2$ be an integer. Assume that ϕ and ψ are as in* Theorem 1. *Then there exists an $f \in A^\phi(B_n)$ with the following property :*

If m is a positive integer, $b \in H^\infty(B_n)$, $g \in H(B_n)^*$ with $g(z_1, 0, \dots, 0) \not\equiv 0$ in U, and

$$h = (f^m + b)g,$$

then some constant mulptiple of h fails in $A^{\psi_m}(B_n)$.

Proof. (cf. [6], pp. 246–247, Proof of Corollary 2. 5.) By Theorem 1, there exists an $f_0 \in H(U)$ which satisfies the following two conditions :

(a) $$\int_U \phi(\log|f_0(z)|)\,(1-|z|^2)^{n-1} d\lambda(z) < \infty \,;$$

(b) If m is a positive integer, $b_0 \in H^\infty(U)$, $g_0 \in H(U)^*$, and $h_0 = (f_0{}^m + b_0)g_0$, then there exists a constant c such that

$$\int_U \psi_m(\log|ch_0(z)|)\,(1-|z|^2)^{n-1}\, d\lambda(z) = \infty.$$

Define

$$f(z_1, \dots, z_n) = f_0(z_1) \qquad \text{for } (z_1, \dots, z_n) \in B_n.$$

By Fubini's theorem and (a),

$$\int_{B_n} \phi(\log|f|)d\lambda = \pi^{n-1}((n-1)!)^{-1} \int_U \phi(\log|f_0(z)|)\,(1-|z|^2)^{n-1}\, d\lambda(z) < \infty,$$

so that $f\in A^{\phi}(B_n)$.

Suppose that m is a positive integer, $b\in H^{\infty}(B_n)$, $g\in H(B_n)^*$ with $g(z_1, 0,\dots, 0,)\neq 0$ in U, and $h=(f^m+b)g$. Define

$$h_0(z_1)=h(z_1, 0, \dots, 0),$$
$$b_0(z_1)=b(z_1, 0, \dots, 0),$$
$$g_0(z_1)=g(z_1, 0, \dots, 0),$$

for $z_1\in U$. Then $b_0\in H^{\infty}(U)$, $g_0\in H(U)^*$, and $h_0=(f_0{}^m+b_0)g_0$. It follows from Fubini's theorem and (b) that for some constant c

$$(8)\quad \int_{B_n}\phi_m(\log|ch(z_1, 0, \dots, 0)|)\,d\lambda(z_1, z_2, \dots, z_n)$$
$$=\pi^{n-1}((n-1)!)^{-1}\int_U\phi_m(\log|ch_0(z_1)|)\,(1-|z_1|^2)^{n-1}\,d\lambda(z_1)=\infty.$$

Fix $z_1\in U$. Put $\rho(z_1)=(1-|z_1|^2)^{1/2}$ and

$$D[r]=\{(z_2, \dots, z_n)\in\mathbf{C}^{n-1};\ |z_2|^2+\dots+|z_n|^2<r^2\}$$

for $r\in(0, \rho(z_1)]$. Define

$$G[z_1](z_2, \dots, z_n)=\phi_m(\log|ch(z_1, z_2, \dots, z_n)|)$$

for $(z_2, \dots, z_n)\in D[\rho(z_1)]$. Since $ch\in H(B_n)^*$ and ϕ_m is a nondecreasing convex function, $G[z_1]$ is plurisubharmonic in $D[\rho(z_1)]$. Hence

$$\pi^{n-1}((n-1)!)^{-1}(1-|z_1|^2)^{n-1}\phi_m(\log|ch(z_1, 0, \dots, 0)|)$$
$$=\lim_{r\uparrow\rho(z_1)}\pi^{n-1}\,((n-1)!)^{-1}r^{2n-2}G[z_1](0, \dots, 0)$$
$$\leqq\lim_{r\uparrow\rho(z_1)}\int_{D[r]}G[z_1](z_2, \dots, z_n)\,d\lambda(z_2, \dots, z_n)$$
$$=\int_{D[\rho(z_1)]}G[z_1](z_2, \dots, z_n)\,d\lambda(z_2, \dots, z_n).$$

It follows form Fubini's theorem that

$$(9)\quad \int_{B_n}\phi_m(\log|ch(z_1, 0, \dots, 0)|)\,d\lambda(z_1, z_2, \dots, z_n)$$
$$=\int_U\pi^{n-1}\,((n-1)!)^{-1}\,(1-|z_1|^2)^{n-1}\,\phi_m(\log|ch(z_1, 0, \dots, 0)|)\,d\lambda(z_1)$$
$$\leqq\int_U d\lambda(z_1)\int_{D[\rho(z_1)]}G[z_1](z_2, \dots, z_n)\,d\lambda(z_2, \dots, z_n)$$
$$=\int_{B_n}\phi_m(\log|ch|)d\lambda.$$

By (8) and (9), we have

$$\int_{B_n} \psi m(\log|ch|)d\lambda = \infty.$$

This completes the proof.

4. Zero sets of functions in the spaces $A^p(B_n)$ and $H^p(B_n)$

Theorem 3. (cf. Theorem C and Theorem E in §2)

(a) *For any $p\in(0, \infty)$ and any integer $n\geqq 1$,*

$$\nu(A^{p-}(B_n)^*)\subsetneqq\nu(A^p(B_n)^*)\subsetneqq\nu(A^{p+}(B_n)^*).$$

(b) *For any $p\in(0, \infty)$ and any integer $n\geqq 2$,*

$$\nu(H^{p-}(B_n)^*)\subsetneqq\nu(H^p(B_n)^*)\subsetneqq\nu(H^{p+}(B_n)^*).$$

Proof. Put

$$\phi(t)=\begin{cases} t^{-1}e^{pt} & (t\geqq p^{-1}) \\ pe & (t< p^{-1}), \end{cases}$$

$$\psi(t)=e^{pt} \qquad (-\infty<t<\infty).$$

Then ϕ and ψ satisfy the assumptions in Theorem 1. Hence Theorem 1 and Theorem 2 give

$$\nu(A^p(B_n)^*)\subsetneqq\nu(A^{p+}(B_n^*)) \qquad (n\geqq 1,\ 0<p<\infty).$$

Likewise, Theorem B gives

$$\nu(H^p(B_n)^*)\subsetneqq\ \nu(H^{p+}(B_n)^*) \qquad (n\geqq 2,\ 0<p<\infty).$$

Theorem 4. (cf. [3], §4, Theorem 5; [4], §4, Theorem 3)

(a) *For any integer $n\geqq 1$,*

$$\nu(\bigcup_{0<p<\infty} A^p(B_n)^*)\subsetneqq\nu(A^0(B_n)^*).$$

(b) *For any integer $n\geqq 2$,*

$$\nu(\bigcup_{0<p<\infty} H^p(B_n)^*)\subsetneqq\nu(N(B_n)^*).$$

Proof. Put

$$\phi(t)=\max(0,\ t) \qquad (-\infty<t<\infty),$$

$$\psi(t)=\begin{cases} \exp(\sqrt{t}) & (t\geqq 1) \\ e & (t<1). \end{cases}$$

By applying Theorem 1, Theorem 2 and Theorem B to these ϕ and ψ, we have

$$\nu(\bigcup_{0<p<\infty} A^p(B_n)^*)\subset\nu(A^\psi(B_n)^*)\subsetneqq \nu(A^\phi(B_n)^*)=\nu(A^0(B_n)^*)$$

for any $n\geqq 1$ and

$$\nu(\bigcup_{0<p<\infty} H^p(B_n)^*)\supset\nu(H^\psi(B_n)^*)\subsetneqq\nu(H^\phi(B_n)^*)=\nu(N(B_n)^*)$$

for any $n\geqq 2$.

Theorem 5. (cf. [4], §4, Theorem 2)

(a) *For any integer* $n\geqq 1$,

$$\nu(\bigcap_{0<p<\infty} A^p(B_n)^*)\supsetneqq\nu(H^\infty(B_n)^*).$$

(*b*) *For any integer* $n\geqq 2$,

$$\nu(\bigcap_{0<p<\infty} H^p(B_n)^*)\supsetneqq\nu(H^\infty(B_n)^*).$$

Proof. Fix $n\geqq 1$. Put

$$\phi(t)=\begin{cases}\exp(t^2) & (t\geqq 0)\\ 1 & (t<0),\end{cases}$$

$$\psi(t)=\begin{cases}\exp(t^3) & (t\geqq 0)\\ 1 & (t<0).\end{cases}$$

Theorem 2 (or Theorem 1) then establishes the existence of an $f\in A^\phi(B_n)$ with the following property :

If $g\in H(B_n)^*$ and $g(z_1, 0, \dots, 0)\not\equiv 0$ in U, then $cfg\notin A^\phi(B_n)$ for some constant c.

Suppose that $\nu_f\in\nu(H^\infty(B_n)^*)$. Then there exists an $h\in H^\infty(B_n)^*$ and a $g\in H(B_n)^*$ such that the zero set of g is empty and $h=fg$. Hence $ch\notin A^\phi(B_n)$ for some constant c. Since $H^\infty(B_n)\subset A^\phi(B_n)$, it follows that $ch\notin H^\infty(B_n)$. This contradicts the fact $h\in H^\infty(B_n)$. Thus $\nu_f\notin\nu(H^\infty(B_n)^*)$. On the other hand, $\nu_f\in\nu(A^\phi(B_n)^*)\subset\nu(\bigcap_{0<p<\infty} A^p(B_n)^*)$. Hence we cbtain (a).

Apply Theorem B instead of Theorem 2 (or Theorem 1). The same reasoning as above now gives (b).

Corollary. (cf. [3], §4, Remark 1 and §5, Remark 3)

(a) *There exists an* $f\in \bigcap_{0<p<\infty} A^p(U)$ *whose zero set does not satisfy the Blaschke condition.*

(b)
$$\bigcap_{0<p<\infty} H^p(U)\subsetneqq \bigcap_{0<p<\infty} A^p(U).$$

Proof. These are immediate consequences of Theorem A and Theorem 5.

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