J. FAC. SCI., SHINSHU UNIVERSITY Vol. 18, No. 1, 1983

The strict inclusion relation between the spaces $H_\varphi(U)$ on the open unit disc U in $\mathbf{C}$

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(Received 7th Oct., 1982)

1. Introduction

Let $n \geq 1$ be an integer. Let $H(B_n)$ denote the space of all holomorphic functions in the open unit ball B_n of the complex n-dimensional Euclidean space $\mathbf{C}^n$. Let $\varphi: (-\infty, \infty) \to [0, \infty)$ be a nondecreasing convex function, not identically 0, and let $H_\varphi(B_n)$ be the class of all $f \in H(B_n)$ whose growth is restricted by the requirement

$$\sup_{0<r<1} \int_{\partial B_n} \varphi(\log|f(rw)|)d\sigma(w) < \infty,$$

where ∂B_n is the boundary of B_n and σ is the Euclidean volume element on the unit sphere ∂B_n in $\mathbf{C}^n$ normalized so that the volume of the sphere is 1. If $\varphi(x) = \max(0, x)$, then $H_\varphi(B_n)$ is called to be the *Nevanlinna class* $N(B_n)$. If $\varphi(x) = e^{px}$, $0<p<\infty$, then $H_\varphi(B_n)$ are called to be the *Hardy classes* $H^p(B_n)$. By $H^\infty(B_n)$ we shall denote the space of all bounded holomorphic functions in B_n.

In [4], W. Rudin proved the following theorem:

Theorem A (Rudin [4], p. 58). *Fix $n \geq 2$. Assume that φ and ψ are nonconstant, nondecreasing, nonnegative convex functions defined on $(-\infty, \infty)$, and that*

$$\lim_{t\to\infty} \psi(t)/\varphi(t) = \infty.$$

Then there exists an $f \in H_\varphi(B_n)$ with the following property:

If $b \in H^\infty(B_n)$, $g \in H(B_n)$, $g \not\equiv 0$, and

$$h = (f+b)g,$$

then some constant multiple of h fails to be in $H_\psi(B_n)$.

In the case $n=1$, this theorem is not valid. Indeed, if $\varphi = e^{px}$, $0<p<\infty$ and $\psi = (2+p^2x^2)e^{px}$, then Theorem A implies that the zero sets of functions in $H^p(B_1)$ differ from the zero sets of functions in $H^q(B_1)$, for any $q>p$. But this is false when $n=1$.

The purpose of the present paper is to prove the following analogue of Theorem

A in the case of $n=1$:
(The open unit disc in $\mathbf{C}$ and the unit circle will be denoted by U and T, in place of B_1 and ∂B_1, respectively.)

Theorem 1. *Assume that φ and ϕ are as in Theorem A. Then there exists an $f\in H_\varphi(U)$ such that $2f\notin H_\phi(U)$.*

Applying Theorem A and Theorem 1, in §3 we shall describe the strict inclusion relation between the Hardy classes $H^p(B_n)$, $0<p\leq\infty$, and the Nevanlinna class $N(B_n)$.

2. Proof of Theorem 1

We need the lemma which was used to prove the Rudin's theorem (Theorem A in §1) in [4].

Lemma (Rudin [4], pp. 59–60). *Suppose*

(i) *μ is a finite positive measure on a set Ω;*

(ii) *v is a real measurable function on Ω, with $0\leq v<1$ a.e., whose essential supremum is 1;*

(iii) *Φ is a continuous nondecreasing real function on $[0, \infty)$, with $\Phi(0)=0$ and $\Phi(t)\to\infty$ as $t\to\infty$;*

(iv) $0<\delta<\infty$.

Then there exist constants $c_k\in(0, \infty)$, for $k=1, 2, 3, \ldots$, such that

$$\int_\Omega \Phi(c_k v^k)\,d\mu=\delta.$$

These c_k also satisfy

$$\lim_{k\to\infty} c_k\alpha^k=0$$

whenever $|\alpha|<1$.

If $0<t<\infty$ and if $Y_k=Y_k(t)$ is the set of all $x\in\Omega$ at which $c_k v^k(x)>t$, then

$$\lim_{k\to\infty}\int_{Y_k}\Phi(c_k v^k)\,d\mu=\delta.$$

Proof of Theorem 1. Without loss of generality, we can assume that

$$\varphi(t)=0 \quad \text{if} \quad t\leq 0. \tag{1}$$

Choose a sequence $\{X_j\}_{j=1,2,\ldots}$ of nonempty connected open subsets of T so that

$$X_j\subset\left\{e^{i\theta}\in T;\ 0<\theta<\frac{\pi}{2}\right\} \tag{2}$$

and $X_j\cap X_k=\phi$ if $j\neq k$. For each $j=1, 2, 3, \ldots$, pick $w_j\in X_j$.

Define

$$S_j(z)=\frac{1}{2}(w_j^{-1}z+1) \qquad (z\in \boldsymbol{C})$$

for each j and

$$D=\left\{z\in \boldsymbol{C};\ \left|z-\frac{1}{2}\right|=\frac{1}{2}\right\}.$$

Then

$$S_j(U)=D,\ S_j(w_j)=1,$$

$$\max_{z\in T}|S_j(z)|=\max_{z\in \overline{U}}|S_j(z)|=1,$$

and

(3) $$|S_j(z)|<1 \text{ for } z\in\overline{U},\ z\neq w_j.$$

Moreover, the following inequalities hold:

(4) $$\frac{1}{2}(1+r)|S_j(z)|\leq|S_j(rz)|\leq|S_j(z)|$$

for $0<r<1$, $z\in\bigcup_{i=1}^{\infty}X_i$. In fact, fix $r\in(0,\ 1)$. Put

(5) $$V(z)=\frac{S_j(rz)}{S_j(z)}=\frac{w_j^{-1}rz+1}{w_j^{-1}z+1}.$$

Then $V(T)=\left\{w\in \boldsymbol{C};\ \mathrm{Re}\ w=\frac{1+r}{2}\right\}$. Hence

(6) $$\min_{z\in T}|V(z)|=\frac{1+r}{2}.$$

Let $z\in\bigcup_{i=1}^{\infty}X_i$. Then, by (2), we can write $z=w_je^{i\theta_j}$ for some θ_j with $-\frac{\pi}{2}<\theta_j<\frac{\pi}{2}$. A simple calculation shows that

(7) $$|V(z)|^2=\frac{1+r^2+2r\cos\theta_j}{2+2\cos\theta_j}<1,$$

since $-\frac{\pi}{2}<\theta_j<\frac{\pi}{2}$ and $0<r<1$. (5), (6) and (7) give (4).

Since $\lim_{t\to\infty}\psi(t)/\varphi(t)=\infty$, there are numbers $t_j>3j$ such that

(8) $$\psi(\log(1+t))>j^3\varphi(\log(1+t)) \quad \text{if } t>t_j.$$

We now apply the lemma, for each positive integer j, with (T, σ) in place of (Ω, μ), and with

$$v_j(z)=|S_j(z)|,$$
$$\Phi(t)=\varphi(\log(1+t)),$$
$$\delta_j=2j^{-2},$$
$$\alpha_j=\max_{z\in T\setminus X_j}|S_j(z)|.$$

Then $0<\alpha_j<1$.

The lemma shows that there exist positive numbers $a_j=c_{k_j}$ (where k_j is a sufficiently large positive integer) such that, setting

$$F_j(z)=a_j(S_j(z))^{k_j} \qquad (z\in\mathbf{C}) \tag{9}$$

we have

$$\int_T\varphi(\log(1+|F_j|))d\sigma=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(\log(1+|F_j(e^{i\theta})|))d\theta=2j^{-2}, \tag{10}$$

$$|F_j(z)|<2^{-j} \text{ on } T\setminus X_j \text{ and for } |z|<1-j^{-1}, \tag{11}$$

$$\int_{Y_j}\varphi(\log(1+|F_j|))d\sigma>j^{-2}, \tag{12}$$

where $Y_j=\{z\in T;\ |F_j(z)|>t_j\}$.

By (11), $Y_j\subset X_j$. By (8) and (12),

$$\int_{Y_j}\phi(\log(1+|F_j|))d\sigma>j. \tag{13}$$

We now define

$$f(z)=\sum_{j=1}^{\infty}F_j(z) \qquad (z\in U). \tag{14}$$

The series converges uniformly on compact subsets of U, by (11). Hence $f\in H(U)$.

To prove that $f\in H_\varphi(U)$, for $N=1, 2, 3, \ldots,$ define

$$M_N(z)=|F_1(z)+\ldots+F_N(z)| \qquad (z\in\mathbf{C}), \tag{15}$$

$$M(z)=\sum_{j=1}^{\infty}|F_j(z)| \qquad (z\in T). \tag{16}$$

Since the sets X_j are disjoint, (11) implies that

$$M(z)\leq\begin{cases}1 & \text{in } T\setminus\bigcup_{j=1}^{\infty}X_j\\ 1+|F_j(z)| & \text{in } X_j.\end{cases}$$

It follows from (1) that

$$\varphi(\log\ M(z))\leq\begin{cases} 0 & \text{in } T\setminus\bigcup_{j=1}^{\infty}X_j \\ \varphi(\log(1+|F_j(z)|)) & \text{in } X_j. \end{cases}$$

Hence (10) implies

$$\int_T\varphi(\log\ M)d\sigma\leq\sum_{j=1}^{\infty}2j^{-2}=3^{-1}\pi^2<4. \tag{17}$$

Since $F_1+\ldots+F_N$ is a holomorphic function in $\mathbf{C}$, $\log\ M_N$ is subharmonic in $\mathbf{C}$, for each N, and so is $\varphi(\log\ M_N)$, because φ is convex and nondecreasing. Moreover, $M_N(z)\leq M(z)$ for $z\in T$, by (15) and (16). It follows from (17) that

$$\int_T\varphi(\log\ M_N(rz))\,d\sigma(z)\leq\int_T\varphi(\log\ M_N)d\sigma<4 \tag{18}$$

for $0<r<1$. If we fix r and let $N\to\infty$, $M_N(rz)\to|f(rz)|$ uniformly on T. Hence (18) gives

$$\int_T\varphi(\log|f(rz)|)d\sigma(z)\leq4\quad(0<r<1).$$

Thus $f\in H_\varphi(U)$.

We turn to proving that $2f\notin H_\varphi(U)$. Fix $j\in\{1, 2, 3,\ldots\}$ and choose r_j so that

$$0<r_j<1\ \text{and}\left\{1-\left(\frac{1+r_j}{2}\right)\right\}^{k_j}||F_j||_\infty<2^{-j}, \tag{19}$$

where $||F_j||_\infty=\max_{z\in T}|F_j(z)|=a_j$. For $z\in Y_j$, by (14), (9), (4), (11) and (19),

$$\begin{aligned}|f(r_jz)| &= |\sum_{i=1}^{\infty}F_i(r_jz)|\\ &\geq|F_j(r_jz)|-\sum_{i\neq j}|F_i(r_jz)|\\ &\geq\left(\frac{1+r_j}{2}\right)^{k_j}|F_j(z)|-\sum_{i\neq j}|F_i(z)|\\ &\geq\left(\frac{1+r_j}{2}\right)^{k_j}|F_j(z)|-(1-2^{-j})\\ &=|F_j(z)|-\left\{1-\left(\frac{1+r_j}{2}\right)^{k_j}\right\}|F_j(z)|-1+2^{-j}\\ &\geq|F_j(z)|-\left\{1-\left(\frac{1+r_j}{2}\right)^{k_j}\right\}||F_j||_\infty-1+2^{-j}\end{aligned}$$

$$>|F_j(z)|-1.$$

Since $|F_j(z)|>t_j>3j\geq 3$ for $z\in Y_j$,

$$|2f(r_jz)|>|F_j(z)|+1$$

for $z\in Y_j$. It follows from (13) that

$$\int_{Y_j}\phi(\log|2f(r_jz)|)d\sigma(z)>j.$$

Thus

$$\int_T\phi(\log|2f(r_jz)|)d\sigma(z)>j \quad (j=1, 2, 3, \dots),$$

so that,

$$\sup_{0<r<1}\int_T\phi(\log|2f(rz)|)d\sigma(z)=\infty.$$

This means $2f\notin H_\phi(U)$. The proof is complete.

3. The strict inclusion relation between the Hardy classes $H^p(B_n)$ and the Nevanlinna class $N(B_n)$

By Theorem A and Theorem 1, we obtain the following

Theorem 2. *Let $n\geq 1$ be an integer. Assume that φ and ϕ are nonconstant, nondecreasing, nonnegative convex functions defined on $(-\infty, \infty)$, and that*

$$\lim_{t\to\infty}\phi(t)/\varphi(t)=\infty.$$

Then there exists an $f\in H_\varphi(B_n)$ such that some constant multiple of f fails to be in $H_\phi(B_n)$.

Remark 1. If ϕ satisfies the growth condition

$$\limsup_{t\to\infty}\phi(t+1)/\phi(t)<\infty,$$

then $H_\phi(B_n)$ is closed under scalar multiplication. (See Rudin [4], p. 58.) In that case, the conclusion of Theorem 2 is simply

$$H_\phi(B_n)\subsetneqq H_\varphi(B_n).$$

Now we apply Theorem 2 to the description of the strict inclusion relation between the classes $H^p(B_n)$, $0<p\leq\infty$ and the class $N(B_n)$.

First we note that

$$H^\infty(B_n)\subset H^q(B_n)\subset H^p(B_n)\subset N(B_n)$$

if $0<p<q<\infty$. For each $p\in(0,\infty)$, we define

$$H^{p-}(B_n)=\bigcup_{p<q<\infty} H^q(B_n),\ H^{p+}(B_n)=\bigcap_{0<q<p} H^q(B_n).$$

Then

$$H^{p-}(B_n)\subset H^p(B_n)\subset H^{p+}(B_n)\qquad (0<p<\infty).$$

Theorem 3.

$$H^{p-}(B_n)\subsetneqq H^p(B_n)\subsetneqq H^{p+}(B_n)\qquad (0<p<\infty).$$

Proof (cf. Rudin [4], p. 59, (c)).

(i) Put

$$\varphi(t)=e^{pt}\qquad (-\infty<t<\infty),$$
$$\phi(t)=\begin{cases} te^{pt} & (t\geq 0)\\ 0 & (t<0).\end{cases}$$

Then φ and ϕ satisfy the assumptions in Theorem 2. Moreover, ϕ satisfies the condition

$$\lim_{t\to\infty}\phi(t+1)/\phi(t)=e^p<\infty.$$

It follows from Remark 1 that

$$H\phi(B_n)\subsetneqq H\varphi(B_n)=H^p(B_n).$$

Since $H^{p-}(B_n)\subset H\phi(B_n)$, this implies

$$H^{p-}(B_n)\subsetneqq H^p(B_n).$$

(ii) Put

$$\varphi(t)=\begin{cases} t^{-1}e^{pt} & (t\geq p^{-1})\\ pe & (t<p^{-1}),\end{cases}$$
$$\phi(t)=e^{pt}\qquad (-\infty<t<\infty).$$

Then Theorem 2 and Remark 1 imply that

$$H^p(B_n)=H\phi(B_n)\subsetneqq H\varphi(B_n)\subset H^{p+}(B_n).$$

Remark 2. In the case $n=1$, some *outer functions* give another proof of Theorem 3. Let f be a positive measurable function on T such that $\log f\in L^1(T)$. Define

$$Q_f(z)=\exp\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log f(e^{it})\,dt\right\}\quad (z\in U).$$

Then $Q_f(z)$ is called the *outer function* with respect to the function f. We note the following theorem:

Theorem B (See e. g. Rudin [3], Theorem 17. 16.). *Fix* $p\in(0, \infty]$. *Let* f *be a positive measurable function on* T *such that* $\log f\in L^1(T)$. *Then* $Q_f\in H^p(U)$ *if and only if* $f\in L^p(T)$.

Now fix $p\in(0, \infty)$. Put

$$f(e^{it})=\begin{cases} t^{-1/p}(-\log t)^{-2/p} & (0<t<e^{-1}) \\ 1 & (t\in[-\pi, \pi]\setminus(0, e^{-1})), \end{cases}$$

$$g(e^{it})=|t|^{-1/p} \qquad (t\in[-\pi, \pi]).$$

Then

$$f\in L^p(T),\ f\notin \bigcup_{p<q<\infty} L^q(T),$$

$$g\notin L^p(T),\ g\in \bigcap_{0<q<p} L^q(T),$$

and

$$\log f\in L^1(T), \quad \log g\in L^1(T).$$

(See Hardy–Littlewood–Pólya [1], §6. 1.) It follows from Theorem B that

$$Q_f\in H^p(U),\ Q_f\notin \bigcup_{p<q<\infty} H^q(U)=H^{p-}(U),$$

$$Q_g\notin H^p(U),\ Q_g\in \bigcap_{0<q<p} H^q(U)=H^{p+}(U).$$

Theorem 4.

$$H^\infty(B_n)\subsetneqq \bigcap_{0<p<\infty} H^p(B_n).$$

Proof (cf. [2], §4, Theorem 2). With

$$\varphi(t)=\begin{cases} \exp(t^2) & \text{if } t\geq 0 \\ 1 & \text{if } t<0, \end{cases}$$

$$\phi(t)=\begin{cases} \exp(t^3) & \text{if } t\geq 0 \\ 1 & \text{if } t<0, \end{cases}$$

Theorem 2 establishes the existence of an $f\in H\varphi(B_n)$ such that $cf\notin H\phi(B_n)$ for some constant c. Note that

$$H^\infty(B_n)\subset H\phi(B_n)\subset H\varphi(B_n)\subset \bigcap_{0<p<\infty} H^p(B_n).$$

Hence $f\in \bigcap_{0<p<\infty} H^p(B_n)$ but $f\notin H^\infty(B_n)$.

Remark 3. In the case $n=1$, as well as in Theorem 3, some outer functions give another proof of Theorem 4. Put

$$f(e^{it})=\begin{cases}-\log t & (0<t<e^{-1}) \\ 1 & (t\in[-\pi,\ \pi]\setminus(0,\ e^{-1})).\end{cases}$$

Then $f\in \bigcap_{0<p<\infty} L^{p}(T)\setminus L^{\infty}(T)$ and $\log f\in L^{1}(T)$. (See [1], §6. 1.) It follows from Theorem B that

$$Q_{f}\in \bigcap_{0<p<\infty} H^{p}(U)\setminus H^{\infty}(U).$$

Theorem 5.

$$\bigcup_{0<p<\infty} H^{p}(B_{n})\subsetneqq N(B_{n}).$$

Proof (cf. [2], §4, Theorem 3). Put

$$\varphi(t)=\max\ (0,\ t) \qquad (-\infty<t<\infty),$$

$$\psi(t)=\begin{cases}\exp\ (\sqrt{t}) & (t\geq 1) \\ e & (t<1).\end{cases}$$

Then φ and ψ satisfy the assumptions in Theorem 2. In addition, ψ satisfies the growth condition

$$\lim_{t\to\infty} \psi(t+1)/\psi(t)=1<\infty.$$

Hence

$$\bigcup_{0<p<\infty} H^{p}(B_{n})\subset H_{\psi}(B_{n})\subsetneqq H_{\varphi}(B_{n})=N(B_{n}).$$

Remark 4. In the case $n=1$, a simple fuction gives another proof of Theorem 5. Put

$$f(z)=\exp\left(\frac{1+z}{1-z}\right) \qquad (z\in U).$$

Then $f\in N(U)$, $|f^{*}|=1$ a. e., and

$$\log\ |f(0)|=1>0=(2\pi)^{-1}\int_{-\pi}^{\pi} \log\ |f^{*}(e^{it})|\,dt.$$

Here f^{*} denotes the radial limits of f. (See Rudin [3], §17. 19.) If $f\in \bigcup_{0<p<\infty} H^{p}(U)$, then

$$\log|f(0)|\leq(2\pi)^{-1}\int_{-\pi}^{\pi}\log|f^*(e^{it})|\,dt.$$

(See [3], Theorem 17.17.) Hence $f\notin\bigcup_{0<p<\infty}H^p(U)$.

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