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On determining sets for $H^p(B_n)$ II

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1. Introduction

In the preceding paper [1], we described the characterization of the determining sets for the Nevanlinna class $N(B_n)$ on the open unit ball B_n of $\mathbf{C}^n$, and showed the existence of various determining sets and non-determining sets for the Hardy classes $H^p(B_n)$, $0<p\leqq\infty$. One of the results in [1] is the following:

Theorem A ([1], Theorem 5 in §5). *Let $n\geqq 3$ be an integer. Then there exists an $f\in\bigcap_{0<p<\infty} H^p(B_n)$ satisfying the following two conditions*:

(a) *The zero set of f is not a determining set for $H^\infty(B_n)$.*

(b) *The zero-divisor ν_f does not equal ν_g for any $g\in H^\infty(B_n)$.*

The purpose of this paper is to prove that the above theorem is still valid when $n=2$. For the proof we shall make use of a result in [2]. (See Theorem D in §2.)

2. Preliminaries

Let $H(B_n)$ denote the space of all holomorphic functions in B_n. Let $f\in H(B_n)$. Suppose $f\not\equiv 0$. We denote by $Z(f)$ the zero set of f:

$$Z(f)=\{z\in B_n;\ f(z)=0\}.$$

ν_f stands for the zero-divisor of f. (For the definition, see e. g. [1], §2.) ν_f is a nonnegative integer-valued function defined in B_n and its support is equal to $Z(f)$.

Let X be a subspace of $H(B_n)$. We define

$$X^*=\{f\in X;\ f\not\equiv 0 \text{ in } B_n\},$$
$$\nu(X^*)=\{\nu_f;\ f\in X^*\}.$$

A zero set E in B_n (i.e. $E=Z(g)$ for some $g\in H(B_n)^*$) is said to be a *determining set* for X if there is no function f in X^* such that $\nu_f\geqq\nu_g$ in B_n.

Next we state some results about the Hardy spaces $H^p(B_n)$, $0<p<\infty$, and the Bergman spaces $A^p(B_n)$, $0<p<\infty$. Suppose $0<p<\infty$. For $f\in H(B_n)$, we define the H^p-norm by

$$||f||_{H^p}=\sup_{0<r<1}\{\int_{\partial B_n}|f(r\zeta)|^p d\sigma(\zeta)\}^{1/p}$$

and define the A^p-norm by

$$||f||_{A^p}=\{\int_{B_n}|f(z)|^p d\lambda(z)\}^{1/p}.$$

Here ∂B_n is the boundary of B_n, σ is the Euclidean volume element on ∂B_n so normalized that $\sigma(\partial B_n)=1$, and λ is the Lebesgue meaure on $\mathbf{C}^n$ so normalized that $\lambda(B_n)=1$. Then

$$H^p(B_n)=\{f\in H(B_n);||f||_{H^p}<\infty\},$$
$$A^p(B_n)=\{f\in H(B_n);||f||_{A^p}<\infty\}.$$

$H^\infty(B_n)$ denotes the space of all bounded holomorphic functions in B_n.

Suppose $n\geqq 2$. Let f and g be functions with domains B_n and B_{n-1}, respectively, and define a restriction operator ρ and an extension operator E by

$$(\rho f)(z')=f(z', 0) \qquad (z'\in B_{n-1}),$$
$$(Eg)(z', z_n)=g(z') \qquad (z=(z', z_n)\in B_n).$$

The following three theorems will be used to prove the main result in §3:

Theorem B ([3], p. 127). *Assume* $n\geqq 2$, $0<p<\infty$.

(a) *The extension* E *is a linear isometry of* $A^p(B_{n-1})$ *into* $H^p(B_n)$.

(b) *The restriction* ρ *is a linear norm-decreasing map of* $H^p(B_n)$ *onto* $A^p(B_{n-1})$.

Theorem C ([3], p. 128). *Suppose* $n\geqq 1$, $0<p<\infty$. *If* $f\in H^p(B_n)$, *then*

$$|f(z)|\leqq 2^{n/p}||f||_{H^p}(1-|z|)^{-n/p} \quad (z\in B_n).$$

Theorem D ([2], Theorem 5 in §4). *For any integer* $n\geqq 1$,

$$\nu(\bigcap_{0<p<\infty} A^p(B_n)^*)\supsetneqq\nu(H^\infty(B_n)^*).$$

3. The main result

Theorem. *Let* $n\geqq 2$ *be an integer. Then there exists an* $f\in\bigcap_{0<p<\infty} H^p(B_n)$ *satisfying the following two conditions*:

(a) $Z(f)$ *is not a determining set for* $H^\infty(B_n)$.

(b) $\nu_f\notin\nu(H^\infty(B_n)^*)$.

Proof. By Theorem D, there is a $g\in\bigcap_{0<p<\infty} A^p(B_{n-1})^*$ such that $\nu_g\notin\nu(H^\infty(B_{n-1})$

*). Define $f=Eg$, where E is the extension operator defined in §2. By Theorem B, $f\in\bigcap_{0<p<\infty}H^p(B_n)^*$. If $\nu_f\in\nu(H^\infty(B_n)^*)$, then there is an $h\in H^\infty(B_n)^*$ with $\nu_h=\nu_f$. It follows that $h=fk$ for some $k\in H(B_n)$ with $Z(k)=\phi$. Put $h'=\rho h$ and $k'=\rho k$, where ρ is the restriction operator defined in §2. Then

$$h'=gk' \text{ in } B_{n-1}.$$

Since $Z(k')=\phi$, $\nu_g=\nu_{h'}$. In addition, $h'\in H^\infty(B_{n-1})$, because $h\in H^\infty(B_n)$. Thus $\nu_g\in\nu(H^\infty(B_{n-1})^*)$. This contradicts the choice of the function g. Hence $\nu_f\notin\nu(H^\infty(B_n)^*)$.

We turn to show that the condition (a) holds. Since $f\in\bigcap_{0<p<\infty}H^p(B_n)$, it follows from Theorem C that

$$|f(z)|\leqq 2^{n/p}\|f\|_{H^p}(1-|z|)^{-n/p} \quad (z\in B_n, 0<p<\infty).$$

Hence

$$|g(z')|\leqq 2^{n/p}\|f\|_{H^p}(1-|z'|)^{-n/p} \quad (z'\in B_{n-1},\ 0<p<\infty).$$

Choose a positive number p with $n<p$. Define

$$F(z)=f(z)z_n^2=g(z')z_n^2$$

for $z=(z',z_n)\in B_n$. Then

$$|F(z)|\leqq 2^{n/p}\|f\|_{H^p}(1-|z'|)^{-n/p}|z_n|^2$$

for $z=(z',z_n)\in B_n$. Since $|z_n|^2<1-|z'|^2<2(1-|z'|)$,

$$(1-|z'|)^{-n/p}<2^{n/p}|z_n|^{-2n/p}.$$

It follows that

$$|F(z)|\leqq 2^{2n/p}\|f\|_{H^p}|z_n|^{2-2n/p}<2^2\|f\|_{H^p}<\infty$$

for $z=(z',z_n)\in B_n$. Hence $F\in H^\infty(B_n)$. By the definition of F, $\nu_F\geqq\nu_f$ in B_n. Therefore, $Z(f)$ is not a determining set for $H^\infty(B_n)$.

Q. E. D.

References

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