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The inclusion relation between the Bergman spaces $A^p(U)$, the Hardy spaces $H^p(U)$ and the Nevanlinna space $N(U)$ on the open unit disc U in $\mathbf{C}$

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1. Introduction

By U and T we shall denote the open unit disc in the complex plane $\mathbf{C}$ and the unit circle, respectively. The space of all holomorphic functions in U will be denoted by $H(U)$. Let $\varphi:(-\infty,\ \infty)\to[0,\ \infty)$ be a nondecreasing convex function, not identically 0, and let $H\varphi(U)$ (resp. $A\varphi(U)$) be the class of all $f\in H(U)$ whose growth is restricted by the requirement

$$\sup_{0<r<1}(2\pi)^{-1}\int_{-\pi}^{\pi}\varphi(\log|f(re^{i\theta})|)d\theta<\infty$$

$$(\text{resp. } \pi^{-1}\int_{0}^{1}\int_{-\pi}^{\pi}\varphi(\log|f(re^{i\theta})|)rdrd\theta<\infty).$$

If $\varphi(x)=\max(0,\ x)$, $H\varphi(U)$ is said to be the *Nevanlinna space* $N(U)$ and $A\varphi(U)$ will be denoted by $BN(U)$. If $\varphi(x)=e^{px}$, $0<p<\infty$, then $H\varphi(U)$ (resp. $A\varphi(U)$) are said to be the *Hardy spaces* $H^p(U)$ (resp. the *Bergman spaces* $A^p(U)$). By $H^\infty(U)$ we shall denote the space of all bounded holomorphic functions in U.

In [3], we proved the following theorem:

Theorem A ([3], §1, Theorem 1). *Assume that φ and ϕ are nonconstant, nondecreasing, nonnegative convex functions defined on $(-\infty,\ \infty)$, and that*

$$\phi(t)/\varphi(t)\to+\infty \text{ as } t\to+\infty.$$

Then there exists an $f\in H\varphi(U)$ such that $2f\notin H\phi(U)$.

If ϕ satisfies the growth condition

$$\limsup_{t\to\infty}\ \phi(t+1)/\phi(t)<\infty,$$

then $H\phi(U)$ is closed under scalar multiplication, so that, the conclusion is simply

$$H\phi(U)\subsetneqq H\varphi(U).$$

On the other hand, J. H. Shapiro [7] proved the following theorem:

Theorem B ([7], Theorem 2.1). *Assume* φ *and* ψ *are strictly positive, convex, increasing, unbounded functions defined on* $(-\infty, \infty)$, *and that*

$$\sup_{-\infty<t<\infty} \varphi(t+1)/\varphi(t)<\infty, \qquad \sup_{-\infty<t<\infty} \psi(t+1)/\psi(t)<\infty,$$

$$\lim_{t\to-\infty} \varphi(t)=0, \qquad \lim_{t\to-\infty} \psi(t)=0,$$

$$\lim_{t\to\infty} \psi(t)/\varphi(t)=\infty.$$

Then there exists an $f\in A_\varphi(U)$ *with the following property:*

If n *is a positive integer,* $b\in H^\infty(U)$, $g\in H(U)$, $g\not\equiv 0$, *and*

$$h=(f^n+b)g,$$

then $h\notin A_{\psi_n}(U)$, *where* $\psi_n(t)=\psi(t/n)$.

(In fact, Shapiro proved this theorem more generally on weighted Bergman spaces.)

Applying Theorem A, we studied in [3] the inclusion relation between the Hardy spaces $H^p(U)$, $0<p\leq\infty$. The purpose of this paper is to study the inclusion relation between the Bergman spaces $A^p(U)$, $0<p<\infty$, the Nevanlinna space $N(U)$ and the Hardy spaces $H^p(U)$, $0<p\leq\infty$. To do so, we need the following generalization of Theorem B:

Theorem 1. *Let* φ *and* ψ *be nonconstant, nondecreasing, nonnegative convex functions defined on* $(-\infty, \infty)$. *Assume that*

$$\lim_{t\to\infty} \psi(t)/\varphi(t+1)=\infty,$$

and that there exists a number $t_0\in(-\infty,\infty)$ *such that* $\varphi(t_0)>0$ *and*

$$\sup_{t\geq t_0} \varphi(t+1)/\varphi(t)<\infty.$$

Then there exists an $f\in A_\varphi(U)$ *with the following property:*

If n *is a positive integer,* $b\in H^\infty(U)$, $g\in H(U)$, $g\not\equiv 0$, *and*

$$h=(f^n+b)g,$$

then some constant multiple of h *fails to be in* $A_{\psi_n}(U)$.

2. Preliminaries

It is easily shown that

$$H^\infty(U)\subset H^q(U)\subset H^p(U)\subset N(U),$$

$$A^\infty(U)\subset A^q(U)\subset A^p(U)\subset BN(U)$$

if $0<p<q<\infty$. Here we write $A^\infty(U)=H^\infty(U)$. For each $p\in(0, \infty)$, we define

$$H^{p-}(U)=\bigcup_{p<q<\infty} H^q(U), \qquad H^{p+}(U)=\bigcap_{0<q<p} H^q(U),$$

$$A^{p-}(U)=\bigcup_{p<q<\infty} A^q(U), \qquad A^{p+}(U)=\bigcap_{0<q<p} A^q(U).$$

Then

$$H^{p-}(U)\subset H^p(U)\subset H^{p+}(U),$$
$$A^{p-}(U)\subset A^p(U)\subset A^{p+}(U).$$

Let $f\in H(U)$. Take a point $a\in U$. Assume $f\not\equiv 0$ in U. Then a power series

$$f(z)=\sum_{k=m}^{\infty} c_k(z-a)^k$$

converges in some neighborhood of a and represents f in this neighborhood. Here $c_m\neq 0$. The integer

$$\nu_f(a)=m\geq 0$$

is called the *zero multiplicity* of f at a. The integer–valued function ν_f defined in U is called the *zero–divisor* of f.

Let μ be a nonnegative integer–valued function defined in U. Then μ is called a *positive divisor* on U if and only if it is locally the zero–divisor of some holomorphic function, that is, for each point $a\in U$ there exists a connected neighborhood V of a and a holomorphic function f in V such that $f\not\equiv 0$ and $\mu=\nu_f$ in V.

We denote by $\mathfrak{D}^+(U)$ the set of all positive divisors on U. Then we have the *divisor map* ν from $H(U)^*$ into $\mathfrak{D}^+(U)$ defined by letting $\nu(f)$ for f in $H(U)^*$ be ν_f. Here, for any subspace X of $H(U)$ we write

$$X^*=\{f\in X;\ f\not\equiv 0 \text{ in } U\}.$$

We recall that $\mu\in\mathfrak{D}^+(U)$ satisfies the *Blaschke condition* if and only if

$$\sum_{z\in U}\mu(z)(1-|z|)<\infty.$$

The set of positive divisors on U which satisfy the Blaschke condition will be denoted by D_0. The following classical theorem will be used in §5 :

Theorem C (See e. g. Duren [1], §2. 2.). For any $p\in(0,\ \infty)$,

$$\nu(H^p(U)^*)=\nu(H^{p-}(U)^*)=\nu(H^{p+}(U)^*)=\nu(H^\infty(U)^*)=\nu(N(U)^*)=D_0.$$

The following is an immediate consequence of Theorem 1 :

Theorem 2. *Assume that φ and ϕ are as in Theorem* 1. *In addition, assume that ϕ satisfies the condition* $\limsup_{t\to\infty}\phi(t+1)/\phi(t)<\infty$. *Then*

$$\nu(A_\psi(U)^*) \subsetneqq \nu(A_\varphi(U)^*).$$

Hence

$$A_\psi(U) \subsetneqq A_\varphi(U).$$

3. Proof of Theorem 1

Our proof is a modification of the Shapiro's proof of Theorem B. (cf. Shapiro [7], pp. 248–251).

Step 1. Without loss of generality, we can assume that

$$\varphi(t)=0 \text{ if } t\leq 0. \tag{1}$$

In fact, when $\varphi(0)>0$, we put

$$\varphi_0(t)=\begin{cases} \varphi(t)-\varphi(0) & (t>0) \\ 0 & (t\leq 0). \end{cases} \tag{2}$$

Then φ_0 has the same properties as φ does. In addition, φ_0 satisfies (1). Because of (2), $\varphi-\varphi_0$ is bounded, hence $A_\varphi(U)=A_{\varphi_0}(U)$.

For $t\geq 0$, define

$$\begin{aligned} &\Phi(t)=\varphi(\log t), \quad \Phi_0(t)=\varphi(\log t+1), \\ &\Psi(t)=\psi(\log t). \end{aligned} \tag{3}$$

Then Φ_0 is a continuous nondecreasing nonnegative function on $[0, \infty)$ and $\Phi_0(t)\to\infty$ as $t\to\infty$. By (1), $\Phi_0(0)=0$. Since $\lim_{t\to\infty} \psi(t)/\varphi(t+1)=\infty$,

$$\lim_{t\to\infty} \Psi(t)/\Phi_0(t)=\infty. \tag{4}$$

Put

$$M=\sup_{t\geq t_0} \varphi(t+1)/\varphi(t).$$

Since φ is nondecreasing, it follows from (3) that

$$\Phi(s+t)\leq M(\Phi(s)+\Phi(t)) \qquad (s\geq s_0,\ t\geq s_0), \tag{5}$$

where $s_0=\exp(t_0)$. And $1<M<\infty$, by the hypothesis.

Step 2. We need the following lemma due to W. Rudin [5]:

Lemma D ([5], pp. 59–60). Suppose

(i) *μ is a finite positive measure on a set Ω;*

(ii) *v is a real measurable function on Ω, with* $0\leq v<1$ a. e., *whose essential supremum is* 1;

(iii) *Λ is a continuous nondecreasing real function on* $[0, \infty)$, *with* $\Lambda(0)=0$ *and* $\Lambda(t)\to\infty$ *as* $t\to\infty$;

(iv) $0<\delta<\infty$.

Then there exist constants $c_k\in(0, \infty)$, *for* $k=1, 2, 3,\dots$, *such that*

$$\int_\Omega \Lambda(c_k v^k)d\mu=\delta.$$

These c_k *also satisfy*

$$\lim_{k\to\infty} c_k\gamma^k=0$$

whenever $|\gamma|<1$.

If $0<t<\infty$ *and if* $Y_k=Y_k(t)$ *is the set of all* $x\in\Omega$ *at which* $c_k v^k(x)>t$, *then*

$$\lim_{k\to\infty}\int_{Y_k} \Lambda(c_k v^k)d\mu=\delta.$$

By λ we shall denote the Lebesgue measure on $\mathbf{C}=\mathbf{R}^2$, so normalized that $\lambda(U)=1$. We now apply Lemma D, for each positive integer k, with (U, λ) in place of (Ω, μ), and with

$$v(z)=|z|,$$
$$\Lambda(t)=\Phi_0(t),$$
$$\delta=(k^2M)^{-1}.$$

Then the following holds:

Lemma 1. *There exist sequences* $\{c_{kn}\}_{n=1,2,3,\dots}$ *of real numbers such that*

(a) $\int_U \Phi_0(|c_{kn}z^n|)d\lambda=(k^2M)^{-1}$;

(b) $0<c_{k1}\le c_{k2}\le c_{k3}\le\dots$, $\lim_{n\to\infty} c_{kn}=\infty$;

(c) $\lim_{n\to\infty} c_{kn}\gamma^n=0$ *whenever* $|\gamma|<1$;

(d) $\lim_{n\to\infty}\int_{\{|c_{kn}z^n|>t\}} \Phi_0(|c_{kn}z^n|)d\lambda=(k^2M)^{-1}$ *for each* $t>0$.

Lemma 2. *There exist four sequences* $\{t_k\}$, $\{a_k\}$, $\{r_k\}$ *and* $\{\rho_k\}$ *of real numbers, and one sequence* $\{n_k\}$ *of integers with*

$$s_0<t_1<t_2<t_3<\dots,\quad \lim_{k\to\infty} t_k=\infty,$$

$$0<a_1<a_2<a_3<\dots,\quad \lim_{k\to\infty} a_k=\infty,$$

$$0<n_1<n_2<n_3<\dots,\ \lim_{k\to\infty} n_k=\infty,$$

$$0<r_1<\rho_1<r_2<\rho_2<\dots,\ \lim_{k\to\infty} r_k=\lim_{k\to\infty} \rho_k=1,$$

such that if $u_k(z)=a_k z^{n_k}$ *and* $R_k=\{z\in\mathbf{C}\,;r_k<|z|\leq\rho_k\}$, *then for* $k\geq 2$ *the following conditions hold*:

(a) $t_k\geq 4\sum_{j=1}^{k-1} a_j$ and $\Psi(t)/\Phi_0(t)>kM$ *if* $t\geq t_k$;

(b) $\int_U \Phi_0(|u_k|)d\lambda=(k^2M)^{-1}$;

(6) (c) $\int_{R_k} \Phi_0(|u_k|)d\lambda>(2k^2M)^{-1}$;

(d) $|u_k(z)|\geq t_k$ *if* $|z|\geq r_k$;

(e) $|u_k(z)|\leq|u_{k-1}(z)|/5$ *if* $r_1\leq|z|\leq\rho_{k-1}$.

Proof. We prove the lemma by induction. Choose any positive integer n_1 and any positive numbers t_1, a_1, r_1, ρ_1, with $s_0<t_1<a_1$, $0<r_1<\rho_1<1$. Suppose $k\geq 2$, and suppose the five sequences have been successfully chosen for all indices less than or equal to $k-1$. By (4), there exists a positive number t_k such that

$$t_k>t_{k-1},\ t_k>k,\ t_k\geq 4\sum_{j=1}^{k-1} a_j,$$

$$\Psi(t)\ /\Phi_0(t)>kM\ \textit{for}\ t\geq t_k.$$

By Lemma 1, there exists a positive integer n_k with $n_{k-1}<n_k$, such that, letting $a_k=c_{kn_k}$, we have

$$a_k>t_k,\ a_k>a_{k-1},\ a_k\rho_{k-1}^{n_k}\leq a_{k-1}r_1^{n_{k-1}}\ /5, \tag{7}$$

$$\int_U \Phi_0(|a_k z^{n_k}|)d\lambda=(k^2M)^{-1},$$

and

$$\int_{\{|a_k z^{n_k}|>t_k\}} \Phi_0(|a_k z^{n_k}|)d\lambda>(2k^2M)^{-1}. \tag{8}$$

Put

$$r_k=(t_k/a_k)^{1/n_k}. \tag{9}$$

Then $\rho_{k-1}<r_k<1$, by (7). Because of (8) and (9), there exists a positive number ρ_k with $r_k<\rho_k<1$ such that

$$\int_{\{r_k<|z|\le\rho_k\}} \Phi_0(|a_k z^{n_k}|)d\lambda > (2k^2M)^{-1}.$$

This completes the proof of the lemma.

Step 3. We now define

$$f(z)=\sum_{k=1}^{\infty} u_k(z) \qquad (z\in U).$$

The series converges uniformly on compact subsets of U, by (6−e). Hence $f\in H(U)$.

Lemma 3.

$$|f|\le 5|u_k|/4+5|u_{k+1}|/4 \ \textit{on}\ \{r_k\le|z|\le\rho_{k+1}\}, \tag{10}$$

$$|f|\ge|u_k|/2 \qquad \textit{on}\ \mathrm{R}_k. \tag{11}$$

Proof. By (6−a) and (6−d),

$$\sum_{j=1}^{k-1}|u_j|\le|u_k|/4 \ \text{on}\ \{|z|\ge r_k\} \tag{12}$$

By (6−e),

$$\sum_{j=k+1}^{\infty}|u_j|\le 5|u_{k+1}|/4 \ \text{on}\ \{r_1\le|z|\le\rho_{k+1}\}, \tag{13}$$

$$\sum_{j=k+1}^{\infty}|u_j|\le|u_k|/4 \ \text{on}\ \{r_1\le|z|\le\rho_k\}. \tag{14}$$

(10) and (11) follow from (12), (13) and (14).

Step 4.

Lemma 4. $f\in A_\varphi(U)$.

Proof.

$$\int_U \varphi(\log|f|)\,d\lambda=\int_U \Phi(|f|)d\lambda$$

$$=\int_{\{|z|\le r_1\}}\Phi(|f|)d\lambda+\sum_{k=1}^{\infty}\int_{\{r_k<|z|\le r_{k+1}\}}\Phi(|f|)d\lambda.$$

Fix $k\in\{1, 2, 3,\dots\}$. By (10),

$$\int_{\{r_k<|z|\le r_{k+1}\}}\Phi(|f|)d\lambda\le\int_{\{r_k<|z|\le r_{k+1}\}}\Phi(5|u_k|/4+5|u_{k+1}|/4)d\lambda.$$

Put

$$E_1=\{z\in\mathbf{C}; r_k<|z|\le r_{k+1},\ 5|u_{k+1}(z)|/4\ge s_0\},$$

$$E_2=\{z\in\mathrm{C};r_k<|z|\leq r_{k+1},5|u_{k+1}(z)|/4<s_0\}.$$

By (6—d),

$$5|u_k(z)|/4\geq|u_k(z)|\geq t_k>s_0 \text{ if } |z|\geq r_k.$$

It follows from (5) that

$$\int_{E_1}\Phi(5|u_k|/4+5|u_{k+1}|/4)d\lambda\leq M\int_{E_1}\Phi(5|u_k|/4)d\lambda+M\int_{E_1}\Phi(5|u_{k+1}|/4)d\lambda$$

$$=M\int_{E_1}\varphi(\log|u_k|+\log(5/4))d\lambda+M\int_{E_1}\varphi(\log|u_{k+1}|+\log(5/4))d\lambda$$

$$\leq M\int_{E_1}\varphi(\log|u_k|+1)d\lambda+M\int_{E_1}\varphi(\log|u_{k+1}|+1)d\lambda$$

$$=M\int_{E_1}\Phi_0(|u_k|)d\lambda+M\int_{E_1}\Phi_0(|u_{k+1}|)d\lambda.$$

On the other hand, by (6—d),

$$\int_{E_2}\Phi(5|u_k|/4+5|u_{k+1}|/4)d\lambda\leq\int_{E_2}\Phi(5|u_k|/4+s_0)d\lambda$$

$$\leq\int_{E_2}\Phi(5|u_k|/4+t_k)d\lambda\leq\int_{E_2}\Phi(9|u_k|/4)d\lambda$$

$$=\int_{E_2}\varphi(\log|u_k|+\log(9/4))d\lambda\leq\int_{E_2}\varphi(\log|u_k|+1)d\lambda$$

$$=\int_{E_2}\Phi_0(|u_k|)d\lambda.$$

Thus

$$\int_{\{r_k<|z|\leq r_{k+1}\}}\Phi(|f|)d\lambda$$

$$\leq M\int_{E_1}\Phi_0(|u_k|)d\lambda+M\int_{E_1}\Phi_0(|u_{k+1}|)\,d\lambda+\int_{E_2}\Phi_0(|u_k|)d\lambda$$

$$\leq M\int_U\Phi_0(|u_k|)d\lambda+M\int_U\Phi_0(|u_{k+1}|)d\lambda.$$

It follows from (6—b) that

$$\int_{\{r_k<|z|\leq r_{k+1}\}}\Phi(|f|)d\lambda\leq k^{-2}+(k+1)^{-2}.$$

Hence

$$\int_U \varphi(\log|f|)d\lambda \leq \int_{\{|z|\leq r_1\}} \Phi(|f|)d\lambda + \sum_{k=1}^{\infty} (k^{-2}+(k+1)^{-2}) < \infty.$$

This means $f \in A_\varphi(U)$.

Step 5. Let n be a positive integer, $b \in H^\infty(U)$, $g \in H(U)^*$, and

$$h = (f^n + b)g.$$

Put

$$\alpha = (2\pi)^{-1}\int_{-\pi}^{\pi} \log|g(r_1 e^{i\theta})|d\theta,$$

$$\beta = \sup_{z\in U} |b(z)|.$$

Then $0 \leq \beta < \infty$. Since $\log|g|$ is subharmonic in U,

(15) $$-\infty < \alpha \leq (2\pi)^{-1}\int_{-\pi}^{\pi} \log|g(re^{i\theta})|d\theta < \infty \qquad (r_1 \leq r < 1).$$

Choose a positive number c so that

(16) $$\log c + \alpha - n\log 4 > 0.$$

Lemma 5. $ch \notin A\phi_n(U)$.

Proof. Define

$$\Psi_n(t) = \phi_n(\log t) \qquad (t \geq 0).$$

Then

$$\int_U \phi_n(\log|ch|)d\lambda = \int_U \Psi_n(|ch|)d\lambda$$

(17) $$\geq \sum_{k=1}^{\infty} \int_{R_k} \Psi_n(|\mathrm{ch}|)d\lambda$$

$$= \sum_{k=1}^{\infty} \int_{r_k}^{\rho_k} 2rdr\ (2\pi)^{-1}\int_{-\pi}^{\pi} \Psi_n(|ch(re^{i\theta})|)d\theta.$$

Fix $r \in (r_k, \rho_k]$. By Jensen's convexity theorem and (15),

$$(2\pi)^{-1}\int_{-\pi}^{\pi} \Psi_n(|ch(re^{i\theta})|)d\theta = (2\pi)^{-1}\int_{-\pi}^{\pi} \phi_n(\log|ch(re^{i\theta})|)d\theta$$

$$\geq \phi_n((2\pi)^{-1}\int_{-\pi}^{\pi} \log|ch(re^{i\theta})|d\theta)$$

(18) $$= \phi_n(\log c + (2\pi)^{-1}\int_{-\pi}^{\pi} \log|g(re^{i\theta})|d\theta + (2\pi)^{-1}\int_{-\pi}^{\pi} \log|(f^n+b)\ (re^{i\theta})|d\theta)$$

$$\geq\phi_n(\log c+\alpha+(2\pi)^{-1}\int_{-\pi}^{\pi}\log|(f^n+b)(re^{i\theta})|d\theta).$$

Since $\lim_{k\to\infty} t_k=\infty$, there exists a positive integer K such that

(19) $$(t_k/4)^n>\beta \qquad \text{if } k\geq K.$$

By (11), (19) and (6–d),

$|f^n+b|\geq|f|^n-\beta>(|u_k|/2)^n-(t_k/4)^n\geq(|u_k|/2)^n-(|u_k|/4)^n\geq(|u_k|/4)^n$ for $k\geq K$, $z\in R_k$. Hence, for $k\geq K$ and $r\in(r_k, \rho_k]$,

(20) $$\begin{aligned}&(2\pi)^{-1}\int_{-\pi}^{\pi}\log|(f^n+b)(re^{i\theta})|d\theta\\ &\geq(2\pi)^{-1}\int_{-\pi}^{\pi}n\log|u_k(re^{i\theta})|d\theta-n\log 4.\end{aligned}$$

By (18), (20) and (16), for $k\geq K$ and $r\in(r_k, \rho_k]$,

$$\begin{aligned}&(2\pi)^{-1}\int_{-\pi}^{\pi}\Psi_n(|ch(re^{i\theta})|)d\theta\\ &\geq\phi_n(\log c+\alpha-n\log 4+(2\pi)^{-1}\int_{-\pi}^{\pi}n\log|u_k(re^{i\theta})|d\theta)\\ &\geq\phi_n((2\pi)^{-1}\int_{-\pi}^{\pi}n\log|u_k(re^{i\theta})|d\theta)\\ &=\phi((2\pi)^{-1}\int_{-\pi}^{\pi}\log|u_k(re^{i\theta})|d\theta)\\ &=\phi(\log(a_k r^{n_k}))=\phi(\log u_k(r))=\Psi(u_k(r))\\ &=(2\pi)^{-1}\int_{-\pi}^{\pi}\Psi(|u_k(re^{i\theta})|)d\theta.\end{aligned}$$

It follows from (17) that

$$\begin{aligned}\int_U\Psi_n(|ch|)d\lambda&\geq\sum_{k=K}^{\infty}\int_{r_k}^{\rho_k}2rdr\,(2\pi)^{-1}\int_{-\pi}^{\pi}\Psi(|u_k(re^{i\theta})|)d\theta\\ &=\sum_{k=K}^{\infty}\int_{R_k}\Psi(|u_k|)d\lambda.\end{aligned}$$

By (6–a), (6–c) and (6–d),

$$\int_{R_k}\Psi(|u_k|)d\lambda\geq kM\int_{R_k}\Phi_0(|u_k|)d\lambda>(2k)^{-1} \qquad (k=1, 2, 3, \ldots).$$

Thus

$$\int_U \Psi_n(|ch|)d\lambda \geq \sum_{k=K}^{\infty} (2k)^{-1} = \infty.$$

This means $ch \notin A_{\psi_n}(U)$. The proof of Theorem 1 is now complete.

4. The inclusion relation between the spaces $A^p(U)$

Theorem 3. *For any* $p \in (0, \infty)$

$$\nu(A^{p-}(U)^*) \subsetneqq \nu(A^p(U)^*) \subsetneqq \nu(A^{p+}(U)^*).$$

Consequently,

$$A^{p-}(U) \subsetneqq A^p(U) \subsetneqq A^{p+}(U) \quad (0 < p < \infty).$$

Proof (cf. Shapiro [7], Corollary 2.2; Horowitz [2], Theorem 4.6; [3], §3, Theorem 3).

(i) Theorem 2 with

$$\varphi(t) = e^{pt} \qquad (-\infty < t < \infty),$$
$$\psi(t) = \begin{cases} te^{pt} & (t \geq 0) \\ 0 & (t < 0), \end{cases}$$

implies that

$$\nu(A^{p-}(U)^*) \subset \nu(A_\psi(U)^*) \subsetneqq \nu(A_\varphi(U)^*) = \nu(A^p(U)^*).$$

(ii) Theorem 2 with

$$\varphi(t) = \begin{cases} t^{-1}e^{pt} & (t \geq p^{-1}) \\ pe & (t < p^{-1}), \end{cases}$$
$$\psi(t) = e^{pt} \qquad (-\infty < t < \infty),$$

implies that

$$\nu(A^p(U)^*) = \nu(A_\psi(U)^*) \subsetneqq \nu(A_\varphi(U)^*) \subset \nu(A^{p+}(U)^*).$$

Theorem 4.

$$H^\infty(U) \subsetneqq \bigcap_{0<p<\infty} A^p(U).$$

Proof. This is an immediate consequence of the following two facts:

(i) $H^p(U) \subset A^p(U) \qquad (0 < p < \infty)$,

(ii) $H^\infty(U) \subsetneqq \bigcap_{0<p<\infty} H^p(U)$.

(See [3], §3, Theorem 4.)

Remark 1. We do not know whether

$$D_0=\nu(H^{\infty}(U)^*)\subsetneqq\nu(\bigcap_{0<p<\infty} A^p(U)^*)$$

is valid or not.

Theorem 5.

$$\nu(\bigcup_{0<p<\infty} A^p(U)^*)\subsetneqq\nu(BN(U)^*).$$

Consequently,

$$\bigcup_{0<p<\infty} A^p(U)\subsetneqq BN(U).$$

Proof (cf. [3], §3, Theorem 5). Put

$$\varphi(t)=\max(0,\ t) \qquad (-\infty<t<\infty),$$

$$\phi(t)=\begin{cases}\exp(\sqrt{t}) & (t\geq 1)\\ e & (t<1).\end{cases}$$

Then φ and ϕ satisfy the assumptions in Theorem 2. Hence

$$\nu(\bigcup_{0<p<\infty} A^p(U)^*)\subset\nu(A_{\phi}(U)^*)\subsetneqq\nu(A_{\varphi}(U)^*)=\nu(BN(U)^*).$$

5. The inclusion relation between the spaces $A^p(U)$, $H^p(U)$ and $N(U)$

Theorem 6. *Suppose* $0<p<\infty$. *Then*

(i) $H^p(U)\subsetneqq A^p(U)$,

(ii) $H^{p-}(U)\subsetneqq A^{p-}(U)$,

(iii) $H^{p+}(U)\subsetneqq A^{p+}(U)$.

Proof. Choose q with $p<q<\infty$. Then

$$H^{\infty}(U)\subset A^q(U)\subset A^p(U),$$

so that,

$$D_0=\nu(H^{\infty}(U)^*)\subset\nu(A^q(U)^*)\subset\nu(A^p(U)^*).$$

On the other hand, by Theorem 3,

$$\nu(A^q(U)^*)\subset\nu(A^{p-}(U)^*)\subsetneqq\nu(A^p(U)^*).$$

Hence

$$D_0\subsetneqq\nu(A^p(U)^*).$$

It follows from Theorem C that

$$\nu(H^p(U)^*)\subsetneqq\nu(A^p(U)^*).$$

Since $H^p(U)\subset A^p(U)$, this implies (i). The same arguments prove (ii) and (iii).

Theorem 7. *For any $p\in(0, \infty)$ and any $q\in(0, \infty)$,*

$$A^p(U)\not\subset H^q(U).$$

Proof. If $A^p(U)\subset H^q(U)$, then

$$\nu(A^p(U)^*)\subset\nu(H^q(U)^*)=D_0.$$

But this is impossible.

Theorem 8. *Suppose $0<2p<q<\infty$.* Then

$$H^p(U)\not\subset A^q(U).$$

Proof. For $z\in U$, $c\in(-\infty, \infty)$, define

$$I_c(z)=(2\pi)^{-1}\int_{-\pi}^{\pi}(1-ze^{it})^{-1-c}dt.$$

When $c<0$, then I_c is bounded in U. When $c>0$, then there exists a positive constant M_c such that

$$I_c(z)\geq M_c(1-|z|)^{-c} \qquad (z\in U).$$

(See Rudin [6], Proposition 1.4.10.)

Choose a with $0<a<p^{-1}-2q^{-1}$. Put $b=p^{-1}-a$. Then $0<2q^{-1}<b<p^{-1}$.

Define

$$f(z)=(1-z)^{-b} \qquad (z\in U).$$

Then, for $r\in(0, 1)$,

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|f(re^{it})|^p dt=(2\pi)^{-1}\int_{-\pi}^{\pi}|1-re^{it}|^{-bp}dt$$

$$=(2\pi)^{-1}\int_{-\pi}^{\pi}|1-re^{it}|^{-1+ap}dt=I_{-ap}(r).$$

Since $-ap<0$,

$$\sup_{0<r<1} I_{-ap}(r)<\infty.$$

Hence $f\in H^p(U)$.

We turn to proving that $f\notin A^q(U)$. Put $c'=bq-1$. Then $c'>1$, since $2q^{-1}<b$. For $r\in(0, 1)$

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|f(re^{it})|^{q}dt=(2\pi)^{-1}\int_{-\pi}^{\pi}|1-re^{it}|^{-bq}dt=I_{c'}(r).$$

Since $c'>1$,

$$\int_{0}^{1}I_{c'}(r)dr\geq\int_{0}^{1}M_{c'}(1-r)^{-c'}dr=\infty.$$

Hence $f\notin A^{q}(U)$.

Remark 2. We do not know whether the conclusion of Theorem 8 is valid, in case $0<p<q\leq 2p$.

Theorem 9.

$$\bigcup_{0<p<\infty}H^{p}(U)\subsetneqq\bigcup_{0<p<\infty}A^{p}(U).$$

Proof. This follows from the fact

$$\nu(\bigcup_{0<p<\infty}H^{p}(U)^{*})=D_{0}\subsetneqq\nu(\bigcup_{0<p<\infty}A^{p}(U)^{*}).$$

Theorem 10.

$$A^{p}(U)\not\subset N(U)\qquad(0<p<\infty).$$

Proof. This follows from the fact

$$\nu(N(U)^{*})=D_{0}\subsetneqq\nu(A^{p}(U)^{*}).$$

Corollary.

$$\bigcup_{0<p<\infty}A^{p}(U)\not\subset N(U).$$

Theorem 11.

$$N(U)\not\subset\bigcup_{0<p<\infty}A^{p}(U).$$

Proof. Define

$$f(z)=\exp\left(\frac{1+z}{1-z}\right)\qquad(z\in U).$$

Then $f\in N(U)$. (See Rudin [4], §17.19.) A simple computation shows

$$\lim_{|z|\to 1}(1-|z|)^{2/p}|f(z)|=\infty\qquad(0<p<\infty).$$

If $f\in A^{p}(U)$, $0<p<\infty$, then

$$\lim_{|z|\to 1} (1-|z|)^{2/p}|f(z)|=0.$$

(See Rudin [6], Theorem 7. 2. 5.) Hence $f \notin \bigcup_{0<p<\infty} A^p(U)$.

Corollary.

$$N(U) \not\subset A^p(U) \qquad (0<p<\infty).$$

Remark 3. We do not know whether

$$\bigcap_{0<p<\infty} H^p(U) \subsetneqq \bigcap_{0<p<\infty} A^p(U)$$

is valid or not.

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Added in proof

We have proved that the statements in Remark 1 and Remark 3 are both valid (Y. Matsugu, "On the zero sets of functions in the Bergman spaces and the Hardy spaces", to appear) :

$$\nu(H^\infty(U)^*) \subsetneqq \nu(\bigcap_{0<p<\infty} A^p(U)^*),$$

$$\bigcap_{0<p<\infty} H^p(U) \subsetneqq \bigcap_{0<p<\infty} A^p(U)^*.$$