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Flat Connections of Differential Operators and Odd Dimensional Characteristic Classes[1)]

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Introduction

It is known that global geometric properties of Fuchs-type operators are formulated as follows: Let $G=GL(n, \mathbf{C})$, G_t and G_ω be the sheaves of germs of constant and holomorphic G-valued functions over M, a complex manifold, $\mathscr{M}$ the sheaf of germs of those matrix valued holomorphic 1-forms θ such that $d\theta+\theta\wedge\theta=0$. Then, set $r(f)=dff^{-1}$, the sequence $0\longrightarrow \mathrm{G}_t\xrightarrow{i}\mathrm{G}_\omega\xrightarrow{r}\mathscr{M}_\omega\longrightarrow 0$ is exact and it derives following exact sequence of cohomology sets

$$H^0(M,\ \mathrm{G}_\omega)\xrightarrow{r^*}H^0(M,\ \mathscr{M}_\omega)\xrightarrow{\delta}H^1(M,\ \mathrm{G}_t)\xrightarrow{i^*}H^1(M,\ \mathrm{G}_\omega).$$

$\theta\in\mathrm{H}^0(M,\ \mathscr{M}_\omega)$ is a global integrable connection on M and $d+\theta$ is a Fuchs type operator. Since there is a bijection χ: $H^1(M,\ \mathrm{G}_t)\longrightarrow\mathrm{Hom}(\pi_1(M),\ G)$, $\chi(\delta(\theta))$ $(\pi_1(M))$ is a subgroup of G. It is the monodromy group of $d+\theta$. If a representation ρ: $\pi_1\ (M)\longrightarrow G$ is given, it is realized as a monodromy representation of some Fuchs type operator if and only if $i^*\chi^{-1}(\rho)=1$, the trivial holomorphic bundle. Same formulation is possible in smooth category to use G_d, the sheaf of germs of smooth G-valued functions, and $\mathscr{M}_d$, the sheaf of germs of those matrix valued smcoth 1-forms θ such that $d\theta+\theta\wedge\theta=0$, instead of G_ω and $\mathscr{M}_\omega$ ([1], [12], [13], [14]).

The notion of connection is extended for an arbitraly differential operator D: $C^\infty(M,\ E_1)\longrightarrow C^\infty(M,\ \mathrm{E}_2)$, M a smooth manifold, E_i, $i=1,\ 2$, the smooth vector bundles, and a smooth vector bundle ξ over M ([3]). The definition is as follows: *Denote* H *the fibre of* ξ, *a collection* $\{\theta_U\}$, $\theta_U: C^\infty(U,\ E_1\otimes\mathrm{H})\longrightarrow C^\infty(U,\ E_2\otimes\mathrm{H})$ *is a differential operator, is called a connection of* D *with respect to* ξ, *if* $\mathrm{ord}\ \theta_U\leqq\mathrm{ord}D-1$ *and set* $D_\theta=\{D_U\otimes 1_H+\theta_U\}$, $D=\{D_U\}$, D_θ *becomes a well defined differential operator from* $C^\infty(M,\ E_1\otimes\mathrm{H})$ *into* $\mathrm{C}^\infty(M,\ E_2\otimes\mathrm{H})$.

To define the curvature operator of a connection of a differential operator is possible (cf. Appendix of this paper), and it relates the theory of non-linear coho-

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mology ([9]). But the notion of a flat connection of a differential operator is given more directly as follows: *A connection* $\{\theta_U\}$ *of* D *with respect to a* G*-bundle* ξ *is called flat if there is a collection* $\{h_U\}$ *of smooth* G*-valued function* h_U *on* U *such that* $\theta_U=\rho_D(h_U)$. *Here* $\rho_D(h_U)$ *is given by*

$$\rho_D(h_U)\varphi=(D_U\otimes 1_{\mathrm{H}})\varphi-(1_{E_1,E_2}\otimes h_U)\ (D_U\otimes 1_{\mathrm{H}})\ ((1_{E_1}\otimes h_U{}^{-1})\varphi).$$

A G-valued function g such that $\rho_D(g)=0$ is called a $c(D)$-class G-valued function. It is shown that a G-valued function g is of $c(D)$-class if and only if its matrix elements are of $c(D)$-class, and there is a system of differential operators $r(D)$ determined by D such that a function f is of $c(D)$-class if and only if $r(D)f=0$. Some examples, such as a real elliptic operator acting on scalar functions, have only constant functions as $c(D)$-class functions. But, some other examples, such as $D=\bar{\partial}$, have nonconstant $c(D)$-class functions (§1). We denote the sheaf of germs of $c(D)$-class G-valued functions by $\mathrm{G}_{c(D)}$ and set $\rho_D(\mathrm{G}_d)=\mathrm{L}_{G,D}$. Then we have the exact sepuence of sheaves

$$0\longrightarrow \mathrm{G}_{c(D)}\xrightarrow{i}\mathrm{G}_d\xrightarrow{\rho_D}\mathrm{L}_{G,D}\longrightarrow 0.$$

From this sequence, we obtain the following exact sequence of cohomology sets

$$H^0(M,\ \mathrm{G}_d)\xrightarrow{\rho_D{}^*}H^0(M,\ \mathrm{L}_{G,D})\xrightarrow{\delta}H^1(M,\ \mathrm{G}_{c(D)})\xrightarrow{i^*}H^1(M,\ \mathrm{G}_d).$$

If $L\in H^0\ (M,\ \mathrm{L}_{G,D})$, $D\otimes 1_{\mathrm{H}}-L$ is a differential operator from $C^\infty(M,\ E_1\otimes\mathrm{H})$ into $C^\infty(M,\ E_2\otimes\mathrm{H})$. We call this operator to be a D-Fuchs type operator. On the other hand, an element of $H^1(M,\ \mathrm{G}_{c(D)})$ is called a $c(D)$-class G-bundle or a D-flat G-bundle. Hence $\delta(L)$ is a differentible trivial $c(D)$-class G-bundle. It is shown that $\delta(L)$ has the minimal structure group as a $c(D)$-class G-bundle. This group is called the monodromy group of $D\otimes 1_{\mathrm{H}}-L$ (§2).

If $G=GL(n,\ \mathbf{C})$, we can define several characteristic classes related to $c(D)$-class G-bundles and the elements of $H^0(M,\ L_{G,D})$. These classes are connected with the exact sequence of cohomologies

$$\cdots\longrightarrow H^{2p-1}(M,\ \mathbf{Z})\xrightarrow{i^*}H^{2p-1}(M,\ \mathbf{C}_{c(D)})\xrightarrow{\exp^*}H^{2p-1}(M,\ \mathbf{C}^*{}_{c(D)})\xrightarrow{\delta}$$

$$\longrightarrow H^{2p}(M,\ \mathbf{Z})\xrightarrow{i^*}H^{2p}(M,\ \mathbf{C}_{c(D)})\longrightarrow\cdots,$$

and the generator of the cohomology ring $H^*(GL(n,\ \mathbf{C}),\ \mathbf{Z})=H^*(U(n),\ \mathbf{Z})$ (cf. [6], [7], [11]). For this purpose, we define a product (denoted by $*$) on $\sum_p H^{2p-1}(M,\ \mathbf{C}^*{}_{c(D)})$ and show δ: $\sum_p H^{2p-1}(M,\ \mathbf{C}^*{}_{c(D)})\longrightarrow\sum_p H^{2p}(M,\ \mathbf{Z})$ is a ring homomorphism (§3, n°8, the product in the right hand side is the cup product). Then our results are summarlized as follows (§§3, 4):

(i). *Denote* $c^p(\xi)$ *the p-th Chern class of a complex vector bundle* ξ, $i^*(c^p(\xi))=0$ *forany p, if* ξ *is a* $c(D)$*-class bundle.*

(ii). *If* ξ *is a* $c(D)$*-class bundle, there is a well defined cohomology class* $b^p(\xi)\in H^{2p-1}(M, \mathbf{C}^*_{c(D)})$ *such that*

$$\delta b^p(\xi)=c^p(\xi).$$

(iii). *If* $L\in H^0(M, \mathrm{L}_{G,D})$ *and D satisfies some assumptions* (cf. §4, n°10), *there is a well defined cohomology class* $\beta^p(L)\in H^{2p-1}(M, \mathbf{C}_{c(D)})$ *such that*

$$\exp^*(\beta^p(L))=(-1)^{p-1}F_{n,p}(b^1(\delta(L),\dots,b^p(\delta(L)).$$

Here $F_{n,p}(s_1,\dots,s_p)=\sum_{i=1}^n X_i^p$, s_q *is the q-th elementary symmetric function of indeterminants* $X_1,\dots,X_n$ *and the product is* $*$*-product.*

(iv). *If* $L=\rho_D(f)$, f *is a smooth G-valued function on M, then*

$$\beta^p(L)=i^*(f^*(c^p)).$$

Here c^p *is the* (2^p-1)*-dimensional generator of* $H^*(GL(n, \mathbf{C}), \mathbf{Z})$.

If $M=\mathbf{C}^*$, $D=d/dz$ and $L=\alpha/z$, $\beta^1(L)$ is $\alpha\langle e\rangle$, $\langle e\rangle$ is the generator of $H^1(\mathbf{C}^*, \mathbf{C})=\mathbf{C}$. In general, $\beta^1(L)$ is determined by the coefficients of the indicial equation in classical case. $\beta^p(L)$ is determined by $\sigma(L)$, the principal symbol of L if D is homogeneous and satisfies the assumption of n°10. If $D=d$ or $\bar{\partial}$, an element of $H^{2p-1}(M, \mathbf{C}_{c(D)})$ is represented by a closed $(2p-1)$-form or a $\bar{\partial}$-closed $(0, 2p-1)$-type form on M. On the other hand, L is a matrix valued 1-form θ on M. In these cases, we have

$$\beta^p(L)=\frac{(-1)^{p-1}}{(2\pi\sqrt{-1})^p}\mathrm{tr}(\overbrace{\theta\wedge\cdots\cdots\wedge\theta}^{2p-1}).$$

We note that (iii) shows the rigidity of $\beta^p(L)$ under the monodromy preserving deformation of L, because if $\delta(L)=\delta(L')$, $\beta^p(L)-\beta^p(L')\in i^*(H^{2p-1}(M, \mathbf{Z}))$ which is a discreet subgroup of $H^{2p-1}(M, \mathbf{C}_{c(D)})$. Therefore $\beta^p(L)$ is an invariant of monodromy preserving deformation (cf. [8], [15], [16]). But in some cases, $\beta^p(L)$, $p\geqq 2$, vanishes. For example, if $L|U=\rho_D(h_U)$ and each h_U is a $\Delta(n, \mathbf{C})$-valued function on U, $\beta^p(L)=0$ if $p\geqq 2$.

The outline of this paper is as follows: In §1, we define and study $c(D)$-class functions and $c(D)$-class G-valued functions. $c(D)$-class G-bundles and D-Fuchs type differential operators are defined in §2. The existence of monodromy group is also shown in §2. §3 is devoted to the definitions of $*$-product and $b^p(\xi)$. The proofs of above (i) and (ii) are also given in this §. The definition of $\beta^p(L)$ and the proofs of (iii) and (iv) are given in §4. In appendix, we give the definition of the curvature operator of a connection of a differential operator.

In this paper, we do not study the singularities of D-Fuchs type operators. From the point of view of the above formulation, the theory of singularities of D-Fuchs type operators seems to be a non-abelian residue theory.

§1. $c(D)$-class functions and $c(D)$-class G-valued functions

1. Let M be a connected paracompact smooth manifold, D: $C^\infty(M, E_1) \longrightarrow C^\infty(M, E_2)$ a differential operator on M. Here E_i, $i=1, 2$, and $C^\infty(U, E_i)$, $i=1, 2$, are the smooth vector bundles over M and the space of its smooth sections on U, an open set of M. If f is a smooth function on U, f acts on each $C^\infty(U, E_i)$ by the scalar multipication. Hence f defines a linear operator $f_{(m)}$ or f on $C^\infty(U, E_i)$.

Definition. *A function f on U is called to be a $c(D)$-class function on U if $f_{(m)}D=Df_{(m)}$. The set of all $c(D)$-class functions on U is denoted by $c(D, U)$.*

Lemma 1. *If $D=\sum_{|\mathbf{I}|\leq k}A_{\mathbf{I}}(x)\partial^{|\mathbf{I}|}/\partial x^{\mathbf{I}}$, $\mathbf{I}=(i_1,\dots,i_n)$, $|\mathbf{I}|=i_1+\dots+i_n$, $\partial^{|\mathbf{I}|}/\partial x^{\mathbf{I}}=\partial^{|\mathbf{I}|}/\partial x_1{}^{i_1}\dots\partial x_n{}^{i_n}$, on U, f belongs in $c(D, U)$ if and only if*

$$\sum_{\mathbf{J}+\mathbf{K}=\mathbf{I},\ |\mathbf{J}|\geq 1}\frac{\mathbf{I}!}{\mathbf{J}!\mathbf{K}!}A_{\mathbf{I}}(x)\frac{\partial^{|\mathbf{J}|}f}{\partial x^{\mathbf{J}}}=0, \qquad |\mathbf{K}|\leq k-1. \tag{1}$$

Proof. Since $Df=fD+\sum_{|\mathbf{K}|\leq k-1}(\sum_{\mathbf{J}+\mathbf{K}=\mathbf{I},\ |\mathbf{J}|\geq 1}(\mathbf{I}!/\mathbf{J}!\mathbf{K}!)A_{\mathbf{I}}(x)\partial^{|\mathbf{J}|}f/\partial x^{\mathbf{J}})\partial^{|\mathbf{K}|}/\partial x^{\mathbf{K}}$, we have the lemma.

Corollary. *If $V\subset U$ and $f\in c(D, U)$, f belongs in $c(D, V)$. Especially, the germ f_x of f at x and the set of germs of $c(D)$-class functions $c(D)_x$ at x are defined.*

Definition. *The system of differential operators on M given by* (1) *is denoted by $r(D)$. $r(D)$ is called maximal if $r(D)f=0$ implies f is a constant.*

Lemma 2. (i). *$c(D, U)$ is a ring by the usual addition and multiplication of functions and contains the ring of constant functions.*

(ii). *$c(D, U)$ is closed by $\mathscr{C}^k$-topology.*

(iii). *If $f\in c(D, U)$ and F is a holomorphic function such that $(\partial^{|\mathbf{I}|}F/\partial x^{\mathbf{I}})(f)$ is defined if $|\mathbf{I}|\leq k$, then $F(f)$ belongs in $c(D, U)$.*

Proof. Since $D(fg)=(Df)g=(fD)g=(fg)D$ if f, $g\in c(D, U)$, $c(D, U)$ is closed under the multiplication. Other parts of (i) and (ii) follow from lemma 1.

If F is holomorphic, there is a series of polynomials $\{F_m\}$ such that $\{F_m(f)\}$ converges to $F(f)$ on some neighborhood $U(x)$ of x, $x\in U$. Since $\partial^{|\mathbf{I}|}G(f)/\partial x^{\mathbf{I}}=P_{\mathbf{I}}(G(f),\dots,(\partial^{|\mathbf{J}|}G/\partial x^{\mathbf{J}})(f),\dots,f,\dots,\partial^{|\mathbf{K}|}f/\partial x^{\mathbf{K}},\dots)$, $\mathbf{J}$, $\mathbf{K}\leq\mathbf{I}$, $\{F_m(f)\}$ converges to $F(f)$ *at least by* $\mathscr{C}^k$-topology. Hence we have (iii).

Corcllary. *$c(D)_x$ is a local ring.*

If g_i is a linear transformation of the fibre of E_i and E_i is trivial on U, g_i acts as a linear operator on $C^\infty(U, E_i)$. This operator is denoted by $g_{i(m)}$ or g_i, $i=1, 2$. Then, since $g_{i(m)}f_{(m)}=f_{(m)}g_{i(m)}$, we have

Lemma 3. *If g_i is inversible, $i=1, 2$, then*

$$c(Dg_1,\ U)=c(D,\ U),\ c(g_2D,\ U)=c(D,\ U).$$

Example 1. If $D=\sum_i A_i(x)\partial/\partial x_i+B(x)$, $r(D)$ is given by $\sum_i A_i(x)\partial/\partial x_i$. If $A_i(x)=(a_i^{jk}(x))$, $r(D)$ is the overdetermined system $\sum_i a_i^{jk}(x)\partial f/\partial x_i=0$, $1\leq j\leq m_1$, $1\leq k\leq m_2$. Here m_1, m_2 are the dimensions of the fibres of E_1, E_2.

Example 2. If $D=\sum_{ij}a_{ij}(x)\partial^2/\partial x_i\partial x_j+\sum_i b_i(x)\partial/\partial x_i+c(x)$, $a_{ij}(x)=a_{ji}(x)$, $r(D)$ is given by $\{2\sum_j a_{ij}(x)\partial/\partial x_j,\ i=1,\dots,\ n,\ (D-c(x))\}$. Hence $r(D)$ is maximal on U if $A(x)=(a_{ij}(x))$ is a regular matrix on each $x\in U$.

Example 3. If D is a scalar valued real elliptic operator, $r(D)$ is maximal.

Since the problem is local, to show this, first we assume D is a constant coefficients operator. Then, since D is a real scalar valued operator, $k\geq 2$ and by a linear change of coordinates, we may assume $D=\partial^k/\partial y_1{}^k+$ *terms with order at most* $k-2$ *in* $\partial/\partial y_1$. Hence $r(D)$ contains $\partial/\partial y_1$ and f is independent to y_1 if $f\in c(D,\ U)$. Set $D=P(\partial/\partial y_1,\dots,\partial/\partial y_n)$, $D'=P(0,\ \partial/\partial y_2,\dots,\partial/\partial y_n)$ is elliptic on the plane $y_1=0$. Therefore $r(D)$ is maximal by induction in this case. For general D, set $D=D(x_0)+(D-D(x_0))$, $D(x_0)$ is a constant coefficients elliptic operator. If $f\in c(D,\ U)$, set

$$D(x_0)f=fD(x_0)+R_0,\ D_1f=fD_1+R_1,\ D_1=D-D(x_0),$$

the coefficients of D_1 vanishes at x_0 and $R_0=-R_1$. Hence the coefficients of R_0 vanishes at x_0 and $df(x_0)=0$ if $f\in c(D,\ U)$, because $r(D(x_0))$ is maximal. Since x_0 is arbitrary, this shows $df=0$ on U. Therefore f is a constant and $r(D)$ is maximal.

Note. Example 1 shows if $D=d$ or $\bar{\partial}$, $r(D)$ is also d or $\bar{\partial}$.

2. Let H be a separable Hilbert space with the O. N. -basis $\{e_\alpha\}$. We denote the inner product $\xi,\ \eta\in$ H by $(\xi,\ \eta)$ and the set of all bounded linear operators of H by $\mathscr{B}$(H). Denote V_i the fibre of E_i, we set

$$\langle v\otimes\xi,\ \eta\rangle=(\xi,\ \eta)v,\ v\in V_i,\ v\otimes\xi\in V_i\otimes H,\ i=1,\ 2.$$

Definition. (i). *A* $\mathscr{B}$(H)*-valued function* $b(x)$ *on* U, *an open set of* M, *is called smooth on* U *if* $(b(x)e_\alpha,\ e_\beta)$ *is a smooth function on* U *for any* $e_\alpha,\ e_\beta\in\{e_\alpha\}$

(ii). *A* $V_i\otimes H$*-valued function* $f(x)$ *on* U *is called smooth on* U *if* $\langle f(x),\ e_\alpha\rangle$ *is a smooth function on* U *for any* $e_\alpha\in\{e_\alpha\}$.

Since O. N. -basis $\{e_\alpha\}$ and $\{e_\alpha'\}$ of H are changed by a unitary operator, these definitions do not depend on the choice of $\{e_\alpha\}$.

If each E_i is trivial on U, D induces a differential operator $D_U: C^\infty(U,\ V_1)\longrightarrow C^\infty(U,\ V_2)$. Hence, denote 1_H the identity map of H, $D_U\otimes 1_H: C^\infty(U,\ V_1\otimes H)\longrightarrow C^\infty(U,\ V_2\otimes H)$ is defined. On the other hand, if $b(x)$ is a smooth $\mathscr{B}$(H)-valued function on U, $1_{V_i}\otimes b(x)$ is a smooth $GL(V_i)\otimes\mathscr{B}$(H)-valued function on U. Hence $1_{V_i}\otimes b(x)=1_{V_i}\otimes b(x)_{(m)}$ is defined as a linear operator on $C^\infty(U,\ V_i\otimes H)$, $i=1,\ 2$.

Lemma 4. *The followings are equivalent.*

(i) $(1_{V_2}\otimes b(x))D_U\otimes 1_H=D_U\otimes 1_H(1_{V_1}\otimes b(x))$.

(ii) $(b(x)e_\alpha,\ e_\beta)D_U=D_U(b(x)e_\alpha,\ e_\beta)$, *for some* O. N.-*basis* $\{e_\alpha\}$ *of* H.

(iii) $(b(x)e_\alpha,\ e_\beta)D_U=D_U(b(x)e_\alpha,\ e_\beta)$, *for all* O. N.-*basis* $\{e_\alpha\}$ *of* H.

Proof. By definition, if $b(x)$ does not depend on x, then

$$(1_{V_2}\otimes b)D_U\otimes 1_H=D_U\otimes 1_H(1_{V_1}\otimes b). \tag{2}$$

Hence (ii) and (iii) are equivalent if (i) and (ii) are equivalent. Since we have

$$\begin{aligned}\langle D_U\otimes 1_H(1_{V_2}\otimes b(x))v(x)\otimes e_\alpha,\ e_\beta\rangle&=D_U((b(x)e_\alpha,\ e_\beta)v(x))\\&=D_U(b(x)e_\alpha,\ e_\beta))v(x),\end{aligned}$$

$$\begin{aligned}\langle(1_{V_2}\otimes b(x))\ (D_U\otimes 1_H)v(x)\otimes e_\alpha,\ e_\beta\rangle&=(b(x)e_\alpha,\ e_\beta)\ (D_Uv(x))\\&=((b(x)e_\alpha,\ e_\beta)\ D_U)v(x),\end{aligned}$$

(i) and (ii) are equivalent and we obtain the lemma.

Corollary. $(1_{V_2}\otimes b(x))D_U\otimes 1_H$ *is equal to* $D_U\otimes 1_H(1_{V_1}\otimes b(x))$ *if and only if* $(b(x)e_\alpha,\ e_\beta)\in c(D,\ U)$ *for any* $e_\alpha,\ e_\beta\in\{e_\alpha\}$.

Definition. (i). *A smooth* $\mathscr{B}$(H)*-valued function on* U *is called a* $c(D)$*-class* $\mathscr{B}$(H)*-valued function on* U *if it satisfies either of* (i), (ii) *or* (iii) *of lemma* 4.

(ii). *Let* G *be a subgroup of* $\mathscr{B}$(H). *Then a* G*-valued function on* U *is called a* $c(D)$*-class* G*-valued function on* U *if it is also a* $c(D)$*-class* $\mathscr{B}$(H)*-valued function.*

Lemma 5. (i). *If* $b(x)$ *is a* $c(D)$*-class* $\mathscr{B}$(H)*-valued function on* U *and* $V\subset U$, $b(x)$ *is a* $c(D)$*-class* $\mathscr{B}$(H)*-valued function on* V.

(ii). *The set of all* $c(D)$*-class* $\mathscr{B}$(H)*-valued functions on* U *is a ring and the set of all* G*-valued functions on* G *is a group.*

(iii). *Denote* $b^*(x)$ *the* $\mathscr{B}$(H)*-valued function defined by* $b^*(x)=(b(x))^*$, *the adjoint operator of* $b(x)$, *where* $b(x)$ *is a* $c(D)$*-class* $\mathscr{B}$(H)*-valued function,* $b^*(x)$ *is a* $c(D)$*-class* $\mathscr{B}$(H)*-valued function if* $c(D,\ U)=\overline{c(D,\ U)}=\{\bar{f}\mid f\in c(D,\ U)\}$, $\bar{f}(x)=\overline{f(x)}$, *the conjugate complex of* $f(x)$.

Proof. By the corollary of lemma 3 and lemma 1, we have (i). By the same reason of lemma 2, (i), we have (ii). Since $(b^*(x)e_\alpha,\ e_\beta)=\overline{(b(x)e_\beta,\ e_\alpha)}$, we have (iii).

Corollary $b(x)h(x)$ *is a* $c(D)$*-class* H*-valued function if* $b(x)$ *is a* $c(D)$*-class* $\mathscr{B}$(H)*-valued function and* $h(x)$ *is a* $c(D)$*-class* H*-valued function. Here* $h(x)$ *is a* $c(D)$*-class* H*-valued function if* $(h(x),\ e_\alpha)\in c(D,\ U)$ *for any* $e_\alpha\in\{e_\alpha\}$.

3. For a system of differential operators S, we denote $\ker(S)_a$ the germ of the elements of ker (S) at a. For $r(D)$, the subsystem consisted by the 1-st order operators is denoted by $r_1(D)$. We also set $r(D)_a=\{\sum_{\mathrm{I}}B_{\mathrm{I}}(a)\partial^{|\mathrm{I}|}/\partial x^{\mathrm{I}}\}$, $r(D)=\sum_{\mathrm{I}}B_{\mathrm{I}}(x)\partial^{|\mathrm{I}|}/\partial x^{\mathrm{I}}$ on U, a neighborhood of a, *etc.*. Similarly, $D(a)$ means $\sum A_{\mathrm{I}}(a)\partial^{|\mathrm{I}|}/\partial x^{\mathrm{I}}$ if $D=\sum A^{\mathrm{I}}(x)\partial^{|\mathrm{I}|}/\partial x^{\mathrm{I}}$ on U. In this n^0, we call $a\in M$ to be a normal point of $r(D)$

if ker $(r_1(D(a)))_a \supset \ker(r_1(D))_a$.

Lemma 6. *If the set of normal points of $r(D)$ contains an open dense set of M, we have*

(3) $$\ker r_1(D) = \ker r(D), \text{ on any open set of } M.$$

Proof. Since the problem is local, we consider the problem in a fixed coordinate neighborhood of M.

By the definition of $r(D)$, if $P(x, \partial/\partial x) \in r(D)$, we have $I_i(x, \partial/\partial x) \in r_1(D)$, where $P(x, \xi) = \sum_i L_i(x, \xi)\xi^{\alpha_i}$, $\xi^{\alpha_i} = \xi_1^{\alpha_{i,1}} \ldots \xi_n^{\alpha_{i,n}}$. Hence we have (3) if D is a constant coefficients operator.

Let a be a normal point of $r(D)$ such that there exists a neighborhood $U(a)$ of a consisted by the normal points of $r(D)$ and set $D = D(a) + D_1$. Then, if $r_1(D)f = 0$, we have $r_1(D(a))f = 0$ on $U(a)$. Hence $(Df - fD)(a) = 0$. Therefore $f \in c(D, U(a))$ and we have the lemma by assumption.

Note. By the proof of example 3, n°1, if D is a scalar valued real elliptic operator, any point of M is a normal point of $r(D)$.

For a smooth $\mathscr{B}(\mathrm{H})$-valued function f on U, we set

$$\delta_D(f) = Df - fD = (D \otimes 1_{\mathrm{H}})(1_{E_1} \otimes f) - (1_{E_2} \otimes f)(D \otimes 1_{\mathrm{H}}).$$

By definition, we have $\delta_D(f) = \sum_{P\mathrm{J} \in r(D)} P_{\mathrm{J}}(x, \partial/\partial x)\partial^{|\mathrm{J}|}/\partial x^{\mathrm{J}}$. We also set

$$\delta_{D,1}(f) = \sum_{P\mathrm{J} \in r_1(D)} P_{\mathrm{J}}\left(x, \frac{\partial}{\partial x}\right)\frac{\partial^{|\mathrm{J}|}}{\partial x^{\mathrm{J}}}.$$

Lemma 6′. *If D satisfies the assumption of lemma 6, $\delta_D(f)$ is equal to 0 if and only if $\delta_{D,1}(f) = 0$.*

Corollary. *Let G be a subgroup of $\mathscr{B}(\mathrm{H})$ and g is a smooth G-valued function on U. Then to set*

$$\rho_D(g) = \delta_D(g)g^{-1} = D \otimes 1_{\mathrm{H}} - (1_{E_2} \otimes g)(D \otimes 1_{\mathrm{H}})(1_{E_1} \otimes g^{-1}),$$

$$\rho_{D,1}(g) = \delta_{D,1}(g)g^{-1},$$

$\rho_D(g) = 0$ is equivalent to $\delta_D(g) = 0$ and if D satisfies the assumption of lemma 6, $\rho_{D,1}(g) = 0$ implies $\rho_D(g) = 0$.

Since δ_D is a derivation and $\delta_D(f) = 0$ if and only if f is a $c(D)$-class $\mathscr{B}(\mathrm{H})$-valued function, we have

(4)i $\rho_D(g) = 0$, *if and only if g is a $c(D)$-class G-valued function,*

(4)ii $\rho_D(gh) = \rho_D(g) + \rho_D(h)^g$, $\rho_D(h)^g = (1_{E_2} \otimes g)\rho_D(h)(1_{E_1} \otimes g^{-1})$,

(4)iii $\rho_D(g^{-1}) = -\rho_D(g)^{g^{-1}}$.

Since $\delta_{D,1}$ is also a derivation, (4)ii and (4)iii are hold for $\rho_{D,1}$. (4)i is hold for

$\rho_{D,1}$ if D satisfies the assumption of lemma 6.

Example. If D is a 1-st order operator, $r(D)$ is equal to $r_1(D)$ and therefore $\rho_D(g)=\rho_{D,1}(g)$. Moreover, if D is homogeneous, we may regard Dg to be a $\mathscr{B}(\mathrm{H})$-valued 1-form and as a 1-form, we have $\rho_D(g)=(Dg)g^{-1}$. Especially, we obtain $\rho_d(g)=dg\cdot g^{-1}$ and $\rho_{\bar\partial}(g)=\bar\partial g\cdot g^{-1}$ (cf. Introduction).

On M, we denote $\mathscr{B}(\mathrm{H})_d$ and G_d the sheaves of germs of smooth $\mathscr{B}(\mathrm{H})$ and G-valued functions over M. The sheaves of germs of $c(D)$-class $\mathscr{B}(\mathrm{H})$ and G valued functions over M are denoted by $\mathscr{B}(\mathrm{H})_{c(D)}$ or $\mathrm{G}_{c(D)}$. ρ_D and δ_D induce the maps ρ_D and δ_D on G_d and $\mathscr{B}(\mathrm{H})_d$. We set

$$\rho_D(\mathrm{G}_d)=\mathrm{L}_{G,D},\quad \delta_D(\mathscr{B}(\mathrm{H})_d)=\mathscr{L}_{\mathscr{B}(\mathrm{H}),\,D}.$$

By definitions, we have the following exact sequences of sheaves.

$$0\longrightarrow \mathrm{G}_{c(D)}\xrightarrow{\ i\ }\mathrm{G}_d\xrightarrow{\ \rho_D\ }\mathrm{L}_{G,\,D}\longrightarrow 0,$$

$$0\longrightarrow \mathscr{B}(\mathrm{H})_{c(D)}\xrightarrow{\ i\ }\mathscr{B}(\mathrm{H})_d\xrightarrow{\ \delta_D\ }\mathscr{L}_{\mathscr{B}(\mathrm{H}),\,D}\longrightarrow 0.$$

Example. For $\mathrm{H}=\mathbf{C}$, the complex number field, denote $\mathbf{C}^*$ the multiplicative group of complex numbers without 0, we have the following commutative diagram of sheaves with exact lines and columns.

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathbf{C}^*_{c(D)} \arrow[r, "i"] \arrow[u] & \mathbf{C}^*_d \arrow[r, "\rho_D"] \arrow[u, "\exp"] & \mathrm{L}_{\mathbf{C}^*,\,D} \arrow[r] \arrow[u, "\mathrm{k}"] & 0 \\
0 \arrow[r] & \mathbf{C}_{c(D)} \arrow[r, "i"] \arrow[u] & \mathbf{C}_d \arrow[r, "\delta_D"] \arrow[u, "\exp"] & \mathscr{L}_{\mathbf{C},\,D} \arrow[r] \arrow[u, "="'] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "="'] \arrow[u, "\iota"] & \mathbf{Z} \arrow[r] \arrow[u, "\iota"] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here $\mathbf{Z}$ is the constant sheaf of integers, ι is the inclusion regarding a constant to be a constant function, exp and k are given by

$$\exp\,(f_x)=(\mathrm{e}^{2\pi\sqrt{-1}f})_x,\quad \mathrm{k}((Df-fD)_x)=\rho_D(\mathrm{e}^{2\pi\sqrt{-1}f})_x,$$

where f_x, *etc.*, mean the germ of f, *etc.*, at x (cf. Introduction).

§2. *D*-flat *G*-bundles and *D*-Fuchs type differential equations

4. Since $\mathrm{G}_{c(D)}$ and G_d are sheaves of groups, the coboundary maps $\delta_i=\delta: C^i(\mathfrak{U},\mathrm{G}_d)\longrightarrow C^{i+1}(\mathfrak{U},\mathrm{G}_d)$ or $C^i(\mathfrak{U},\mathrm{G}_{c(D)})\longrightarrow C^{i+1}(\mathfrak{U},\mathrm{G}_{c(D)})$, $i=0,1$, are defined. Here $\mathfrak{U}$ is an open covering of M. For $C^i(\mathfrak{U},\mathrm{L}_{G,D})$, $i=0,1$, we define $\delta^{L_i}=\delta^L: C^i(\mathfrak{U},\mathrm{L}_{G,D})\longrightarrow C^{i+1}(\mathfrak{U},\mathrm{L}_{G,D})$ by

(5) $$\delta^L \rho_D = \rho_D \delta.$$

Explicitly, δ^{L_1} and δ^{L_2} are given by

$$\delta^{L_1}(L)_{U,\,V} = L_U - L_V{}^{g_{UV}},\ \ L_U = \rho_D(h_U),\ \ g_{UV} = h_U h_V{}^{-1},$$

$$\delta^{L_2}(L)_{U,V,W} = L_{U,\,V} + L_{V,W}{}^{g_{UV}} + L_{W,U}{}^{g_{UW}},\ \ L_{U,\,V} = \rho_D(g_{UV}).$$

Note. δ^L may not be defined on $C^i(\mathfrak{U},\ \mathrm{L}_{G,\,D})$. But if $\{L\} \in C^i(\mathfrak{U},\ \mathrm{L}_{G,D})$, there exists a refinement $\mathfrak{V}$ of $\mathfrak{U}$ such that δ^L is defined for $t^{\mathfrak{U}}_{\mathfrak{W}}(\{L\})$ if $\mathfrak{W}$ is a refinement of $\mathfrak{V}$. Here $t^{\mathfrak{U}}_{\mathfrak{W}}: C^i(\mathfrak{U},\ \mathrm{L}_{G,\,D}) \longrightarrow C^i(\mathfrak{W},\ \mathrm{L}_{G,\,D})$ is the map induced by the refinement.

We set $B^i(\mathfrak{U},\ \mathrm{L}_{G,\,D}) = \ker\delta^{L_i} = \{\{L\} \mid \{L\} \in C^i(\mathfrak{U},\ \mathrm{L}_{G,\,D}),\ \delta^{L_i}(\{L\}) = 0\}$, $i = 0.1$, and $\mathrm{H}^0(\mathfrak{U},\ \mathrm{L}_{G,\,D}) = B^0(\mathfrak{U},\ \mathrm{L}_{G,D})$. On $B^1(\mathfrak{U},\ \mathrm{L}_{G,\,D})$, we define an equivalence relation $\sim$ by

$$\{L_{U,\,V}\} \sim \{L_{U,\,V}'\}\ \ if\ L_{U,\,V} - L_{U,\,V}' = \rho_D(h_U) - \rho_D(h_V)^{h_U g_{UV} h_V{}^{-1}},$$

$$L_{U,V} = \rho_D(g_{UV}),\ \ for\ some\ \{h_U\} \in C^0(\mathfrak{U},\ \mathrm{G}_d).$$

We denote $H^1(\mathfrak{U},\ \mathrm{L}_{G,\,D})$ the quotient set of $B^1(\mathfrak{U},\ \mathrm{L}_{G,\,D})$ by this relation. Then, to set $H^1(M,\ \mathrm{L}_{G,D}) = lim[H^1(\mathfrak{U},\ \mathrm{L}_{G,\,D}),\ t^{\mathfrak{U}}_{\mathfrak{W}}]$, we have the following exact sequence of cohomology sets.

(6) $$0 \longrightarrow H^0(M,\ \mathrm{G}_{c(D)}) \xrightarrow{i^*} H^0(M,\ \mathrm{G}_d) \xrightarrow{\rho_D{}^*} H^0(M,\ \mathrm{L}_{G,\,D}) \xrightarrow{\delta}$$
$$\longrightarrow H^1(M,\ \mathrm{G}_{c(D)}) \xrightarrow{i^*} H_1(M,\ \mathrm{G}_d) \xrightarrow{\rho_D{}^*} H^1(M,\ \mathrm{L}_{G,\,D}).$$

Here $\delta: H^0(M,\ \mathrm{L}_{G,\,D}) \longrightarrow H^1(M,\ \mathrm{G}_{c(D)})$ is given by

$$\delta(L) = \{g_{UV}\},\ \ g_{UV} = h_V{}^{-1} h_V,\ \ L|U = \rho_D(h_U).$$

Definition. (i). *An element of $H^1(M,\ \mathrm{G}_{c(D)})$ is called a $c(D)$-class G-bundle.*

(ii). *A smooth G-bundle in i^*-image is called a D-flat G-bundle.*

(iii). *A connection $\{\theta_U\}$ of D with respect to ξ, a smooth G-bundle, is called a D-flat connection if there exists $\{h_U\} \in C^0(U,\ \mathrm{G}_d)$ such that*

$$\theta_U = \rho_D(h_U),\ \ for\ any\ U \in \mathfrak{U}.$$

Proposition 1. *For any $\xi \in H^1(M,\ \mathrm{G}_d)$, the followings are equivalent.*

(i). *ξ is a D-flat G-bundle.*

(ii). *D allows 0 as a connection with respect to ξ.*

(iii). *D has a D-flat connection with respect to ξ.*

Proof. If $\xi = \{g_{UV}\} \in H^1(M,\ \mathrm{G}_{c(D)})$, we have $D_U \otimes 1_{\mathrm{H}}(g_{UV,\,1} \otimes g_{UV}) = g_{UV,\,2} \otimes g_{UV}$ $(D_V \otimes 1_{\mathrm{H}})$, where $\{g_{UV,\,i}\}$ is the transition function of E_i. Hence (ii) follows from (i). If D allows 0 as a connection with respect to ξ, $\{-\rho_D(h_U)\}$ is a connection of D with respect to $\{h_U{}^{-1} g_{UV} h_V\}$ ([3]). Hence (iii) follows from (ii). If (iii) is hold, we have

$$(1_{V_2}\otimes h_U)(D_U\otimes 1_{\mathrm{H}})(1_{V_1}\otimes h_U^{-1}g_{UV})=(1_{V_2}\otimes g_{UV}h_V)(D_V\otimes 1_{\mathrm{H}})(1_{V_1}\otimes h_V^{-1}).$$

Hence $\{h_U^{-1}g_{UV}h_V\}$ is a $c(D)$-lcass G-bundle and (i) follows from (iii).

Corollary. *A G-bundle ξ is D-flat if and only if D has a D-flat connection with respect to ξ.*

By proposition 1, (ii), if ξ is a $c(D)$-class G-bundle, D is lifted to a differential operator $C^\infty(M,\ E_1\otimes\xi)\longrightarrow C^\infty(M,\ E_2\otimes\xi)$ with connection 0. This lift of D is denoted by $D\otimes 1_\xi$. By definition and proposition 1, $D\otimes 1_\xi$ is defined if and only if ξ is a $c(D)$-class G-bundle.

Example. If $r(D)$ is maximal, D-flat is flat in the usual sence. On the other hand, if $D=\bar{\partial}$, a G-bundle ξ is D-flat if and only if G is a complex Lie group and ξ is a holomorphic G-bundle.

5. If $L\in H^0(M,\ \mathrm{L}_{G,D})$, $L: C^\infty(M,\ \mathrm{E}_1\otimes\mathrm{H})\longrightarrow C^\infty(M,\ E_2\otimes\mathrm{H})$ is a differential operator of order at most $k-1$. Hence $D\otimes 1_{\mathrm{H}}-L: C^\infty(M,\ \mathrm{E}_1\otimes\mathrm{H})\longrightarrow C^\infty(M,\ \mathrm{E}_2\otimes\mathrm{H})$ is a differential operator such that

$$\sigma(D\otimes 1_{\mathrm{H}}-L)=\sigma(D)\otimes 1_{\mathrm{H}}. \tag{7}$$

Here $\sigma(D)$, *etc.*, means the principal symbol of D, *etc.*, On the other hand, since $L\in H^0(M,\ \mathrm{L}_{G,D})$, we obtain

$$(D\otimes 1_{\mathrm{H}}-L)\,|\,U=D^{h_u}=(1_{V_2}\otimes h_U)(D_U\otimes 1_{\mathrm{H}})(1_{V_1}\otimes h_U^{-1}),\quad L\,|\,U=\rho_D(h_U). \tag{8}$$

(8) shows the commutativity of the diagram

$$\begin{array}{ccc} C^\infty(M,\ \mathrm{E}_1\otimes\delta(L)) & \xrightarrow{D\otimes 1_{\delta(L)}} & C^\infty(M,\ E_2\otimes\delta(L)) \\ t_{\delta(L)}\uparrow\cong & & t_{\delta(L)}\uparrow\cong \\ C^\infty(M,\ E_1\otimes\mathrm{H}) & \xrightarrow{D\otimes 1_{\mathrm{H}}-L} & C^\infty(M,\ E_2\otimes\mathrm{H}). \end{array}$$

Here $t_{\delta(L)}$ is the map given by the smooth trivialization of $\delta(L)$. Explicitly, $t_{\delta(L)}$ is given by

$$t_{\delta(L)}(\{f_U\otimes\varphi\})=f_U\otimes h_U\varphi,\quad \delta(L)=\{h_Uh_V^{-1}\}, \tag{9}$$
$$\varphi \text{ is a smooth } \mathrm{H}\text{-valued function.}$$

Using $t_{\delta(L)}$, (8) is rewritten as

$$t_{\delta(L)}(D\otimes 1_{\mathrm{H}}-L)t_{\delta(L)}^{-1}=D\otimes 1_{\delta(L)}. \tag{8$'$}$$

Definition. *A differential operator of the form $D\otimes 1_{\mathrm{H}}-L$ is called a D-Fuchs type differential operator and $\delta(L)$ is called its monodromy bundle.*

Lemma 7. *$\delta(L)=\delta(L')$ if and only if there exists a smooth G-valued function f on M such that*

$$L'=\rho_D(f)+L^f,\quad L^f=(1_{E_2}\otimes f)L(1_{E_1}\otimes f^{-1}). \tag{10}$$

Proof. By the exactness of (6), set $L=\rho_D(h_U)$ and $L'=\rho_D(h_U')$, we have

$$h_U'=fh_Uc_U,\ c_U \text{ is a } c(D)\text{-class } G\text{-valued function on } U,$$
$$f\in H^0(M,\ \mathrm{G}_d).$$

This shows (10).

If $r(D)$ is maximal, there is a bijection $\chi: H^1(M,\ \mathrm{G}_{c(D)})\longrightarrow \mathrm{Hom}(\pi_1(M),\ G)$. We call $\chi(\delta(L))$ the monodromy representation of $D\otimes 1_{\mathrm{H}}-L$ and $\chi(\delta(L))(\pi_1(M))$ the monodromy group of $D\otimes 1_{\mathrm{H}}-L$ (cf. Introduction). For $D=d/dz$, $\mathrm{H}=\mathbf{C}^n$, the n-dimensional complex vector space, and M is a Riemann surface, these definitions are same as usual definitions.

Definition. *The least structure group of $\delta(L)$ as a $c(D)$-class undle is called the monodromy group of $D\otimes 1_{\mathrm{H}}-L$.*

In the rest of this §, we construct the monodromy group of $D\otimes 1_{\mathrm{H}}-L$ under the assumption that G is a Lie group.

Definition. *Denote $\pi_F: M_F\longrightarrow M$ the projection of a smooth G-bundle with the fibre F over M, if D can be lifted on $C^\infty(M_F,\ \pi_F{}^*(E_1))$ with connection 0, we denote $\pi_F{}^*(D)$ this lift of D.*

Let F be a smooth right G-manifold with a G-invariant measure $d\mu$ constructed by G-invariant vector fields over F. Then, denote $U(\mathrm{L}^2(F))$ the group of unitary operators on $\mathrm{L}^2(F)=\mathrm{L}^2(F,\ d\mu)$, there is a unitary representation $\kappa: G\longrightarrow U(\mathrm{L}^2(F))$ given by the G-action on F, and the following diagram is commutative.

$$\begin{tikzcd} \mathrm{U}(\mathrm{L}^2(F))_{c(D)} \arrow[r] & \mathrm{U}(\mathrm{L}^2(F))_d \\ \mathrm{G}_{c(D)} \arrow[u, "\kappa^*"] \arrow[r] & \mathrm{G}_d. \arrow[u, "\kappa^*"'] \end{tikzcd} \tag{11}$$

Lemma 8. *Let ξ be a D-flat G-bundle, θ a connection of associate F-bundle of ξ, $\kappa(\xi)$ the associate $\mathrm{L}^2(F)$-bundle of ξ defined by θ* (cf. [3]). *Then, to denote M_F the tatal space of the associate F-bundle of ξ, $\pi_F{}^*(D)$ is defined.*

Proof. By the commutativity of (11) and proposition 1, $\kappa(\xi)$ is D-flat. Hence D can be lifted on $C^\infty(M_F,\ \pi_F{}^*(E_1))$ with connection 0 (cf. [3]). Therefore we get the lemma.

Corollary. (i). *If $D\otimes 1_{\mathrm{H}}-L$ is a D-Fuchs type operator and M_F is the associate F-bundle of $\delta(L)$ which satisfies the above assumptions, then $\pi_F{}^*(D\otimes 1_{\mathrm{H}}-L)$ is defined.*

(ii). *Under the same assumptions, if M_F is the principal bundle, $\pi_F{}^*(\xi)$ is trivial as a $c(\pi_F{}^*(D))$-class bundle.*

(iii). *Under the same assumptions, if $\pi_F{}^*(\delta(L))$ is a trivial $c(\pi_F{}^*(D))$-bundle then there is a smooth G-valued function f on M_F such that*

$$\pi_F{}^*(D\otimes 1_{\mathrm{H}}-L)=\pi_F{}^*(D)^f. \tag{12}$$

Proof. Since $\pi_F{}^*(D)$ is defined, $\pi_F{}^*(D_U)$ is equal to $\pi_F{}^*(D_V)$ on $\pi_F{}^{-1}(U)\cap\pi_F{}^{-1}(V)$. Then, since $(D\otimes 1_{\mathrm{H}}-L)\,|\,U=D^{h_U}$, $\pi_F{}^*(D\otimes 1_{\mathrm{H}}-L)$ is given by

$$(12)' \qquad \pi_F{}^*(D\otimes 1_{\mathrm{H}}-L)\,|\,\pi_F{}^{-1}(U)=(\pi_F{}^*(D_U))^{\pi_F{}^*(h_U)}.$$

This shows (i). The trivialization of $\pi_F{}^*(\xi)$ is given by

$$(13) \qquad \{h_U(x,\ g)\,|\,h_U(x,\ g)=g\in G,\ x\in U\subset M\}.$$

Hence we have (ii). (iii) follows from (12)'.

6. In this n^0, we use same notations and assumptions as in lemma 8. Since M_F is a right G-space, we set $f^g(u)=f(ug)$, $u\in M_F$, $g\in G$. Here f is a function on M_F. The set of $c(\pi_F{}^*(D))$-class G-valued functions on M_F is denoted by $\mathrm{G}_{c(D),\,M_F}$ and we set

$$B^1(G,\ \mathrm{G}_{c(D),\,M_F})=\{\chi : G\longrightarrow \mathrm{G}_{c(D),\,M_F}\,|\,\chi_{gh}=\chi_h\chi_g^h,\ \chi_g^h(u)$$
$$=\chi_g(uh),\ \chi_g=\chi(g)\}.$$

We call χ and $\chi'\in B^1(G,\ \mathrm{G}_{c(D),\,M_F})$ to be equivalent if $\chi_g{}'=h^{-1}\chi_g h^g$ for some $h\in\mathrm{G}_{c(D),\,MF}$ and denote $H^1(G,\ \mathrm{G}_{c(D),\,M_F})$ the quotient set of $B^1(G,\ \mathrm{G}_{c(D),\,M_F})$ by this relation.

Since a constant function is a $c(\pi_F{}^*(D))$-class function invariant under the action of G, there is a map $\iota_F : \mathrm{Hom}(G,\ G)\longrightarrow H^1(G,\ \mathrm{G}_{c(D),\,M_F})$. Here $\mathrm{Hom}\,(G,\ G)$ means the set of Lie homomorphisms of G. We set

$$\ker\ \iota_F=\{\kappa\,|\,\iota_F(\kappa)=\iota_F(1),\ 1_g=g \textit{ for all } g\in G\}.$$

Definition. *$\bar{\chi}\in H^1(G,\ \mathrm{G}_{c(D),M_F})$ is called to have (smooth) representative function if there exists a smooth G-valued function f on M_F such that $f^g=f\chi_g$, $\chi\in\bar{\chi}$, $g\in G$. This f is called a representative function subordinate to $\bar{\chi}$.*

If $\chi\sim\chi'$, and χ has a representative function f, set $\chi_g{}'=h^{-1}\chi_g h^g$, $f'=fh$ is a representative function subordinate to χ'. Hence this definition does not depend on the choice of a representative of $\bar{\chi}$.

We set

$$\delta(H^0(M,\ \mathrm{L}_{G,\,D}))_{M_F}=\{\delta(L)\,|\,\pi_F{}^*(\delta(L)) \textit{ is trivial},\}$$

$$H^1(G,G_{c(D),\,M_F})_f=\{\bar{\chi}\in H^1(G,\ \mathrm{G}_{c(D),\,M_F})\,|\,\bar{\chi} \textit{ has a smooth representative function}\}.$$

Lemma 9. (i). *There is a bijection $\chi : \delta(H^0(M,\ \mathrm{L}_{G,\,D}))_{M_F}\longrightarrow H^1(G,\ \mathrm{G}_{c(D),\,M_F})_f$ and if M_F is the associate F-bundle of $\delta(L)$, we obtain*

$$(14) \qquad \chi(\delta(L))\in\ker\ \iota_F.$$

(ii). *$\iota_F(\kappa)$ belongs in $\ker\ \iota_F$ if and only if there exists $f\in\mathrm{G}_{c(D),\,M_F}$ such that*

(15) $$f^g=\kappa_g^{-1}fg.$$

Proof. If $\pi_F{}^*(D)^f$ comes from an operator on M, set $f^g=f\chi_g$, $\chi=\{\chi_g\}$ defines an element of $H^1(G, \mathrm{G}_{c(D),M_F})$. If $\pi_F{}^*(D)^f=\pi_F{}^*(D)^{f'}$, set $f^g=f\chi_g$, $f'^g=f'\chi_g{}'$, χ and χ' define same element of $H^1(G, \mathrm{G}_{c(D),M_F})$. Hence χ is 1 to 1 by lemma 7. If $\bar{\chi}\in H^1(G, \mathrm{G}_{c(D),M_F})_{f'}$, there exists f such that $f^g=f\chi_g$, $\chi\in\bar{\chi}$. Then $\pi_F{}^*(D)^f$ comes from a D-Fuchs type operator on M and χ is onto. If M_F is the associate F-bundle of $\delta(L)$, the trivialization of $\pi_F{}^*(\delta(L)$ given by (13) gives $\iota_F(1)$. This shows (14). (ii) follows from the definition of the equivalence in the definition of $H^1(G, \mathrm{G}_{c(D),M_F})$.

Lemma 10. (i). *If $\kappa\in\ker\iota_F$, there exists a smooth G-valued function f on M_F such that*

(16) $$\pi_F{}^*(D\otimes 1_{\mathrm{H}}-L)=\pi_F{}^*(D)^f,\ f^g=f\kappa(g),$$

and the structure group of $\delta(L)$ is reduced to $\kappa(G)$ as a $c(D)$-class bundle.

(ii). *If the structure group of $\delta(L)$ is reduced to G_0 as a $c(D)$-class G-bundle, there exists $\kappa\in\ker\iota_F$ such that $\kappa(G)=G_0$.*

Proof. Since κ has a representative function f, we have (16) by (15). (15) also shows the second assertion of (i). Since a $c(D)$-class reduction of the structure group of $\delta(L)$ gives a representative function on M_F, we obtain (ii) by lemma 9, (ii).

Definition. *We call κ_1 κ_2 in* Hom (G, G) *(resp. in* ker ι_F*), if $\kappa_2=\kappa\kappa_1$ for some $\kappa\in$*Hom (G, G) *(resp. $\kappa\in$*ker ι_F*) and $\kappa_1\sim\kappa_2$ if κ_1 κ_2 and κ_2 κ_1.*

Lemma 11. (i). *If κ_1 and κ_2 belong in* ker ι_F, *there composition $\kappa_1\kappa_2$ also belongs in* ker ι_F.

(ii). *If $\kappa_1\kappa_2$ and κ_2 belong in* ker ι_F, *κ_1 belongs in* ker ι_F.

Proof. If $f_1{}^g=\kappa_{1,g}{}^{-1}f_1g$ and $f_2{}^g=\kappa_{2,g}{}^{-1}f_2g$, we have

$$(\kappa_2(f_1)f_2)^g=\kappa_2(\kappa_1(g))^{-1}\kappa_2(f_1)f_2g.$$

This shows (i). Similarly, if $f^g=\kappa_1(\kappa_2(g))^{-1}fg$ and $f_2{}^g=\kappa_{2,g}{}^{-1}f_2g$, we have $(\mathrm{K}_1(f_2)^{-1}f)^g=\kappa_{1,g}{}^{-1}\ \kappa_1(f_2)^{-1}fg$, which shows (ii).

Corollary. (i). *If κ_1 and κ_2 are in* ker ι_F *and $\kappa_1\kappa_2$ in* Hom (G, G), *$\kappa_1>\kappa_2$ in* kerι_F.

(ii). *If $\kappa_1\sim\kappa_2$, $\kappa_1(G)$ is isomorphic to $\kappa_2(G)$.*

Proof. (i) follows from lemma 11, (ii). If $\kappa_1\sim\kappa_2$, we have $\kappa_1=\kappa\kappa_2$ and $\kappa_2=\kappa'\kappa_1$. Hence dim κ_1 (G)=dim κ_2 (G) and there are discreet subgroups N_1 of κ_1 (G) and N_2 of κ_2 (G) such that

$$\kappa_1(G)/N_1\cong\kappa_2(G),\ \kappa_2(G)/N_2\cong\kappa_1(G),$$

because $\kappa_1(G)$ and $\kappa_2(G)$ are Lie groups. Hence there are isomorphisms $\hat{\kappa}:\widetilde{\kappa_1(G)}\longrightarrow\widetilde{\kappa_2(G)}$, where $\widetilde{\kappa_1(G)}$ and $\widetilde{\kappa_2(G)}$ are the universal covering groups of κ_1 (G) and κ_2 (G),

and $\hat{\kappa}' : \widetilde{\widehat{\kappa_1(G)}} \longrightarrow \widetilde{\widehat{\kappa_1(G)}}$ such that $\hat{\kappa}$ maps $\pi_1\ (\kappa_1(G))$ isomorphic into $\pi_1\ (\kappa_2(G))$ and $\hat{\kappa}'$ maps $\pi_1\ (\kappa_2(G))$ isomorphic into $\pi_1\ (\kappa_1(G))$. Since $\kappa_1\ (G)$ and $\kappa_2\ (G)$ are Lie groups, this shows $\hat{\kappa} : \pi_1\ (\kappa_1(G)) \cong \pi_1\ (\kappa_2(G))$ and we have (ii).

Lemma 12. ker ι_F *has the least element in the above semiorder.*

Proof. Let $\{\kappa_\alpha\}$ be an increasing system in ker ι_F and set $\kappa_\alpha(G) = G_\alpha$. Then there are Lie epimorphisms κ_α^β, $\beta < \alpha$ and Lie monomorphisms ι_α such that

$$\kappa_\alpha^\beta \kappa_\alpha = \kappa_\beta^\beta, \quad \kappa_\alpha^\beta : G_\alpha \longrightarrow G_\beta, \quad \iota_\alpha : G_\alpha \longrightarrow G, \quad \kappa_\alpha^\beta \iota_\alpha = \iota_\beta.$$

Hence $lim[G_\alpha : \kappa_\alpha^\beta] = G_0$, $\kappa_0 : G \longrightarrow G_0$ and $\iota_0 : G_0 \longrightarrow G$ are defined. Since κ_α^β and ι_α are Lie maps, κ_0 is a Lie epimorphism and ι_0 is a Lie monomorphism.

By lemma 9, (ii), there exists $f_\alpha \in G_{c(D),\, M_F}$ such that $(f_\alpha)^g = (\kappa_\alpha)_g^{-1} f_{\alpha g}$ for any α. Then, since $(\kappa_\alpha^\beta f_\alpha)^g = (\kappa_\beta)_g^{-1} f_\beta g$, set

$$f_0 = \iota_0 \{(\kappa_\beta^\alpha f_\alpha)\},$$

$f_0 \in G_{c(D),\, M_F}$. Because each κ_α^β is a smooth map and $f_0{}^g = (\kappa_0)_g^{-1} f_{0g}$. Hence by Zorn's lemma, there exist minimum elements in ker ι_F. But if κ_1 and κ_2 are different minimum elements in ker ι_F, $\kappa_1\kappa_2$ and $\kappa_2\kappa_1$ are in ker ι_F by lemma 11, (i). Hence $\kappa_1 > \kappa_2$ and $\kappa_2 > \kappa_1$. Therefore $\kappa_1 \sim \kappa_2$ and ker ι_F has the least element.

Definition. *The least element of* ker ι_F *is called the monodromy homomorphism (or representation) of* $D \otimes 1_{\mathrm{H}} - L$.

By lemma 10, (ii), lemma 12 and the definition of the monodromy groups of D-Fuchs type operators, we obtain

Theorem 1. *If G is a Lie group, a D-Fuchs type operator has the monodromy group.*

Proof. Since $D \otimes 1_{\mathrm{H}} - L$ has the monodromy homomorphism and the image of G by the monodromy homomorphism is the least structure group of $\delta(L)$ as a $c(D)$-class bundle, we have the theorem.

§3. Characteristic classes related to $c(D)$-class bundles

7. In this § and next §, we assume $\mathrm{H} = \mathbf{C}^n$ and $G = GL(n, \mathbf{C})$.

By the commutative diagram in n°3, example, we have the following commutative diagram with exact lines

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(M, \mathbf{C}^*_{c(D)}) \arrow[r, "i^*"] & H^1(M, \mathbf{C}^*_d) \arrow[r, "\rho_D{}^*"] & H^1(M, \mathrm{L}_{\mathbf{C}^*,\, D}) \arrow[r, "\delta_1"] & H^2(M, \mathbf{C}^*_{c(D)}) \arrow[r] & \cdots \\
 & & 0 \arrow[u] \arrow[r] & H^1(M, \mathscr{L}_{\mathbf{C},\, D}) \arrow[u, "\mathrm{k}^*"] \arrow[r, "\delta_2"] \arrow[r, equal] & H^2(M, \mathbf{C}_{c(D)}) \arrow[u, "="'] \arrow[r] & 0
\end{tikzcd}$$

Lemma 13 (i). *$\xi\in H^1(M, \mathbf{C}_d{}^*)$ is in i^*-image if and only if $\delta_2 k^{*-1}\rho_D{}^*(\xi)=0$.*

(ii). *Let $ch: H^1(M, \mathbf{C}^*{}_d) \xrightarrow{\cong} H^2(M, \mathbf{Z})$ be the isomorphism given by $ch(\xi)=c^1(\xi)$, the first Chern class of ξ, and $\iota: \mathbf{Z}\longrightarrow \mathbf{C}_{c(D)}$ the inclusion, then*

$$\delta_2 k^{*-1}\rho_D{}^*(\xi)=\iota^* ch(\xi),\ \ \xi\in H^1(M, \mathbf{C}^*{}_d). \tag{17}$$

Proof. (i) follows from the definition. By the definition of k, we have

$$\delta_2 k^{*-1}\rho_D{}^*(\xi)=\frac{1}{2\pi\sqrt{-1}}(\log g_{UV}+\log g_{VW}+\log g_{WU}),\ \ \xi=\{g_{UV}\}.$$

Since this right hand side represents $ch(\xi)$, we get (17).

Definition. *Let ξ be a $GL(n, \mathbf{C})$-bundle over M, denote $ch(\xi)$ its tatal Chern class, then we call $\iota^*(ch(\xi))$ the (tatal) $c(D)$-characteristic class of ξ. The component of $\iota^*(ch(\xi))$ in $H^{2p}(M, \mathbf{C}_{c(D)})$ is called p-th $c(D)$-characteristic class of ξ.*

Example. If $r(D)$ is maximal, $c(D)$-characteristic class is the (tatal) complex Chern class. If M is a compact Kaehler manifold and $D=\bar{\partial}$, p-th $c(D)$-characteristic class is the $(0, 2p)$-component of p-th complex Chern class.

In the rest, we denote the flag manifold $GL(m, \mathbf{C})/\Delta(m, \mathbf{C})=\mathrm{U}(m)/T^m$ by $F=F(m)$. The associate Flag bundle of a ($c(D)$-class) $GL(m, \mathbf{C})$-bundle ξ is denoted by $M_F=\{M_F, F, M, \pi_F\}$.

Lemma 14. *Under the above notations, if ξ is a $c(D)$-class bundle, $\pi_F{}^*: H^*(M, \mathbf{C}_{c(D)})\longrightarrow H^*(M_F, \mathbf{C}_{c(\pi_F{}^*(D))})$ is a monomorphism.*

Proof. If $\pi_F{}^{-1}(U)=U\times F$, $(\mathbf{C}_{c(D)}|U)\otimes \mathbf{C}_d(F)$ is dense in $\mathbf{C}_{c(\pi_F{}^*(D)}|\pi_F{}^{-1}(U)$, that is $H^0(U, \mathbf{C}_{c(D)})\otimes H^0(\mathrm{F}, \mathbf{C}_d)$ is dense by the $\mathscr{C}^\infty$-topology in $H^0(\pi_F{}^{-1}(U), \mathbf{C}_{c(\pi_F{}^*(D))})$. Since $\mathbf{C}_d(F)$ is a fine sheaf, $H^*(M_F, \mathbf{C}_{c(\pi_F{}^*(D))})$ is calculated by a covering of the form $\{\pi_F{}^{-1}(U)\}$ by Leray's theorem. Then, taking the invariant measure $d\mu$ on F such that $\int_F d\mu=1$, we set

$$\int_F\{g_{i_0,\dots,i_p}\}=\{\int_F g_{i_0,\dots,i_p}d\mu\},$$

$$g_{i_0,\dots,i_p} \text{ is defined on } \pi_F{}^{-1}(U_{i_0})\cap\dots\cap\pi_F{}^{-1}(U_{i_p})=\pi_F{}^{-1}(U_{i_0}\cap\dots\cap U_{i_p}).$$

By definition, $\int_F$ defines a homomorphism from $H^*(M_F, \mathbf{C}_{c(\pi_F{}^*(D))})$ into $H^*(M, \mathbf{C}_{c(D)})$ and $\int_F\pi_F{}^*$ is the identity. Hence we get the lemma.

Corollary. *Under the same assumptions, $c(D)$-characteristic class of ξ vanishes if and only if $c(\pi_F{}^*(D))$-characteristic class of $\pi_F{}^*(\xi)$ vanishes.*

Proof. Since $\pi_F{}^*$ in both sides in the following commutative diagram are monomorphisms, we have the lemma.

$$\begin{array}{ccc} H^*(M_F,\ \mathbf{Z}) & \xrightarrow{\iota^*} & H^*(M_F,\ \mathrm{C}_{c(\pi_F{}^*(D))}) \\ \pi_F{}^* \uparrow & & \pi_F{}^* \uparrow \\ H^*(M,\ Z) & \xrightarrow{\iota^*} & H^*(M,\ \mathrm{C}_{c(D)}). \end{array}$$

Proposition 2. *If ξ is a $c(D)$-class $GL(m,\ \mathbf{C})$-bundle, its $c(D)$-characteristic class vanishes.*

Proof. By lemma 13, the proposition is true if $m=1$. Set $m=q+1$ and assume the proposition is true for $c(D)$-class $GL(r,\ \mathbf{C})$-bundle if $r\leqq q$.

On M_F, $\pi_F{}^*(\xi)$ is an extension bundle of a $c(\pi_F{}^*(D))$-class $GL(q,\ \mathbf{C})$-bundle η_q and a $c(\pi_F{}^*(D))$-class complex line bundle η_1. Since $\mathrm{C}_{c(\pi_F{}^*(D))}$ is a sheaf of rings by lemma 2, (i), $\iota^*(\mathrm{ch}(\eta_1))\cup\iota^*(ch(\eta q))$ is defined and we have

$$\iota^*(ch(\pi_F{}^*(\xi)))=\iota^*(ch(\eta_1))\cup\iota^*(ch(\eta_q))=0,$$

by inductive assumption. Hence we obtain the proposition by corollary of Lemma 14.

Note. For flat bundles and holomorphic bundles, this proposition is known. In fact, a vector bundle is flat if and only if its curvature form is equal to 0 and therefore its complex Chern class is equal to 0. On the other hand, a vector bundle is equivalent to a holomorphic bundle if and only if (0, 2)-type part of its curvature form is equal to 0. Hence (0, $2p$)-type part of the Chern class of a holomorphic vector bundle is equal to 0.

8. For $\{g_{i_0,\dots,i_p}\}\in C^p(\mathfrak{U},\ \mathbf{C}^*_{c(D)})$ and $\{h_{i_0,\dots,i_q}\}\in C^q(\mathfrak{U},\ \mathbf{C}^*_{c(D)})$, we set

$$(g*h)_{i_0,\dots,i_{p+q+1}}=\exp\Big[\frac{1}{2\pi\sqrt{-1}}\log\ g_{i_0,\dots,i_p}\ (\delta\log h)_{i_p,\dots,i_{p+q+1}}\Big],$$

$$(\delta\log h)_{i_0,\dots,i_{q+1}}=\sum_{j=0}^{q+1}(-1)^j\log\ h_{i_0,\dots,i_{j-1},i_{j+1},\dots,i_{q+1}}.$$

Here we assume $\mathfrak{U}$ is sufficiently fine and $\log\ g_{i_0,\dots,i_p}$ or $\log\ h_{i_0,\dots,i_q}$ are determined as 1-valued functions. The choice of the branch of logarithm is arbitraly, and therefore this definition of $(g*h)$ depend on the choice of the branch of logarithm.

Lemma 15. (i). *If $\{g_{i_0,\dots,i_p}\}$ and $\{h_{i_0,\dots,i_q}\}$ are both cocycles, $\{(g*h)_{i_0,\dots,i_{p+q+1}}\}$ is a cocycle and its cohomology class in $H^{p+q+1}(M,\ \mathbf{C}^*_{c(D)})$ does not depend on the choice of the branch of logarithm.*

(ii). *If either of $\{g_{i_0,\dots,i_p}\}$ or $\{h_{i_0,\dots,i_q}\}$ is a coboundary and the other is a cocycle, $\{(g*h)_{i_0,\dots,i_{p+q+1}}\}$ is a coboundary.*

Proof. Since we have

$$\log\ g_{i_1,\dots,i_{p+1}}\ (\delta\log h)_{i_{p+1},\ \dots,i_{p+q+2}}-$$

$$-\log\ g_{i_0,i_2,\dots,i_{p+1}}\ (\delta\log h)_{i_{p+1},\dots i_{p+q+2}}+\cdots+$$

$$+(-1)^p \log\ g_{i_0,\dots,i_{p-1},i_{p+1}}\ (\delta\log h)_{i_{p+1},\dots,i_{p+q+2}}+$$
$$+(-1)^{p+1}\log\ g_{i_0,\dots,i_p}\ (\delta\log h)_{i_p,i_{p+2},\dots,i_{p+q+2}}+\cdots+$$
$$+(-1)^{p+q+2}\log\ g_{i_0,\dots,i_p}\ (\delta\log h)_{i_p,\dots,i_{p+q+1}}$$
$$=(\delta\log\ g)_{i_0,\dots,i_{p+q+1}}\ (\delta\log h)_{i_{p+1},\dots,i_{p+q+2}}+$$
$$+(-1)^p\log\ g_{i_0,\dots,i_p}\ \{\delta(\delta\log h)\}_{i_p,\dots,i_{p+q+2}},$$

$\{(g_*h)_{i_0,\dots,i_{p+q+1}}\}$ is a cocycle if $\{g_{i_0,\dots,i_p}\}$ and $\{h_{i_0,\dots,i_q}\}$ are both cocycles. If we take other branches of logarithm in the definition of (g_*h), denote $\log'$ other branches of log, we get

$$(g_*h)_{i_0,\dots,i_{p+q+1}}\ \{(g_*h)'_{i_0,\dots,i_{p+q+1}}\}^{-1}$$
$$=\exp[\frac{1}{2\pi\sqrt{-1}}\ \{(\log\ g_{i_0,\dots,i_p}-\log' g_{i_0,\dots,i_p})\ (\delta\log h)_{i_p,\dots,i_{p+q+1}}+$$
$$+\log'\ g_{i_0,\dots,i_p}\ \{(\delta\log h)_{i_p,\dots,i_{p+q+1}}-(\delta\log' h)_{i_p,\dots,i_{p+q+1}}\}\}].$$

Since $(1/2\pi\sqrt{-1})\ (\delta\log h)_{i_p,\dots,i_{p+q+1}}$ is an integer if $\{h_{i_0,\dots,i_p}\}$ is a cocycle, we get by this formula

$$(g_*h)_{i_0,\dots,i_{p+q+1}}\ \{(g_*h)'_{i_0,\dots,i_{p+q+1}}\}^{-1}$$
$$=g_{i_0,\dots,i_p}{}^{(n_{i_{p+1},\dots,i_{p+q+1}}-n_{i_p,i_{p+2},\dots,i_{p+q+1}}+\cdots+(-1)^{q+1}n_{i_p,\dots,i_{p+q}})},$$

where each $n_{i_0,\dots i_q}$ is an integer. Then, to define $f_{i_0,\dots,i_{p+q}}$ by

$$f_{i_0,\dots,i_{p+q}}=g_{i_0,\dots,i_p}{}^{n_{i_p,\dots,i_{p+q}}},$$

we get

$$(\delta f)_{i_0,\dots,i_{p+q+1}}$$
$$=(g_{i_1,\dots,i_p}\ g_{i_0,i_2,\dots,i_{p+1}}{}^{-1}\cdots g_{i_0,\dots,i_{p-1},i_{p+1}}{}^{(-1)^p})^{n_{i_{p+1},\dots,i_{p+q+1}}}$$
$$\cdot g_{i_0,\dots,i_p}{}^{((-1)^{p+1}n_{i_pi_{p+2},\dots,i_{p+q+1}}+\cdots+(-1)^{p+q+1}n_{i_p,\dots,i_{p+q}})}$$
$$=g_{i_0,\dots,i_p}{}^{(-1)^p n_{i_{p+1},\dots,i_{p+q+1}}}g_{i_0,\dots,i_p}{}^{((-1)^{p+1}n_{i_p,i_{p+2},\dots,i_{q+q+1}}+\cdots+(-1)^{p+q+1}n_{i_p,\dots,i_{p+q}})},$$

if $\{g_{i_0,\dots,i_p}\}$ is a coboundary. Hence we obtain the second assertion of (i).

If $\{h_{i_0,\dots i_q}\}$ is a coboundary, we also get

$$(g_*h)_{i_0,\dots,i_{p+q+1}}=g_{i_0,\dots,i_p}{}^{(n_{i_{p+1},\dots,i_{p+q+1}}-n_{i_p,i_{p+2},\dots,i_{p+q+1}}+\cdots+(-1)^{q+1}n_{i_p,\dots,i_{p+q}})},$$

because $\{(1/2\pi\sqrt{-1})\ (\delta h)_{i_0,\dots,i_{q+1}}\}$ is an integral coboundary in this case. Hence

$\{(g_*h)_{i_0,\dots,i_{p+q+1}}\}$ is a coboundary if $\{g_{i_0,\dots,i_p}\}$ is a cocycle. But since

$$\begin{aligned}&\delta(\log g_{i_0,\dots,i_p}\log h_{i_p,\dots,i_{p+q}})_{i_0,\dots,i_{p+q+1}}\\&=(\delta\log g)_{i_0,\dots,i_{p+1}}\log h_{i_{p+1},\dots,i_{p+q+1}}+\\&+(-1)^p\log g_{i_0,\dots,i_p}(\delta\log h)_{i_p,\dots,i_{p+q+1}},\end{aligned}$$

we may define g_*h by

$$\begin{aligned}&(g_*h)_{i_0,\dots,i_{p+q+1}}\\&=(-1)^{p+1}\left[\frac{1}{2\pi\sqrt{-1}}(\delta\log g)_{i_0,\dots,i_{p+1}}\log h_{i_p,\dots,i_{p+q+1}}\right].\end{aligned}$$

Hence $\{(g_*h)_{i_0,\dots,i_{p+q+1}}\}$ is a coboundary if $\{g_{i_0,\dots,i_p}\}$ is a coboundary and $\{h_{i_0,\dots,i_q}\}$ is a cocycle. Therefore we obtain (ii).

Definition. *If $c_p\in H^p(M,\mathbf{C}^*_{c(D)})$ and $c_q\in H^q(M,\mathbf{C}^*_{c(D)})$ are the cohomology classes of cocycles $\{g_{i_0,\dots,i_p}\}$ and $\{h_{i_0,\dots,i_q}\}$, we denote $c_p{}_*c_q$ the cohomology class of $\{(g_*h)_{i_0,\dots,i_{p+q+1}}\}$ in $H^{p+q+1}(M,\mathbf{C}^*_{c(D)})$ and call the $_*$-product of c_p and c_q.*

Lemma 16. (i). *$\sum_p H^p(M,\mathbf{C}^*_{c(D)})$ is a ring by the $_*$-product. That is, we have*

$$c_1{}_*(c_2{}_*c_3)=(c_1{}_*c_2)_{*3},\quad c_1{}_*c_2=(-1)^{p+1}c_2{}_*c_1,\quad c_1\in H^p(M,\mathbf{C}^*_{c(D)}),$$

$$c_1{}_*(c_2c_3)=(c_1{}_*c_2)\ (c_1{}_*c_3),\ cc' \text{ is the usual product in } \sum_p H^p(M,\mathbf{C}^*_{c(D)}).$$

(ii). *Let $\delta:\sum_p H^{p-1}(M,\mathbf{C}^*_{c(D)})\longrightarrow\sum_p H^p(M,\mathbb{Z})$ be the coboundary homomorphism, we have*

$$\delta(c_1{}_*c_2)=(\delta c_1)\cup\delta(c_2). \tag{18}$$

Proof. Since we have

$$\begin{aligned}&\delta(\log g_{i_0,\dots,i_p}(\delta\log h)_{i_p,\dots,i_{p+q+1}})\\&=\delta\log g_{i_0,\dots,i_{p+1}}(\delta\log h)_{i_{p+1},\dots,i_{p+q+2}},\\&\quad\log g_{i_0,\dots,i_p}(\delta\log h)_{i_{p+1},\dots,i_{p+q+1}}-\\&\quad-(-1)^{p+1}(\delta\log g_{i_0,\dots,i_{p+1}})\log h_{i_{p+1},\dots,i_{p+q+1}}\\&=(-1)^{p+1}(\delta(\log g_{i_0,\dots,i_p}\log h_{i_p,\dots,i_{p+q}})_{i_0,\dots,i_{p+p+1}}),\\&\quad\delta\log f_{i_0,\dots,i_{p+1}}(\log gh)_{i_{p+1},\dots,i_{p+q+1}}\\&=\delta\log f_{i_0,\dots,i_{p+1}}(\log g_{i_{p+1},\dots,i_{p+q+1}}+\log h_{i_{p+1},\dots,i_{p+q+1}}),\end{aligned}$$

we obtain (i) by lemma 15.

By the definition of $_*$-product, we get

$$\frac{1}{2\pi\sqrt{-1}}(\delta \log (g_* h))_{i_0,\dots,i_{p+q+1}}$$

$$=\frac{1}{2\pi\sqrt{-1}}(\delta \log g)_{i_0,\dots,i_{p+1}}\frac{1}{2\pi\sqrt{-1}}(\delta \log h)_{i_{p+1},\dots,i_{p+q+2}}.$$

Since this right hand side represents $\delta(c_1)\cup\delta(c_2)$, we obtain (ii).

Corollary. $\delta:\sum_p H^{p-1}(M, \mathbf{C}^*_{c(D)})\longrightarrow\sum_p H^p(M, \mathbf{Z})$ *is a ring homomorphism, where the products are* $*$*-product and cup-product. Especially,* $\sum_p H^{2p-1}(M, \mathbf{C}^*_{c(D)})$ *is a commutative ring.*

Note. We know $\delta:\sum_p H^{p-1}(M, \mathbf{C}^*_d)\cong\sum_p H^p(M, \mathbf{Z})$. In this case, we have $c_1 * c_2=\delta^{-1}(\delta(c_1)\cup\delta(c_2))$ by (18).

9. As in $n°7$, we fix a $c(D)$-class $GL(q, \mathbf{C})$-bundle ξ and its associate $F(q)$-bundle $M_F=\{M_F, F(q), M, \pi_F\}$. Then we have the following commutative diagram with exact lines.

$$\begin{array}{ccccccc}
H^{2p-1}(M_F, \mathbf{Z}) & \xrightarrow{\iota^*} & H^{2p-1}(M_F, \mathbf{C}_{c(\pi_F{}^*(D))}) & \xrightarrow{\exp^*} & H^{2p-1}(M_F, \mathbf{C}^*_{c(\pi_F{}^*(D))}) & \longrightarrow & \\
\uparrow{\scriptstyle \pi_F{}^*} & & \uparrow{\scriptstyle \pi_F{}^*} & & \uparrow{\scriptstyle \pi_F{}^*} & & \\
H^{2p-1}(M, \mathbf{Z}) & \xrightarrow{\iota^*} & H^{2p-1}(M, \mathbf{C}_{c(D)}) & \xrightarrow{\exp^*} & H^{2p-1}(M, \mathbf{C}^*_{c(D)}) & \longrightarrow &
\end{array}$$

$$\xrightarrow{\delta} H^{2p}(M_F, \mathbf{Z})$$

$$\uparrow{\scriptstyle \pi_F{}^*}$$

$$\xrightarrow{\delta} H^{2p}(M, \mathbf{Z}).$$

In this diagram, each $\pi_F{}^*$ is a monomorphism except $\pi_F{}^*: H^{2p-1}(M, \mathbf{C}^*_{c(D)})\longrightarrow H^{2p-1}(M_F, \mathbf{C}^*_{c(\pi_F{}^*(D))})$. Hence $\pi_F{}^*: H^{2p-1}(M, \mathbf{C}^*_{c(D)})\longrightarrow H^{2p-1}(M_F, \mathbf{C}^*_{c(\pi_F{}^*(D))})$ is also a monomorphism. On the other hand, if $c\in H^{2p-1}(M_F, \mathbf{C}^*_{c(\pi_F{}^*(D))})$ is in δ-kernel, set $c=\exp^*(b)$, $b\in H^{2p-1}(M_F, \mathbf{C}_{c(\pi_F{}^*(D))})$, $\int_F b$ is defined. Since $\int_F \iota^*(a)$, $a\in H^{2p-1}(M_F, \mathbf{Z})$, is in ι^*-image by the definition of $\int_F, \exp^*(\int_F b)\in H^{2p-1}(M, \mathbf{C}^*_{c(D)})$ is determined by c. Hence we may define $\int_F c$ by

$$\int_F c=\exp^*(\int_F b),\ c=\exp^*(b).$$

On M_F, $\pi_F{}^*(\xi)$ is an m-fold extension of $c(\pi_F{}^*(D))$-class $\mathbf{C}^*$-bundles $\eta_1,\dots,\eta_q$ as a $c(\pi_F{}^*(D))$-class bundle. Then, regard each η_i to be an element of $H^1(M_F, \mathbf{C}^*_{c(\pi_F{}^*(D))})$, we have

$$(19)' \qquad \pi_F{}^*(c^p(\xi))=\sum\delta(\eta_{i_1})\cup\dots\cup\delta(\eta_{i_p}),\ p\leq q.$$

Here $c^p(\xi)$ is the p-th integral Chern class of ξ and $\sum X_{i_1}\dots X_{i_p}$ is the p-th ele-

mentary symmetric function of indeterminants $X_1, \ldots, X_q$.

By lemma 16, $\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p} \in H^{2p-1}(M_F,\ \mathbf{C}^*{}_{c(\pi_F{}^*(D))})$ is defined and we have

$$\delta(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}) = \pi_F{}^*(c^p(\xi)). \tag{20}$$

Since $c^p\ (\xi)$ is in δ–image by proposition 2, there is an element $b^p \in H^{2p-1}(M,\ \mathbf{C}^*{}_{c(D)})$ such that

$$\delta(\pi_F{}^*(b^p)) = \delta(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}).$$

Hence $\int_F (\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}) - \pi_F{}^*(b^p)$ is defined. If $\delta(\pi_F{}^*(b')) = \delta(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p})$, we get

$$\begin{aligned} &\int_F \{(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}) - \pi_F{}^*(b)\} - \int_F \{(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}) - \pi_F{}^*(b')\} \\ &= \int_F \pi_F{}^*(b' - b) = b' - b. \end{aligned}$$

Because $\pi_F{}^*$ is a monomorphism. Hence $b^p + \int_F \{\Pi\, (\eta_{i_1}{}^* \ldots {}^*\eta_{i_p}) - \pi_F{}^*(b^p)\} \in H^{2p-1}(M,\ \mathbf{C}^*{}_{c(D)})$ does not depend on the choice of b^p.

Definition. *For a $c(D)$–class $GL(q,\ \mathbf{C})$–bundle ξ, we define $b^p(\xi) \in H^{2p-1}(M,\ \mathbf{C}^*{}_{c(D)})$ by*

$$b^p(\xi) = b^p + \int_F \{(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}) - \pi_F{}^*(b^p)\},\quad \delta(\pi_F{}^*(b^p)) = \delta(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}). \tag{21}$$

We also set $b(\xi) = \sum_{p \geq 1} b^p(\xi)$.

By the definition of $b^p(\xi)$ and (20), we obtain

Theorem 2. (i). *$b^p(\xi) = 0$ if $p > q$ and we have*

$$\delta(b^p(\xi)) = c^p(\xi),\ \textit{the p–th integral Chern class of } \xi. \tag{19}$$

(ii). *If $M_Y = \{M_Y,\ Y,\ M,\ \pi_Y\}$ is a $c(D)$–class bundle over M with the smooth fibre Y, and ξ is a $c(D)$–class $GL(q,\ \mathbf{C})$–bundle over M, then*

$$\pi_Y{}^*(b^p(\xi)) = b^p(\pi_Y{}^*(\xi)).$$

(iii). *If ξ is a $c(D)$–class extension of $c(D)$–class bundles η_1 and η_2, then*

$$1 + b(\xi) = (1 + b(\eta_1))_*(1 + b(\eta_2)).$$

(iv). *If $\xi = \delta(L)$, $b^p(\xi)$ is in $\exp^*$–image and if the monodromy group of $D \otimes 1_{\mathbf{C}^q} - L$ is contained in $GL(q_0,\ \mathbf{C})$, $q_0 < q$, then $b^p(\xi) = 0$, $p > q_0$.*

Note. In some cases, for example $D = d$ or $\bar{\partial}$, $\mathbf{C}^*{}_{c(D)}$ is also defined on M_F and $\pi_F{}^* : H^{2p-1}(M,\ \mathbf{C}^*{}_{c(D)}) \cong H^{2p-1}(M_F,\ \mathbf{C}^*{}_{c(D)})^W$, the invariant subgroup of $H^{2p-1}(M_F,\ \mathbf{C}^*{}_{c(D)})$ under the action of Weyl group. In these cases, we can pefine $b^p(\xi)$ by

$$b^p(\xi) = \pi_F{}^{*-1}(\Pi\, \eta_{i_1}{}^* \ldots {}^*\eta_{i_p}).$$

§4. Characteristic classes related to D-Fuchs type operators

10. We denote the tangent and cotangent bundles of M by $T=T(M)$ and $T^*=T^*(M)$. Their fibres at x are denoted by T_x and T^*_x. Set $T^{\mathbf{C}}=T\otimes\mathbf{C}$, *etc.*, the subspace of $T^{\mathbf{C}}{}_x$ spanned by $r_1(D(x))$ is denoted by $T^{\mathbf{C},D}{}_x$ and set $T^{\mathbf{C},D}=\cup_{x\in M}T^{\mathbf{C},D}{}_x$. For $T^{\mathbf{C},D}$, we assume there is an open covering $\{U\}$ of M such that on each U, there is a system of smooth vector fields $\{X^U{}_1,\dots,X^U{}_m\}$ as follows: (i). $\{X^U{}_1(x),\dots,X^U{}_m(x)\}$ spannes $T^{\mathbf{C},D}{}_x$ if $x\in U$. (ii). $\{X^U{}_1(x),\dots,X^U{}_m(x)\}$ *are linear independent if x is in some dense open subset of U.* Under these assumptions, there is a constant m such that dim $T^{\mathbf{C},D}{}_x\leqq m$ and $T^{\mathbf{C},D}$ is a vector bundle over some open dense subset M_0 of M. To fix an Hermitian structure of $T^{\mathbf{C}}$, we can determin the dual space $T^{*\mathbf{C},D}{}_x$ of $T^{\mathbf{C},D}{}_x$ as the subspace of $T^{*\mathbf{C}}{}_x$ for each $x\in M$. Set $T^{*\mathbf{C},D}=\cup_{x\in M}T^{*\mathbf{C},D}{}_x$, $T^{*\mathbf{C},D}|M_0$ is the dual bundle of $T^{\mathbf{C},D}|M_0$ and contained in $T^{*\mathbf{C}}|M_0$. In the rest, we assume $\{X^U{}_1,\dots,X^U{}_m\}$ to be an O. N. -basis of $T^{\mathbf{C},D}{}_x$ if $x\in M_0$, for the given Hermitian structure. Their dual basis are denoted by $\{X^{U*}{}_1,\dots,X^{U*}{}_m\}$.

Definition. *For a smooth function f on U, we set*

$$d^Df(x)=\sum_{i=1}^{m}(X^U{}_if)\ (x)X^{U*}{}_i(x),\quad x\in U.$$

By definition, d^D is defined on M and does not depend on the choice of $\{X^U{}_1,\dots,X^U{}_m\}$. Set $\Lambda^pT^{*\mathbf{C},D}=\cup_{x\in M}\Lambda^pT^{*\mathbf{C},D}{}_x$, d^D induces a differential operator $d^D:C^\infty(M,\ \Lambda^pT^{*\mathbf{C},D})\longrightarrow C^\infty(M,\ \Lambda^{p+1}T^{*\mathbf{C},D})$ for any p. Therefore, denote the sheaf of germs of smooth sections of $\Lambda^pT^{*\mathbf{C},D}$ by $\mathrm{C}^{p,D}{}_d$, we have the following exact sequence of sheaves

$$0\longrightarrow \mathbf{C}_{c(D)}\xrightarrow{i}\mathbf{C}_d\xrightarrow{d^D}\mathrm{C}^{1,D}{}_d\xrightarrow{d^D}\dots\xrightarrow{d^D}\mathrm{C}^{p,D}{}_d\xrightarrow{d^D}\dots \tag{22}$$
$$\xrightarrow{d^D}\mathrm{C}^{n,D}{}_d\longrightarrow 0.$$

By the definitions of d^D and $\mathrm{C}^{1,D}{}_d$, the sequence $0\longrightarrow\mathbf{C}_{c(D)}\xrightarrow{i}\mathbf{C}_d\xrightarrow{d^D}\mathrm{C}^{1,D}{}_d$ is exact if and only if (3) is hold for D. d^Dd^D is not equal to 0 unless the Lie algebra spanned by $\{X^U{}_1,\dots,X^U{}_m\}$ is abelian.

Note. If D is homogeneous, $r_1(D)$ is determined by $\sigma(r(D))$, the principal symbol of $r(D)$. Hence d^D is determined by $\sigma(r(D))$.

Assumption. *In this §, we assume that there is an Hermitian structure on $T^{\mathbf{C}}$ such that the sequence (22) is exact.*

Under this assumption, denote the kernel sheaf of d^D in $\mathrm{C}^{p,D}{}_d$ by $\mathrm{B}^{p,D}{}_d$, we have the isomorphism

$$H^p(M,\ \mathrm{C}_{c(D)})\cong H^0(M,\ \mathrm{B}^{p,D}{}_d)/d^DH^0(M,\mathrm{C}^{p-1,D}{}_d),\quad p\leqq 1. \tag{23}$$

Because the sheaves $\mathbf{C}_d$, $\mathrm{C}^{1,D}{}_d, \ldots$, are fine.

Example. If $r_1(D)$ is maximal, D satisfies the assumption and the sequence (22) is the de Rham complex. Similarly, if $r(D)=r_1(D)=\bar{\partial}$, D satisfies the assumption and the sequence (22) is the Dolbeauldt complex.

Lemma 17. *If D satisfies the assumption, $M_Y=\{M_Y, Y, M, \pi_Y\}$ is a $c(D)$-class bundle over M with the fibre Y, a smooth manifold, then $\pi_Y{}^*(D)$ also satisfies the assumption.*

Proof. By assumption, denote T_Y the fibre of the tangent bundle of Y, we have $T^{\mathrm{C},\ \pi Y^{*(D)}}=\pi_Y{}^*(T^{\mathrm{C},D}{}_{\pi_Y(x)})\otimes T_Y$. Hence $d^{\pi Y^{*(D)}}=\pi_Y{}^*(d^D)\otimes 1_Y$ at $\mathbf{C}_d$. There fore we have the lemma.

By the definition of d^D and the assumption on D, d^D has same formal properties as d. For example, d^D is linear, $d^Dd^D=0$ and

$$d^D(\varphi\wedge\psi)=d^D\varphi\wedge\psi+(-1)^p\varphi\wedge d^D\psi,\quad \varphi\in C^\infty(U,\ \Lambda^pT^{*\mathrm{C},D}).$$

11. In the sence of de Rham, the $(2p-1)$-dimensional generator ω^p of $H^*(GL\ (n,\ \mathbf{C}),\ \mathbf{C})=H^*(U(n),\ \mathbf{C})$ is given by

$$\omega^p(T)=\mathrm{tr}(dTT^{-1}\wedge\cdots\wedge dTT^{-1})$$

$$=\sum_{i_1,\ldots,i_{2p-1},j_1,\ldots,j_{2p-1}}\zeta^{j_1,i_2}\cdots\zeta^{j_{2p-2},i_{2p-1}}\zeta^{j_{2p-1},i_1}\cdot dz_{i_1,j_1}\wedge\cdots\wedge dz_{i_{2p-1},j_{2p-1}},$$

$$T=(z_{i,j}),\quad T^{-1}=(\zeta^{i,j}),$$

([5], [10]). Hence if $f: U\longrightarrow GL(n,\ \mathbf{C})$ is a smooth map, we have

$$f^*(\omega^p)=\mathrm{tr}(dff^{-1}\wedge\cdots\wedge dff^{-1}). \tag{24}$$

We also set

$$f^{*D}(\omega^p)=\mathrm{tr}(d^Dff^{-1}\wedge\cdots\wedge d^Dff^{-1}). \tag{24$'$}$$

Example. If $D=\bar{\partial}$, $f^{*D}(\omega^p)$ is the type $(0,\ 2p-1)$-part of $f^*(\omega^p)$.

Lemma 18. *If $\log f$ is defined, we have*

$$f^*(\omega^p)=\mathrm{tr}\ (d\log f\wedge\cdots\wedge d\log f), \tag{25}$$

$$f^{*D}(\omega^p)=\mathrm{tr}(d^D\log f\wedge\cdots\wedge d^D\log f). \tag{25$'$}$$

Proof. Since $dff^{-1}=(d(fC))\ (fC)^{-1}$ and $d^Dff^{-1}=(d^D(fC))\ (fC)^{-1}$ for any constant matrix C, we may assume $f-I$ is inversible and $\log f$ is given by the Taylor series $\sum_{m\geq 1}(-1)^{m-1}(1/m)\ (f-I)^m$ on U, an open set of M. Then, since $f-I$ is inversible by assumption, we get

$$\mathrm{tr}[(f-I)^{k_0}\ df(f-I)^{k_1}\wedge\cdots\wedge df(f-I)^{k_{2p-2}}\wedge df(f-I)^{k_{2p-1}-k_0}]$$

$$=\mathrm{tr}\ [df(f-I)^{k_1}\wedge\cdots\wedge df(f-I)^{k_{2p-1}}],$$

for any integers $k_0, k_1, \ldots, k_{2p-1}$. Therefore we obtain

$$\mathrm{tr}(d\log f\wedge\cdots\wedge d\log f)$$
$$=\sum_{k_1,\ldots,k_{2p-1}}(-1)^{k_1+\cdots+k_{2p-1}}\,\mathrm{tr}[df(f-I)^{k_1}\wedge\cdots\wedge df(f-I)^{k_{2p-1}}],$$

because tr is linear. Since $f^{-1}=\sum_{k\geqq 0}(-1)^k(f-I)^k$ under our assumption, this right hand side is equal to tr $(dff^{-1}\wedge\cdots\wedge dff^{-1})$. Therefore we obtain (25). (25)′ is obtained by the same way, because d^D has same formal properties as d.

Corollary. *$f^{*D}(\omega^p)$ is d^D-closed.*

Lemma 19. *Let $L=\{\rho_D(h_U)\}$ be an element of $H^0(M, \mathrm{L}_{G,D})$. Then to set*

$$L^*(\omega^p)\,|\,U=h_U{}^{*D}(\omega^p), \tag{26}$$

$L^*(\omega^p)$ is a d^D-closed $(2p-1)$-form on M and does not depend on the choice of $\{h_U\}$.

Proof. Since $\rho_D(h_U)=\rho_D(h_V)$ on $U\cap V$, we get $h_U{}^{*D}(\omega^p)=h_V{}^{*D}(\omega^p)$ on $U\cap V$. On the other hand, if $\rho_D(h_U)=\rho_D(h_U')$, h_U' is written as h_Uf_U, where f_U is a $c(D)$-class $GL(n, \mathbf{C})$-valued function. Hence $h_U{}^{*D}(\omega^p)$ is equal to $h_U'^{*D}(\omega^p)$. Therefore we have the lemma.

Lemma 20. *Set $\langle L^*(\omega^p)\rangle$ the cohomology class of $L^*(\omega^p)$ in $H^{2p-1}(M, \mathbf{C}_{c(D)})$, we have*

$$\langle L^*(\omega^p)\rangle$$
$$=\{(-1)^{p-1}\mathrm{tr}[\log g_{i_0,i_1}\ (\delta\log g)_{i_1,i_2,i_3}\cdots(\delta\log g)_{i_{2p-3},i_{2p-2},i_{2p-1}}]\},$$
$$g_{ij}=h_{Ui}{}^{-1}h_{Uj},\quad (\delta\log g)_{ijk}=\log g_{jk}-\log g_{ik}+\log g_{ij}.$$

Proof. Since we can take the open covering $\{U\}$ sufficiently fine, we may assume $\log h_U$ is defined for any $U\in\{U\}$. Then, by lemma 18, to set

$$L^*(\Omega^q)=\overbrace{d^D\ \log\ h_U\wedge\cdots\wedge d^D\ \log\ h_U}^{q},$$

we have

$$\mathrm{tr}\ L^*(\Omega^{2p-1})=\mathrm{tr}\ L^*(\omega^p),\quad L^*(\Omega^q)=(-1)^{q-1}d^D[L^*(\Omega^{q-1})\ \log h_U].$$

Moreover, by the same calculation as in the proof of lemma 18, we get

$$\mathrm{tr}[L^*(\Omega^q)\wedge d^D(\log h_U{}^{-1}h_V)]$$
$$=\mathrm{tr}[L^*(\Omega^q)\wedge d^D\ \log\ h_V-L^*(\Omega^q)\wedge d^D\ \log\ h_U].$$

Hence the Čech cocycle represents the class of $L^*(\Omega^p)$ in $H^1(M, \mathrm{B}^{2p-2,D}{}_d)$ is $\{\mathrm{tr}\ [L^*(\Omega^{2p-2})\log g_{ij}]\}$. Then, since $\delta\{(\delta\log\ g\}_{i_0,i_1,i_2,i_3}=0$, we get

$$\log h_{i_1}\ (\delta\log\ g)_{i_1,i_2,i_3}-\log h_{i_0}\ (\delta\log g)_{i_0,i_2,i_3}+$$
$$+\log h_{i_0}(\delta\log g)_{i_0,i_1,i_3}-\log h_{i_0}(\delta\ \log\ g)_{i_0,i_1,i_2}$$

$$=(\log h_{i_1}-\log h_{i_0})\ \delta\log g)_{i_1,\,i_2,\,i_3}.$$

Hence in $H^2(M,\ \mathrm{B}^{2p-3,D}{}_d)$, $L^*(\omega^p)$ is represented by $\{-\mathrm{tr}[L^*(\Omega^{2p-3})\log g_{i_0,\,i_1}(\delta\log g)_{i_1,\,i_2,\,i_3}]\}$. Since $(\delta\log g)_{i_1,\,i_2,\,i_3}$ is a constant matrix, we can repeat this process. Therefore we have the lemma because $(-1)^{(p-1)(2p-1)}=(-1)^{p-1}$.

Corollary. *Denote c^p the $(2p-1)$-dimensional generatorof $H^*(GL(n,\ \mathbf{C}),\ \mathbf{Z})=H^*(U(n),\ \mathbf{Z})$, we have*

$$\iota^*(c^p)=\frac{(-1)^{p-1}}{(2\pi\sqrt{-1}\,)^p}\langle\omega^p\rangle.$$

Proof. Since $(\delta g)_{ijk}=I$, the identity matrix, $(\delta\log g)_{ijk}=2\pi\sqrt{-1}\,N_{ijk}$, where N_{ijk} is a matrix with integral proper values, for any i, j, k. On the other hand, $\log g_{ij}=2\pi\sqrt{-1}\,N_{ij}$ if $h_{U_i}=h_{U_j}$ on $U_i\cap U_j$. Hence $f^*(\omega^p)$ is represented by a cocycle of the form $\{(-1)^{p-1}(2\pi\sqrt{-1})^p\ n_{i_0,\ldots,i_{2p-1}}\}$ in $H^{2p-1}(M,\ \mathbf{C})$, where $n_{i_0,\ldots,i_{2p-1}}$ is an integer for any $(i_0,\ i_1,\ldots,i_{2p-1})$ and $f\colon M\longrightarrow GL(n,\ \mathbf{C})$ is a smooth map. On the other hand $\iota^*(c^p)$ is represented by $a_p\omega^p$ where a_p is a constant, $\iota^*(f^*(c^p))$ is represented by $\{(-1)^{p-1}a_p(2\pi\sqrt{-1})^p n_{i_0,\ldots,i_{2p-1}}\}$ and it is an integral class. Since we can take f and M arbitrally, $(-1)^{p-1}a_p(2\pi\sqrt{-1})^p$ should be equal to 1. Therefore we obtain the corollary.

12. Definition. *We define $\beta^p(L)\in H^{2p-1}(M,\ \mathbf{C}_{c(D)})$ by*

$$\beta^p(L)=\frac{(-1)^{p-1}}{(2\pi\sqrt{-1})^p}\langle L^*(\omega^p)\rangle.$$

Theorem 3. (i). *If $L\in H^0(M,\ \mathrm{L}_{C^*,\,D})$, then*

(27) $\beta^1(L)=\delta k^{*-1}(L)$.

(ii). *Let $F_{q,\,pq}\ (Y_1,\ldots,Y_p)=\sum a_{i_1,\ldots,i_p}\ Y_1{}^{i_1}\ldots Y_p{}^{i_p}$ be the polynomial $F_{q,\,p}(s_1,\ldots,s_p)=\sum_{i=1}X_i{}^p$, where s_r is the r-th elementary symmetric function of indeterminants $X_1,\ldots,X_q$, and set*

$$F_{q,\,p}(b_1,\ldots,b_p)=\Pi\ [(\overbrace{b_1*\ldots*b_1}^{i_1})*\ldots*(\overbrace{b_p*\ldots*b_p}^{i_p})]^{a_{i_1,\ldots,i_p}},$$

$$b_r\in H^{2r-1}(M,\ \mathrm{C}^*{}_{c(D)}).$$

Then we have

(28) $$\exp^*(\beta^p(L))=(-1)^{p-1}F_{q,\,p}(b^1(\delta(L)),\ldots,b^p(\delta(L))).$$

(iii). *If $L=\rho_D(f)$, f is a smooth $GL(n,\ \mathbf{C})$-valued function on M, then*

(29) $$\beta^p(L)=\iota^*(f^*(c^p)).$$

(iv). *If* $L|U=\rho_D(h_U)$, h_U *is a smooth* $\Delta(q, \mathbf{C})$*-valued function on* U, *for each* $U\in\{U\}$, *then*

$$\beta^p(L)=0,\ p\geqq 2. \tag{30}$$

(v). *If* $M_Y=\{M_Y, Y, M, \pi_Y\}$ *is a* $c(D)$*-class bundle over* M *with the smooth fibre* Y, *set* $\pi_Y{}^*(L)=\{\rho_{\pi_Y{}^*(D)}(\pi_Y{}^*(h_U))\}$, *we have*

$$\beta^p(\pi_Y{}^*(L))=\pi_Y{}^*(\beta^p(L)).$$

(vi). *If* D *is homogeneous and satisfies the assumption in* $n°10$, $\beta^p(L)$ *is determined by* $\sigma(L)$, *the principal symbol of* L.

Proof. If $L=\{\rho_D(h_U)\}\in H_0(M, \mathrm{L}_{\mathbf{C}^*, D})$, $\delta k^{*-1}(L)$ is given by $(1/2\pi\sqrt{-1})(\log h_U-\log h_V)$. Hence we have (i) by lemma 20. (iii) also follows from lemma 20 and (v) follows from the definitions of $\beta^p(L)$, $\pi_Y{}^*(L)$ and lemma 17.

To show (ii), first we assume $\delta(L)=\{g_{ij}\}$ is a $\Delta(q, \mathbf{C})$-bundle. Then $\delta(L)$ is a q-fold extension of $c(D)$-class $\mathbf{C}^*$-bundles $\eta_1,\dots,\eta_q$ and the transition function of each η_m is given by the m-th diagonal element $\{g_{ij,m}\}$ of $\{g_{ij}\}$. Since g_{ij} is a $\Delta(q, \mathbf{C})$-valued function, $\log g_{ij}$ is a $\Delta(q, \mathbf{C})$-valued function whose m-th diagonal element is $\log g_{ij,m}$. Hence we have

$$\begin{aligned}&\mathrm{tr}[\log g_{i_0, i_1}(\delta\log g)_{i_1, i_2, i_3}\cdots(\delta\log g)_{i_{2p-3}, i_{2p-2}, i_{2p-1}}]\\ &=\sum_{m=1}^{q}\log g_{i_0, i_1, m}(\delta\log g)_{i_1, i_2, i_3, m}\cdots(\delta\log g)_{i_{2p-3}, i_{2p-2}, i_{2p-1}, m}.\end{aligned}$$

Therefore we obtain

$$\exp^*(\beta^p(L))=\left\{\sum_{m=1}^{q}\overbrace{\eta_m{}^*\cdots{}^*\eta_m}^{p}\right\}^{(-1)^{p-1}}.$$

Hence by the definitions of $b^p(\xi)$ and $F_{q,p}$, we have (28) by lemma 16.

To show (ii) in general, we use the commutative diagram

$$\begin{array}{ccc} H^{2p-1}(M_F, \mathbf{C}_{c(\pi_F{}^*(D))}) & \xrightarrow{\exp^*} & H^{2p-1}(M_F, \mathbf{C}^*_{c(\pi_F{}^*(D))}) \\ \pi_F{}^*\uparrow & & \pi_F{}^*\uparrow \\ H^{2p-1}(M, \mathbf{C}_{c(D)}) & \xrightarrow{\exp^*} & H^{2p-1}(M, \mathbf{C}^*_{c(D)}), \end{array}$$

where M_F is the associate $F(q)$-bundle of $\delta(L)$. Since $\pi_F{}^*(D)$ satisfies the assumption of n°10 by lemma 17, $\beta^p(\pi_F{}^*(L))$ is defined and since $\pi_F{}^*(\delta(L))$ is a $c(D)$-class $\Delta(q, \mathbf{C})$-bundle, we have

$$\exp^*(\beta^p(\pi_F{}^*(L)))=(-1)^{p-1}F_{q,p}[b^1(\delta(\pi_F{}^*(L))),\dots,b^p(\delta(\pi_F{}^*(L)))].$$

But since $\delta(\pi_F{}^*(L))=\pi_F{}^*(\delta(L))$ by the definition of $\pi_F{}^*(L)$, we have by (v) and theorem 2, (ii)

$$\pi_F{}^*(\exp^*(\beta^p(L)))=\pi_F{}^*[(-1)^{p-1}F_{q,p}(b^1(\delta(L)),\dots,b^p(\delta(L)))],$$

because by the definition of $_*$-product, we get $\pi_F{}^*(a_*b)=\pi_F{}^*(a)_*\pi_F{}^*(b)$. Then, since each $\pi_F{}^*$ is a monomorphism, we obtain (ii).

If h_U is a $\Delta(q,\ \mathbf{C})$-valued function, $d^D h_U h_U{}^{-1}$ is a $\Delta(q,\ \mathbf{C})$-valued 1-form. Hence to set $d^D h_U h_U{}^{-1}=(\varphi_{ij})$, we get

$$\mathrm{tr}(L^*(\Omega^r))=\sum_{i=1}^{q}\overbrace{\varphi_{i,i}\wedge\cdots\wedge\varphi_{i,i}}^{r}=0,\ r\geqq 2.$$

This shows (iv).

If D is homogeneous, $\sigma(L)$ is determined by $r_1(D)$. Hence we have (vi).

Corollary. *If $\delta(L)=\delta(L')$, $\beta^p(L)-\beta^p(L')$ is in ι^*-image for all p.*

Note 1. By (ii), we have

$$b^1(\delta(L))=\exp^*(\beta^1(L)). \tag{28$'$}$$

On the other hand, since the diagram

$$\begin{array}{ccccccc}
H^0(M,\ \mathbf{C}_d) & \longrightarrow & H^0(M,\ \mathscr{L}_{\mathbf{C},\,D}) & \xrightarrow{\delta_2} & H^1(M,\ \mathbf{C}_{c(D)}) & \longrightarrow & 0 \\
 & & k^{*-1}\uparrow= & & \exp^*\uparrow & & \\
 & & H^0(M,\ \mathrm{L}\mathbf{C}^*,\,D) & \xrightarrow{\delta_1} & H^1(M,\ \mathbf{C}^*{}_{c(D)}) & & \\
 & & & & \delta\downarrow & & \\
 & & & & H^2(M,\ \mathbf{Z}), & &
\end{array}$$

is commutative, we can define, $\beta^1(L)$ by (i) without any assumption about D and it satisfies (28)$'$.

Note 2. *If $d^D=d$* or $\bar{\partial}$, we can define $\pi_Y{}^*(\beta^p(L))$ and $\beta^p(\pi_Y{}^*(L))$ (resp. $\pi_Y{}^*(b^p(\xi))$ and $b^p(\pi_Y{}^*(\xi))$) as the elements of $H^{2p-1}(M_Y,\ \mathbf{C})$ or $\mathrm{H}^{2p-1}(M_Y,\ \mathbf{C}\omega)$ (resp. $H^{2p-1}(M_Y,\ \mathbf{C}^*)$ or $H^{2p-1}(M_Y,\ \mathbf{C}^*\omega)$) and for these elements, theorem 3, (v) (resp. theorem 2, (ii)) hold.

Appendix. Curvature operators of connections of diferential operators

In this appendix, we assume $E_1=E_2=E$, that is D is defined on $C^\infty(M,\ E)$ and maps into itself. For a differential operator $L: C^\infty(U,\ E\otimes\mathrm{H})\longrightarrow C^\infty(U,\ E\otimes\mathrm{H})$ with order at most $k-1$, $k=\mathrm{ord}\,D$, we set

$$\Theta_D(L)=(D\otimes 1_{\mathrm{H}})L+L(D\otimes 1_{\mathrm{H}})-L^2,$$

and call the curvature operator of L with respect to D. By definition, if $-L=\{-L_U\}$ is a connection of D with respect to ξ, a G-bundle with the fibre H ([3]), set $D_L=\{D\otimes 1_{\mathrm{H}}-L_U\}: C^\infty(M,\ E\otimes\xi)\longrightarrow C^\infty(M,\ E\otimes\xi)$, we have

$$D_L{}^2|U=D_U{}^2\otimes 1_{\mathrm{H}}-\Theta_D(L_U).$$

Hence if L is flat, that is $L=\rho_D(h)$, we obtain

$$\Theta_D(L)=\rho_{D^2}(h).$$

Example 1. Let $C^\infty(M, E_1)\xrightarrow{D_1}C^\infty(M, E_2)\xrightarrow{D_2}\cdots\xrightarrow{D_m}C^\infty(M, E_{m+1})$ be a differential complex, ξ a G-bundle with the fibre H, $-\theta_i$ is a connection of D_i with respect to ξ, $1\leqq i\leqq m$. Then, to set $E=E_1\oplus\cdots\oplus E_{m+1}$, $D(f_1\oplus\cdots\oplus f_{m+1})=0\oplus D_1f_1\oplus\cdots\oplus D_mf_m$ and $\theta(f_1\oplus\cdots\oplus f_{m+1})=0\oplus\theta_1f_1\oplus\cdots\oplus\theta_mf_m$, θ is a connection of D with respect to ξ and $\Theta_D(\theta)=-(D_\theta)^2$. Therefore the series $C^\infty(M, E_1\otimes\xi)\xrightarrow{D_1,\theta_1}C^\infty(M, E_2\otimes\xi)\xrightarrow{D_2,\theta_2}\cdots\xrightarrow{D_m,\theta_m}C^\infty(M, E_{m+1}\otimes\xi)$ is a differential complex if and only if the curvature operator of θ with respect to D vanishes. To vanish the curvature operator of θ, it is sufficient there exist $h_U\in C^\infty(U, G_d)$ such that $\theta_{i,U}=\rho_{Di}(h_U)$, $1\leqq i\leqq m$, for all U.

Example 2. In the above example, if $D_i=d$ or $\bar\partial$ for each i, $\Theta_D(\theta)$ is equal to $d\theta-\theta\wedge\theta$ or $\bar\partial\theta-\theta\wedge\theta$.

Lemma 1. *We have*

(1)i $\quad\Theta_D(cL)=c\Theta_D(L)+(c-c^2)L^2$, *$c$ is a constant G-valued function,*

(1)ii $\quad\Theta_D(L_1+L_2)=\Theta_D(L_1)+\Theta_D(L_2)-(L_1L_2+L_2L_1)$,

(1)iii $\quad\Theta_D(L^g)=[\Theta_D(L_1)]^g+[\Theta_D(g)L^g+L^g\Theta_D(g)]$.

Corollary 1. *If $\Theta_D(L_1)=\Theta_D(L)^g+\rho_{D^2}(g)$, then there exists a differential operator P such that $L^g+\rho_D(g)=L_1+P$, $\Theta_D(P)=L_1P+PL_1$.*

Corollary 2. (i). *If $L=\{L_U\}$ is a connection of D with respect to $\xi=\{g_{UV}\}$, then*

(2) $$\Theta_D(L_U)=\Theta_D(L_V)^{g_{UV}}+\rho_{D^2}(g_{UV}),\ \text{on } U\cap V.$$

(iii). *If* (2) *is hold for $L=\{L_U\}$*, then

(3) $$\Theta_D(L_U+P_{UV})=\Theta_D(L_U)\ \text{on } U\cap V, P_{UV}=(D_U-L_U)-(D_V-L_V)^{g_{UV}}.$$

Proof. If $L=\{L_U\}$ is a connection of D with respect to ξ, we have $(D_U-L_U)^2=(D_V{}^{g_{UV}}-L_V{}^{g_{UV}})^2$ on $U\cap V$. Since $(D_U-L_U)^2=D_U{}^2-\Theta_D(L_U)$ and $(D_V{}^{g_{UV}}-L_V{}^{g_{UV}})^2=D_U{}^2-\rho_{D^2}(g_{UV})-[\Theta_D(L_V)]^{g_{UV}}$, we get (*i*). Since $(D_U-L_U)^2-(D_V{}^{g_{UV}}-L_V{}^{g_{UV}})^2=0$ if (2) is hold, set $P_{UV}=(D_U-L_U)-(D_V-L_V)^{g_{UV}}$, we get (3) by (1)ii.

Corollary 3. *If $\Theta_D(L)=\rho_{D^2}(h)$, L is equal to $\rho_D(h)+P$, where $\Theta_{D^h}(P)=0$.*

Definition. *Let $L, L':C^\infty(U, E\otimes \mathrm{H})\longrightarrow C^\infty(U, E\otimes \mathrm{H})$ be differential operators of order at most $k-1$, we call $L\sim L'$ mod. Θ_D if there exists a smooth G-valued function g on U such that $\Theta_D(L)=\Theta_D(L')^g+\rho_{D^2}(g)$.*

By lemma 1, $L\sim L'$ is an equivalence relation and it induces an equivalence relation on $\mathscr{D}^{k-1}_{E\otimes \mathrm{H}}$, the sheaf of germs of differential operators $L:C^\infty(U, E\otimes \mathrm{H})\longrightarrow C^\infty(U, E\otimes \mathrm{H})$ of order at most $k-1$. The quotient sheaf of $\mathscr{D}^{k-1}_{E\otimes \mathrm{H}}$ by this

relation is denoted by $\widetilde{\Theta}_D \mathscr{D}^{k-1}_{E\otimes \mathrm{H}}$. The map from $\mathscr{D}^{k-1}_{E\otimes \mathrm{H}}$ onto $\widetilde{\Theta}_D \mathscr{D}^{k-1}_{E\otimes \mathrm{H}}$ induced by the relation $L\sim L'$ is denoted by $\widetilde{\Theta}_D$. The kernel sheaf of $\widetilde{\Theta}_D$ is denoted by $\widetilde{\mathrm{L}}_{G,\,D}$. $\widetilde{\mathrm{L}}_G, D$ containes $\mathrm{L}_{G,\,D}$.

Definition *For* $\xi=\{g_{UV}\}\in H^1(M,\ \mathrm{G}_d)$, $\{L_U\}\in C^\infty(U,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ *and* $\{L_{UV}\}\in C^1(U,\ \mathscr{D}^{k-1}_{E\otimes H},)$ *we set*

$$\delta_\xi\{L\}_{UV}=\mathrm{L}_U-\mathrm{L}_V{}^{g_{UV}},\quad \delta_\xi\{L\}_{UVW}=L_{UV}+L_{VW}{}^{g_{UV}}+L_{WU}{}^{g_{UW}}.$$

Lemma 2. $\delta_\xi(\delta_\xi\{L\})_{UVW}=0$ *and if* $\{(\delta_\xi\ L)_{UVW}\}=0$ *and there is a partition of unity by smooth functions subordinate to* $\{\mathfrak{U}\}$, $\{L_{UV}\}=\{\delta_\xi(R)_{UV}\}$ *for some* $\{R_U\}\in C^\infty(\mathfrak{U},\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ (cf. [3]).

Proof. $\delta_\xi(\delta_\xi\{L\}_{UVW}=0$ follows from the definitions. If $(\delta_\xi\ L)_{UVW}=0$, we have $L_{UU}=0$ and $L_{UV}=-L_{VU}{}^{g_{UV}}$. Hence set $R_U=\sum_{W\cap U\neq\phi} e_W L_{UW}$, $\{e_W\}$ is the Partition of unity subordinate to $\mathfrak{U}$, we have $\delta_\xi(R)_{UV}=L_{UV}$.

Denote L_U the section of $\mathscr{D}^{k-1}_{E\otimes \mathrm{H}}$ on U and set $\mathfrak{U}=\{U\}$, an open covering of M, we set

$$H_D{}^0(\mathfrak{U},\ \widetilde{\mathrm{L}}_{G,\,D})=\{\{L_U\}\mid(\delta_\xi L)_{UV}=\rho_D(g_{UV}),\ \textit{for some}\ \xi=\{g_{UV}\}\in H^1(M,\ \mathrm{G}_d),\ L_U\ \textit{is a section of}\ \widetilde{\mathrm{L}}_{G,D}\ \textit{on}\ U\}.$$

$$H_0{}^D(\mathfrak{U},\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})=\{\{L_U\}\mid(\delta_\xi L)_{UV}=\rho_D(g_{UV})\ \textit{for some}\ \xi=\{g_{UV}\}\in H^1(M,\ \mathrm{G}_d)\}.$$

$$H^0(\mathfrak{U},\ \widetilde{\Theta}_D\mathscr{D}^{k-1}_{E\otimes \mathrm{H}})=\{\{\Theta_D L_U\}\mid\delta_\xi(\Theta_D L)_{UV}=\rho_{D^2}(g_{UV})\ \textit{for some}\ \xi=\{g_{UV}\}\in H^1(M,\ \mathrm{G}_d)\}.$$

We define $H_0{}^D(M,\ \widetilde{\mathrm{L}}_{G,\,D})$, $H_0{}^D(M,\mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ and $H^0(M,\ \widetilde{\Theta}_D\mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ as the limits of these sets. We also set

$$B^1{}_{\Theta D}(\mathfrak{U},\mathscr{D}^{k-1}_{E\otimes \mathrm{H}})=\{\{R_{UV}\}\mid R_{UV}=(\delta_\xi L)_{UV}\ \textit{for some}\ \xi\}=\{g_{UV}\}\in H^1(M,\ \mathrm{G}_d)$$

$$\textit{and}\ \Theta_D(R_{UV})=\rho_{D^2}(g_{UV})-[\{\rho_D(g_{UV})-R_{UV}\}\,L_V{}^{g_{UV}}+L_V{}^{g_{UV}}\{\rho_D(g_{UV})-R_{UV}\}].$$

We call $\{R_{UV}\}$ and $\{R_{UV'}\}\in B^1{}_{\Theta D}(\mathfrak{U},\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ to be equivalent if

$$R_{UV}=(\delta_\xi L)_{UV},\quad R_{UV'}=(\delta_\xi(L+Q))_{UV},\quad \Theta_D(Q_U)=L_UQ_U+Q_UL_U.$$

The quotient set of $B^1{}_{\Theta D}$ $(\mathfrak{U},\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ by this relation is denoted by $H^1{}_{\Theta D}$ $(\mathfrak{U},\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$. Its limit set is denoted by $H^1{}_{\Theta D}(M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$. Then by lemma 1 and lemma 2, we have the following exact sequence of cohomology sets

$$0\longrightarrow H_D{}^0\ (M,\ \widetilde{L}_{G,\,D})\xrightarrow{\ i\ }H_D{}^0(M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})\xrightarrow{\widetilde{\Theta}_D}H^0(M,\ \widetilde{\Theta}_D\mathscr{D}^{k-1}_{E\otimes \mathrm{H}})\xrightarrow{\ \delta\ } \tag{4}$$

$$\longrightarrow H^1{}_{\theta D}(M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}}) \xrightarrow{i} H^1(M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}}) = \{0\}.$$

Note. By the definition of δ^L (n°4), there is an inclusion map $\iota : H^1(M,\ \mathrm{L}_{G,\,D}) \longrightarrow H^1{}_{\theta D}\ (M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ and we have the commutative diagram

$$\begin{tikzcd}
 & H^1{}_{\theta D}\ (M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}}) \arrow[r, "i_1"] & H^1\ (M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}}) = \{0\}. \\
H^1(M,\ \mathrm{G}_d) \arrow[r, "\rho_D{}^*"] & H^1(M,\ \mathrm{L}_{G,\,D}) \arrow[u, "\iota"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

In this diagram, the explicit trivialization of $i_2\rho_D{}^*(\xi)$ is the connection of D with respect to ξ. If the category is not smooth (for example, holomorphic category or topological category), $H^1(M,\ \mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ may not be equal to $\{0\}$ and $i_2\rho_D{}^*(\xi)$ gives the obstruction class to have a connection in this category (cf. [2], [4]).

Definition. Regard $\{L_U\} \in H_0{}^D(\mathscr{D}^{k-1}_{E\otimes \mathrm{H}})$ *to be a connection of D with respect to ξ, we call $\widetilde{\Theta}_D(\{L_U\})$ to be the curvature operator of $\{L_U\}$.*

Theorem. *A $c(D)$-class G-bundle ξ has a connection of D with respect to ξ with the curvature operator equal to* 0. *Conversly, if $\widetilde{L}_{G,\,D} = \mathrm{L}_{G,\,D}$, a G-bundle ξ is of $c(D)$ -class if D has a connection with respect to ξ with a curvature operator equal to* 0.

Proof. Since a $c(D)$-class G-bundle ξ allows $\{0\}$ as a connection of D with respect to ξ, we have the first assertion. If $\widetilde{L}_{G,\,D} = L_{G,\,D}$, we have $\rho_D(g_{UV}) = \rho_D(h_U) - \rho_D(h_V)^{g_{UV}}$ if $\xi = \{g_{UV}\}$ has a connection of D with respect to ξ with the curvature operator is equal to 0. Hence $\{g_{UV}\}$ is in δ-image in the sequence (6) of n°4. Therefore we obtain the theorem.

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