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On bounded symmetric domains of exceptional type

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It is known that there exist two bounded symmetric domains of exceptional type up to holomorphic diffeomorphism. One of them is of 16 dimension (called of type E_6) and the other is of 27 dimension (called of type E_7). M. Ise [7] and M. Koecher [8] gave a realization of type E_6 (resp. type E_7) as a bounded domain of $\mathfrak{C}^C \times \mathfrak{C}^C$ (resp. $\mathfrak{J}^C$), using eigenvalues of Hermitian mappings.

In this paper we give these another realizations. For this purpose, first we find a realization D of the non-compact Hermitian symmetric space $E_{6,\sigma}/U(1)\,Spin(10)$ (resp. $E_{7,\iota}/U(1)E_6$) and then give the Harish-chandra imbedding $\Psi : D \rightarrow \mathfrak{C}^C \times \mathfrak{C}^C$ (resp. $\mathfrak{J}^C$). By the images of these imbeddings Ψ we can realize the symmetric space $E_{6,\sigma}/U(1)Spin(10)$ (resp. $E_{7,\iota}/U(1)E_6$) as a bounded domain in the vector space $\mathfrak{C}^C \times \mathfrak{C}^C$ (resp. $\mathfrak{J}^C$). As consequence of these results, we have our main Theorems 17 and 28.

I. Preliminaries.

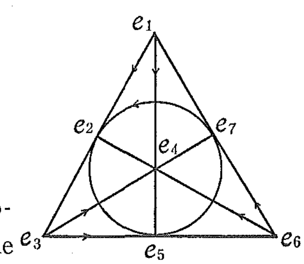
§1. Cayley algebra $\mathfrak{C}$, Jordan algebra $\mathfrak{J}$ and Freudenthal's manifold $\mathfrak{M}^C$.

Let $\mathfrak{C}$ denote the Cayley division algebra over the field of real numbers $\boldsymbol{R}$. This algebra $\mathfrak{C}$ has a basis $\{e_0, e_1, e_2, \dots, e_7\}$ with the following multiplication relations :

$e_0 = 1,\ e_i^2 = -1,\ i = 1, 2, \dots, 7,$

$e_i e_j = -e_j e_i,\ i \neq j,\ i, j = 1, 2, \dots, 7,$

$e_1 e_2 = e_3,\ e_2 e_5 = e_7,\ e_4 e_2 = e_6, \dots .$



Let $\mathfrak{C}^C$ be the complexification of $\mathfrak{C}$ over the field of complex numbers $\boldsymbol{C}$. In $\mathfrak{C}^C$, the inner product (x, y) and the positive definite Hermitian inner product $\langle x, y\rangle$ are defined respectively by

$$(x, y) = \frac{1}{2}(\bar{x}y + \bar{y}x) \quad (\bar{x} \text{ is the conjugate of } x \text{ with respect to } \mathfrak{C}),$$

$$\langle x, y\rangle = (\tilde{x}, y) \quad (\tilde{x} \text{ is the conjugate of } x \text{ with respect to } \boldsymbol{C}),$$

and we denote (x, x) by $|x|^2$ briefly.

Let $\mathfrak{J}=\mathfrak{J}(3, \mathfrak{C})$ denote the exceptional Jordan algebra of all 3×3 Hermitian matrices X with entries in $\mathfrak{C}$:

$$X=X(\xi, x)=\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i\in \boldsymbol{R}, \quad x_i \in \mathfrak{C}$$

with respect to the multiplication $X\circ Y=\frac{1}{2}(XY+YX)$ and $\mathfrak{J}^C$ the complexification of $\mathfrak{J}$ over $\boldsymbol{C}$. In $\mathfrak{J}^C$, the inner product (X, Y), the positive definite Hermitian inner product $\langle X, Y\rangle$, the crossed product $X\times Y$, the cubic form (X, Y, Z) and the determinant $\det X$ are defined respectively by

$$(X, Y)=\mathrm{tr}(X\circ Y)=\sum_{i=1}^{3}(\xi_i\eta_i+2(x_i, y_i)),$$

$$\langle X, Y\rangle=(\tau X, Y)=(\overline{X}, Y),$$

$$X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X, Y))E),$$

$$(X, Y, Z)=(X\times Y, Z)=(X, Y\times Z),$$

$$\det X=\frac{1}{3}(X, X, X)$$

where $X=X(\xi, x)$, $Y=Y(\eta, y)$, $\tau:\mathfrak{J}^C\to\mathfrak{J}^C$ is the complex conjugation (τX is often denoted by $\overline{X}$) and E the 3×3 unit matrix.

Let $\mathfrak{J}_-$ be the totality of 3×3 skew-Hermitian matrices A with entries in $\mathfrak{C}$:

$$A=\begin{pmatrix} z_1 & a_3 & -\bar{a}_2 \\ -\bar{a}_3 & z_2 & a_1 \\ a_2 & -\bar{a}_1 & z_3 \end{pmatrix}, \quad z_i,\ a_i\in\mathfrak{C}, \quad z_i=-\bar{z}_i$$

and $\mathfrak{J}_-{}^C$ the complexification of $\mathfrak{J}_-$.

For $X\in\mathfrak{J}^C$ and $A\in\mathfrak{J}_-{}^C$, we define mappings $\widetilde{X}$, $\widetilde{A}:\mathfrak{J}^C\to\mathfrak{J}^C$ respectively by

$$\widetilde{X}(Y)=X\circ Y, \qquad Y\in\mathfrak{J}^C,$$

$$\widetilde{A}(Y)=[A, Y]=AY-YA, \qquad Y\in\mathfrak{J}^C$$

In $\mathfrak{J}^C$ and $\mathfrak{J}_-{}^C$ we adopt the following notations :

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\ E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\ E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$F_1(x)=\begin{pmatrix}0&0&0\\0&0&x\\0&\bar{x}&0\end{pmatrix},\ F_2(x)=\begin{pmatrix}0&0&\bar{x}\\0&0&0\\x&0&0\end{pmatrix},\ F_3(x)=\begin{pmatrix}0&x&0\\\bar{x}&0&0\\0&0&0\end{pmatrix},$$

$$A_1(y)=\begin{pmatrix}0&0&0\\0&0&y\\0&-\bar{y}&0\end{pmatrix},\ A_2(y)=\begin{pmatrix}0&0&-\bar{y}\\0&0&0\\y&0&0\end{pmatrix},\ A_3(y)=\begin{pmatrix}0&y&0\\-\bar{y}&0&0\\0&0&0\end{pmatrix}.$$

We define a mapping $\sigma:\mathfrak{J}^C\to\mathfrak{J}^C$ by

$$\sigma\begin{pmatrix}\xi_1&x_3&\bar{x}_2\\\bar{x}_3&\xi_2&x_1\\x_2&\bar{x}_1&\xi_3\end{pmatrix}=\begin{pmatrix}\xi_1&-x_3&-\bar{x}_2\\-\bar{x}_3&\xi_2&x_1\\-x_2&\bar{x}_1&\xi_3\end{pmatrix}$$

and an inner product $\langle X,\ Y\rangle_\sigma$ on $\mathfrak{J}^C$ by

$$\langle X,\ Y\rangle_\sigma=\langle\sigma X,\ Y\rangle.$$

Now, we define subspaces $\mathfrak{J}_\times$, $\mathfrak{J}_1$ and $\mathfrak{J}_\sigma$ of $\mathfrak{J}^C$ respectively by

$$\mathfrak{J}_\times=\{X\in\mathfrak{J}^C\,|\,X\times X=0\},$$

$$\mathfrak{J}_1=\{X\in\mathfrak{J}^C\,|\,X\times X=0,\ \langle X,\ X\rangle=1\},$$

$$\mathfrak{J}_\sigma=\{X\in\mathfrak{J}^C\,|\,X\times X=0,\ \langle X,\ X\rangle_\sigma=1\}.$$

And we define equivalence relations $\sim$ in $\mathfrak{J}_\times$, $\mathfrak{J}_1$ and $\mathfrak{J}_\sigma$ as follows.
For $X,\ Y\in\mathfrak{J}_\times$,

$$X\sim Y\iff\zeta X=Y\qquad\text{for some }\zeta\in\boldsymbol{C}^*=\{\zeta\in\boldsymbol{C}\,|\,\zeta\neq0\},$$

and for $X,\ Y\in\mathfrak{J}_1$ (similarly for $\mathfrak{J}_\sigma$),

$$X\sim Y\iff\theta X=Y\qquad\text{for some }\theta\in\mathrm{U}(1)=\{\theta\in\boldsymbol{C}\,|\,|\theta|=1\}.$$

We denote the totality of equivalence classes of these spaces by $[\mathfrak{J}_\times]$, $[\mathfrak{J}_1]$ and $[\mathfrak{J}_\sigma]$, respectively. For $X\in\mathfrak{J}_\times$, we denote its equivalence class by $[X]\in[\mathfrak{J}_\times]$ and so on.

We define a 56 dimensional vector space $\mathfrak{P}^C$ by

$$\mathfrak{P}^C=\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus\boldsymbol{C}\oplus\boldsymbol{C}.$$

In $\mathfrak{P}^C$, the positive definite Hermitian inner product $\langle P,\ Q\rangle$, the skew-symmetric inner product $\{P,\ Q\}$ and the inner product $\langle P,\ Q\rangle_\iota$ are defined respectively by

$$\langle P,\ Q\rangle = \langle X,\ Z\rangle + \langle Y,\ W\rangle + \bar{\xi}\zeta + \bar{\eta}\omega,$$

$$\{P,\ Q\} = (X,\ W) - (Z,\ Y) + \xi\omega - \zeta\eta,$$

$$\langle P,\ Q\rangle_\iota = \langle X,\ Z\rangle - \langle Y,\ W\rangle + \bar{\xi}\zeta - \bar{\eta}\omega$$

where $P = (X,\ Y,\ \xi,\ \eta)$ and $Q = (Z,\ W,\ \zeta,\ \omega)$. An element $P = (X,\ Y,\ \xi,\ \eta) \in \mathfrak{P}^C$ is often denoted by $P = X + \dot{Y} + \xi + \dot{\eta}$ briefly. For example $1 = (0,\ 0,\ 1,\ 0)$, $\dot{1} = (0,\ 0,\ 0,\ 1)$.

We define subspaces $\mathfrak{M}^C$ (called a Freudenthal's manifold), $\mathfrak{M}_1$ and $\mathfrak{M}_\iota$ of $\mathfrak{P}^C$ respectively by

$$\mathfrak{M}^C = \{P = (X,\ Y,\ \xi,\ \eta) \in \mathfrak{P}^C \mid X \times X = \eta Y,\ Y \times Y = \xi X,\ (X,\ Y) = 3\xi\eta,\ P \neq 0\},$$

$$\mathfrak{M}_1 = \{P \in \mathfrak{M}^C \mid \langle P,\ P\rangle = 1\},$$

$$\mathfrak{M}_\iota = \{P \in \mathfrak{M}^C \mid \langle P,\ P\rangle_\iota = 1\}.$$

And we define equivalence relations $\sim$ in $\mathfrak{M}^C$, $\mathfrak{M}_1$ ad $\mathfrak{M}_\iota$ as follows.
For $P = (X,\ Y,\ \xi,\ \eta),\ Q \in \mathfrak{P}^C$, in $\mathfrak{M}^C$

$$P \sim Q \Longleftrightarrow (aX,\ aY,\ a\xi,\ a\eta) = Q \qquad \text{for some } a \in \boldsymbol{C}^*$$

and in $\mathfrak{M}_1$ (similarly in $\mathfrak{M}_\iota$)

$$P \sim Q \Longleftrightarrow (\theta X,\ \theta Y,\ \theta\xi,\ \theta\eta) = Q \qquad \text{for some } \theta \in \mathrm{U}(1).$$

We denote the totality of equivalence classes of these spaces by $[\mathfrak{M}^C]$, $[\mathfrak{M}_1]$ and $[\mathfrak{M}_\iota]$, respectively. For $(X,\ Y,\ \xi,\ \eta) \in \mathfrak{M}^C$, we denote its equivalence class by $[X,\ Y,\ \xi,\ \eta]$ (or $[X + \dot{Y} + \xi + \dot{\eta}]) \in [\mathfrak{M}^C]$ and so on.

§2. Lie groups E_6, $E_{6,\sigma}$ of type E_6 and their Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_{6,\sigma}$ [10], [12].

A simply connected compact simple Lie group E_6 of type E_6 is defined to be the group of linear isomorphisms of $\mathfrak{J}^C$ leaving the determinant $\det X$ and the Hermitian inner product $\langle X,\ Y\rangle$ invariant:

$$E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C) \mid \det \alpha X = \det X,\ \langle \alpha X,\ \alpha Y\rangle = \langle X,\ Y\rangle\}$$
$$= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau\alpha\tau(X \times Y),\ \langle \alpha X,\ \alpha Y\rangle = \langle X,\ Y\rangle\}.$$

A connected non-compact simple Lie group $E_{6,\sigma}$ of type $E_{6(-14)}$ is defined to be the group of linear isomorphisms of $\mathfrak{J}^C$ leaving the determinant $\det X$ and the inner product $\langle X,\ Y\rangle_\sigma$ invariant:

$$E_{6,\sigma} = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C) \mid \det \alpha X = \det X,\ \langle \alpha X,\ \alpha Y\rangle_\sigma = \langle X,\ Y\rangle_\sigma\}$$
$$= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau\sigma\alpha\sigma\tau(X \times Y),\ \langle \alpha X,\ \alpha Y\rangle_\sigma = \langle X,\ Y\rangle_\sigma\}.$$

A subgroup $U(1)$ of the group $E_{6,\sigma}$ defined by

$$U(1)=\left\{\phi(\theta)\;\middle|\;\phi(\theta)X(\xi,\ x)=\begin{pmatrix}\theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2\\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1\\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3\end{pmatrix},\ \theta\in \mathrm{U}(1)\right\}$$

is isomorphic to the group $\mathrm{U}(1)$, and we identify $U(1)$ with $\mathrm{U}(1)$. A subgroup $H=\{\alpha\in E_{6,\sigma}\,|\,\alpha E_1=E_1\}$ is isomorphic to the spinor group $Spin(10)$, and we identify H with $Spin(10)$. These groups $U(1)$ and $Spin(10)$ are also subgroups of the group E_6. The group $E_{6,\sigma}$ has the following polar decomposition :

$$E_{6,\sigma}\simeq U(1)Spin(10)\times \boldsymbol{R}^{32}$$

where a subgroup $U(1)Spin(10)$ of $E_{6,\sigma}$ is isomorphic to the group $(U(1)\times Spin(10))/\boldsymbol{Z}_4$ ($\boldsymbol{Z}_4=\{(\phi(1),\ 1),\ (\phi(-1),\ -1),\ (\phi(\sqrt{-1}),\ -\sqrt{-1}),\ (\phi(-\sqrt{-1}),\ \sqrt{-1})\}$).

A connected complex Lie group $E_6{}^{\boldsymbol{C}}$ of type E_6 is given by

$$E_6{}^{\boldsymbol{C}}=\{\alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{J}^{\boldsymbol{C}},\ \mathfrak{J}^{\boldsymbol{C}})\,|\,\det\alpha X=\det X\},$$

and its Lie algebra $\mathfrak{e}_6{}^{\boldsymbol{C}}$ is

$$\mathfrak{e}_6{}^{\boldsymbol{C}}=\{\phi\in \mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{J}^{\boldsymbol{C}},\ \mathfrak{J}^{\boldsymbol{C}})\,|\,(\phi X,\ X,\ X)=0\}.$$

Let $\mathfrak{D}_0$ be a Lie algebra generated by $\{z_1\widetilde{E}_1+z_2\widetilde{E}_2+z_3\widetilde{E}_3\,|\,z_i\in\mathfrak{C},\ z_i=-\bar{z}_i,\ \sum_{i=1}^{3}z_i=0\}$ and $\mathfrak{D}_0{}^{\boldsymbol{C}}$ the complexification of $\mathfrak{D}_0$. Then $\mathfrak{e}_6{}^{\boldsymbol{C}}$ has a decomposition as a vector space

$$\mathfrak{e}_6{}^{\boldsymbol{C}}=\mathfrak{D}_0{}^{\boldsymbol{C}}+\{\widetilde{A}_1(y_1)+\widetilde{A}_2(y_2)+\widetilde{A}_3(y_3)\,|\,y_i\in\mathfrak{C}^{\boldsymbol{C}}\}+\{\widetilde{X}\,|\,X\in\mathfrak{J}^{\boldsymbol{C}},\ \mathrm{tr}(X)=0\}.$$

For $A,\ A_i\in\mathfrak{J}_{-0}{}^{\boldsymbol{C}}=\{A\in\mathfrak{J}_-{}^{\boldsymbol{C}}\,|\,\mathrm{tr}(A)=0\}$, $X,\ X_i\in\mathfrak{J}_0{}^{\boldsymbol{C}}=\{X\in\mathfrak{J}^{\boldsymbol{C}}\,|\,\mathrm{tr}(X)=0\}$ $(i=1,\ 2)$, the Lie bracket on $\mathfrak{e}_6{}^{\boldsymbol{C}}$ is given as follows.

$$[\widetilde{A}_1,\ \widetilde{A}_2]=[A_1,\ A_2]^{\sim},\ [\widetilde{X}_1,\ \widetilde{X}_2]=-\frac{1}{4}[X_1,\ X_2]^{\sim},$$

$$[\widetilde{A},\ \widetilde{X}]=[A,\ X]^{\sim}.$$

The Lie algebras $\mathfrak{e}_6$ and $\mathfrak{e}_{6,\sigma}$ of the groups E_6 and $E_{6,\sigma}$ are respectively

$$\mathfrak{e}_6=\{\phi\in\mathfrak{e}_6{}^{\boldsymbol{C}}\,|\,\langle\phi X,\ Y\rangle+\langle X,\ \phi Y\rangle=0\},$$

$$\mathfrak{e}_{6,\sigma}=\{\phi\in\mathfrak{e}_6{}^{\boldsymbol{C}}\,|\,\langle\phi X,\ Y\rangle_\sigma+\langle X,\ \phi Y\rangle_\sigma=0\}.$$

The automorphism group F_4 of $\mathfrak{J}$ is a simply connected compact simple Lie group of type F_4 :

$$F_4=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})\,|\,\alpha(X\circ Y)=\alpha X\circ\alpha Y\},$$

and its Lie algebra $\mathfrak{f}_4$ is

$$\mathfrak{f}_4 = \{\delta \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y\}.$$

Any element ϕ of $\mathfrak{e}_6$ is represented by

$$\phi = \delta + \sqrt{-1}\widetilde{X}$$

where $\delta \in \mathfrak{f}_4$ and $X \in \mathfrak{J}_0 = \{X \in \mathfrak{J} \mid \mathrm{tr}(X) = 0\}$. And any element ϕ of $\mathfrak{e}_{6,\sigma}$ is represented by

$$\phi = d + \begin{pmatrix} 0 & \sqrt{-1}y_3 & -\sqrt{-1}\bar{y}_2 \\ -\sqrt{-1}\bar{y}_3 & 0 & \sqrt{-1}y_1 \\ \sqrt{-1}y_2 & -\sqrt{-1}\bar{y}_1 & 0 \end{pmatrix}^{\sim} + \begin{pmatrix} \sqrt{-1}\xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \sqrt{-1}\xi_2 & \sqrt{-1}x_1 \\ x_2 & \sqrt{-1}\bar{x}_1 & \sqrt{-1}\xi_3 \end{pmatrix}^{\sim}$$

where $d \in \mathfrak{D}_0$, x_i, $y_i \in \mathfrak{C}$ and $\xi_i \in \boldsymbol{R}$, $\sum_{i=1}^{3} \xi_i = 0$, $i = 1, 2, 3$.

Now we shall calculate the Killing form of $\mathfrak{e}_6{}^C$. To do this, we prepare the following

Lemma 1 ([1] P. 36). *Let $\mathfrak{g}^C$ be a simple Lie algebra over $\boldsymbol{C}$ and B the Killing form of $\mathfrak{g}^C$. If B' is a nondegenerate symmetric bilinear form on $\mathfrak{g}^C$ and invariant under the adjoint representation* ad *of $\mathfrak{g}^C$, then there exists $c \in \boldsymbol{C}$ such that $B = cB'$.*

From [6] Proposition 1, a set $\{[\widetilde{X}, \widetilde{Y}] \mid X, Y \in \mathfrak{J}^C\}$ generates $\mathfrak{f}_4{}^C$ (which is the complexification of $\mathfrak{f}_4$) additively. Hence we define an inner product on $\mathfrak{f}_4{}^C$ as follows. For $\delta_1 = \sum_i [\widetilde{X}_i, \widetilde{Y}_i]$, $\delta_2 = \sum_j [\widetilde{Z}_j, \widetilde{W}_j] \in \mathfrak{f}_4{}^C$,

$$(\delta_1, \delta_2) = \sum_{i,j} ([\widetilde{X}_i, \widetilde{Y}_i]\, W_j, Z_j).$$

From [6] Proposition 2, this inner product (δ_1, δ_2) is symmetric and independent of expressions of δ_1, δ_2. Since any element ϕ of $\mathfrak{e}_6{}^C$ is represented by $\phi = \delta + \widetilde{X}$, $\delta \in \mathfrak{f}_4{}^C$, $X \in \mathfrak{J}_0{}^C$, we define an inner product on $\mathfrak{e}_6{}^C$ by

$$(\phi_1, \phi_2) = (\delta_1, \delta_2) - (X_1, X_2)$$

where $\phi_i = \delta_i + \widetilde{X}_i$, $i = 1, 2$.

Proposition 2. *The Killing form B of $\mathfrak{e}_6{}^C$ is given by*

$$B(\phi_1, \phi_2) = -12(\phi_1, \phi_2) \qquad \phi_1, \phi_2 \in \mathfrak{e}_6{}^C.$$

Proof. First we show that the inner product (ϕ_1, ϕ_2) is $\mathrm{ad}\,\mathfrak{e}_6{}^C$-invariant.

For $\phi = \delta + \widetilde{X}$, $\phi_i = \delta_i + \widetilde{X}_i$ (δ, $\delta_i \in \mathfrak{f}_4{}^C$, X, $X_i \in \mathfrak{J}_0{}^C$, $i = 1, 2$), it holds that

$$\begin{aligned}([\phi, \phi_1], \phi_2) &= ([\delta, \delta_1]+[\widetilde{X}, \widetilde{X}_1]+(\delta X_1)^{\sim}-(\delta_1 X)^{\sim}, \delta_2+\widetilde{X}_2)\\ &= ([\delta, \delta_1]+[\widetilde{X}, \widetilde{X}_1], \delta_2)-(\delta X_1-\delta_1 X, X_2)\\ &= -(\delta_1, [\delta, \delta_2])-(X_1, \delta_2 X)+(X_1, \delta X_2)-(\delta_1, [\widetilde{X}, \widetilde{X}_2])\\ &= -(\delta_1+\widetilde{X}_1, [\delta, \delta_2]+[\widetilde{X}, \widetilde{X}_2]+(\delta X_2)^{\sim}-(\delta_2 X)^{\sim})\\ &= -(\phi_1, [\phi, \phi_2]),\end{aligned}$$

i. e., the inner product (ϕ_1, ϕ_2) is $\mathrm{ad}\mathfrak{e}_6{}^C$-invariant. Therefore from Lemma 1, there exists $c\in \boldsymbol{C}$ such that $B(\phi_1, \phi_2)=c(\phi_1, \phi_2)$. we can determine $c=-12$ putting $\phi_1=\phi_2=\widetilde{A}_2(1)=-4[\widetilde{E}_1, \widetilde{F}_2(1)]$. Thus $B(\phi_1, \phi_2)=-12(\phi_1, \phi_2)$.

§3. Lie groups E_7, $E_{7,\iota}$ of type E_7 and their Lie algebras $\mathfrak{e}_7$, $\mathfrak{e}_{7,\iota}$ [4], [5].

A simply connected compact simple Lie group E_7 of type E_7 is defined to be the group of linear isomorphisms of $\mathfrak{P}^C$ leaving the manifold $\mathfrak{M}^C$, some skew-symmetric inner product $\{P, Q\}$ and the Hermitian inner product $\langle P, Q\rangle$ invariant :

$$E_7=\{\alpha\in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C)\,|\,\alpha\mathfrak{M}^C=\mathfrak{M}^C,\ \{\alpha\dot{1}, \alpha 1\}=1,\ \langle\alpha P, \alpha Q\rangle=\langle P, Q\rangle\}.$$

A connected non-compact simple Lie group $E_{7,\iota}$ of type $E_{7(-25)}$ is defined to be the group of linear isomorphisms of $\mathfrak{P}^C$ leaving the manifold $\mathfrak{M}^C$, some skew-symmetric inner product $\{P, Q\}$ and the inner prodcut $\langle P, Q\rangle_\iota$ invariant :

$$E_{7,\iota}=\{\alpha\in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C)\,|\,\alpha\mathfrak{M}^C=\mathfrak{M}^C,\ \{\alpha 1, \alpha\dot{1}\}=1,\ \langle\alpha P, \alpha Q\rangle_\iota=\langle P, Q\rangle_\iota\}.$$

A subgroup $H=\{\alpha\in E_{7,\iota}\,|\,\alpha 1=1,\ \alpha\dot{1}=\dot{1}\}=\left\{\beta=\begin{pmatrix}\beta & 0 & 0 & 0\\ 0 & \tau\beta\tau & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\ \middle|\ \beta\in E_6\right\}$

is isomorphic to the group E_6, hence we identify H with E_6. A subgroup $U(1)$ of $E_{7,\iota}$ defined by

$$U(1)=\left\{\theta=\begin{pmatrix}\theta^{-1}1 & 0 & 0 & 0\\ 0 & \theta 1 & 0 & 0\\ 0 & 0 & \theta^3 & 0\\ 0 & 0 & 0 & \theta^{-3}\end{pmatrix}\ \middle|\ \theta\in \mathrm{U}(1)\right\}$$

is isomorphic to the group $\mathrm{U}(1)$, hence we identify $U(1)$ with $\mathrm{U}(1)$. These groups

E_6 and $U(1)$ are also subgroups of the group E_7. The group $E_{7,\iota}$ has the following polar decomposition :

$$E_{7,\iota} \simeq U(1)E_6 \times \boldsymbol{R}^{54}$$

where a subgroup $U(1)E_6$ of $E_{7,\iota}$ is isomorphic to the group $(U(1)\times E_6)/\boldsymbol{Z}_3$ $(\boldsymbol{Z}_3=\{(1,\ 1),\ (\omega,\ \omega 1),\ (\omega^2,\ \omega^2 1)\},\ \omega\in \boldsymbol{C},\ \omega^3=1,\ \omega\neq 1)$.

A connected complex Lie group $E_7{}^C$ of type E_7 is given by

$$E_7{}^C=\{\alpha\in \mathrm{Iso}_C(\mathfrak{P}^C,\ \mathfrak{P}^C)\,|\,\alpha\mathfrak{M}^C=\mathfrak{M}^C,\ \{\alpha P,\ \alpha Q\}=\{P,\ Q\}\}.$$

We define a bilinear symmetric mapping $\times$: $\mathfrak{P}^C\times\mathfrak{P}^C\to\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus \boldsymbol{C}$ by

$$P\times Q=\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}\times\begin{pmatrix}Z\\ W\\ \zeta\\ \omega\end{pmatrix}=\begin{pmatrix}2X\times Z-\eta W-\omega Y\\ 2Y\times W-\xi Z-\zeta X\\ (X,\ W)+(Y,\ Z)-3(\xi\omega+\zeta\eta)\end{pmatrix}.$$

The Lie algebra $\mathfrak{e}_7{}^C$ of $E_7{}^C$ is

$\mathfrak{e}_7{}^C=\{\Phi\in \mathrm{Hom}_C(\mathfrak{P}^C,\ \mathfrak{P}^C)\,|\,\Phi P\times P=0$ for all $P\in\mathfrak{M}^C,\ \{\Phi\dot{1},\ \dot{1}\}+\{\dot{1},\ \Phi\dot{1}\}=0\}$
and any element Φ of $\mathfrak{e}_7{}^C$ is represented by the form :

$$\Phi=\Phi(\phi,\ A,\ B,\ \rho)=\begin{pmatrix}\phi-\frac{1}{3}\rho 1 & 2B & 0 & A\\ 2A & \phi'+\frac{1}{3}\rho 1 & B & 0\\ 0 & A & \rho & 0\\ B & 0 & 0 & -\rho\end{pmatrix}$$

where $\phi\in\mathfrak{e}_6{}^C$, ϕ' is the skew transpose of ϕ with respect to the inner product $(X,\ Y)$: $(\phi X,\ Y)+(X,\ \phi' Y)=0$, A, $B\in\mathfrak{J}^C$, $\rho\in\boldsymbol{C}$ and the action of Φ on $\mathfrak{P}^C$ is defined by

$$\Phi(\phi,\ A,\ B,\ \rho)\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}=\begin{pmatrix}\phi X-\frac{1}{3}\rho X+2B\times Y+\eta A\\ 2A\times X+\phi' Y+\frac{1}{3}\rho Y+\xi B\\ (A,\ Y)+\rho\xi\\ (B,\ X)-\rho\eta\end{pmatrix}.$$

For $\Phi_i=\Phi(\phi_i,\ A_i,\ B_i,\ \rho_i)\in\mathfrak{e}_7{}^C$ $(i=1,\ 2)$, the Lie bracket $[\Phi_1,\ \Phi_2]$ is given by

$$[\Phi(\phi_1,\ A_1,\ B_1,\ \rho_1),\ \Phi(\phi_2,\ A_2,\ B_2,\ \rho_2)]=\Phi(\phi,\ A,\ B,\ \rho)$$

$$\begin{cases} \Phi = [\phi_1, \ \phi_2] + 2A_1 \vee B_2 - 2A_2 \vee B_1, \\ A = (\phi_1 + \frac{2}{3}\rho_1 1)A_2 - (\phi_2 + \frac{2}{3}\rho_2 1) \ A_1, \\ B = (\phi_1{}' - \frac{2}{3}\rho_1 1)B_2 - (\phi_2{}' - \frac{2}{3}\rho_2 1)B_1, \\ \rho = (A_1, \ B_2) - (B_1, \ A_2) \end{cases}$$

where $(A \vee B)\ (X) = \frac{1}{2}(B,\ X)A + \frac{1}{6}(A,\ B)X - 2B \times (A \times X)$.

The Lie algebras $\mathfrak{e}_7$ and $\mathfrak{e}_{7,\iota}$ of the groups E_7 and $E_{7,\iota}$ are respectively

$$\mathfrak{e}_7 = \{\Phi \in \mathfrak{e}_7{}^C \,|\, \langle \Phi P,\ Q\rangle + \langle P,\ \Phi Q\rangle = 0\},$$

$$\mathfrak{e}_{7,\iota} = \{\Phi \in \mathfrak{e}_7{}^C \,|\, \langle \Phi P,\ Q\rangle_\iota + \langle P,\ \Phi Q\rangle_\iota = 0\}.$$

Any element Φ of $\mathfrak{e}_7$ is represented by

$$\Phi = \Phi(\phi,\ A,\ -\overline{A},\ \rho), \qquad \phi \in \mathfrak{e}_6,\ A \in \mathfrak{J}^C,\ \rho \in \boldsymbol{C},\ \rho + \overline{\rho} = 0,$$

and any element Φ of $\mathfrak{e}_{7,\iota}$ is represented by

$$\Phi = \Phi(\phi,\ A,\ \overline{A},\ \rho), \qquad \phi \in \mathfrak{e}_6,\ A \in \mathfrak{J}^C,\ \rho \in \boldsymbol{C},\ \rho + \overline{\rho} = 0.$$

Now we shall calculate the Killing form of $\mathfrak{e}_7{}^C$. We define an inner product $(\Phi_1,\ \Phi_2)$ on $\mathfrak{e}_7{}^C$ by

$$(\Phi_1,\ \Phi_2) = 2(\phi_1,\ \phi_2) - 4(A_1,\ B_2) - 4(A_2,\ B_1) - \frac{8}{3}\rho_1\rho_2$$

where $\Phi_i = \Phi(\phi_i,\ A_i,\ \mathrm{B}_i,\ \rho_i)$, $i = 1,\ 2$.

Proposition 3. *The Killing form B of $\mathfrak{e}_7{}^C$ is given by*

$$B(\Phi_1,\ \Phi_2) = -9(\Phi_1,\ \Phi_2), \qquad \Phi_1,\ \Phi_2 \in \mathfrak{e}_7{}^C.$$

Proof. First we shall show that the inner product $(\Phi_1,\ \Phi_2)$ is $\mathrm{ad}\mathfrak{e}_7{}^C$-invariant. For $\Phi = \Phi(\phi,\ A,\ B,\ \rho)$, $\Phi_i = \Phi(\phi_i,\ A_i,\ B_i,\ \rho_i)$, $i = 1,\ 2$, it holds that

$$\begin{aligned}([\Phi,\ \Phi_1],\ \Phi_2) &= 2([\phi,\ \phi_1] + 2A \vee B_1 - 2A_1 \vee B,\ \phi_2) - 4(\phi A_1 + \frac{2}{3}\rho A_1 - \phi_1 A \\ &\quad - \frac{2}{3}\rho_1 A,\ B_2) - 4(A_2,\ \phi' B_1 - \frac{2}{3}\rho B_1 - \phi_1{}' B + \frac{2}{3}\rho_1 B) - \frac{8}{3}(A,\ B_1)\rho_2 \\ &\quad + \frac{8}{3}(B,\ A_1)\rho_2 \\ &\underset{(*)}{=} -2(\phi_1,\ [\phi,\ \phi_2] + 2A \vee B_2 - 2A_2 \vee B) + 4(A_1,\ \phi' B_2 \\ &\quad - \frac{2}{3}\rho B_2 - \phi_2{}' B + \frac{2}{3}\rho_2 B) + 4(\phi A_2 + \frac{2}{3}\rho A_2 - \phi_2 A - \frac{2}{3}\rho_2 A,\ B)\end{aligned}$$

$$+\frac{8}{3}\rho_1(A,\ B_2)-\frac{8}{3}\rho_1(B,\ A_2)=-(\Phi_1,\ [\Phi,\ \Phi_2])$$

$$((*)\ (\phi,\ A\vee B)=-(\phi A,\ B)),$$

i. e., the inner product $(\Phi_1,\ \Phi_2)$ is $\mathrm{ad}\mathfrak{e}_7{}^C$-invariant. Therefore, from Lemma 1 there exists $c\in C$ such that $B(\Phi_1,\ \Phi_2)=c(\Phi_1,\ \Phi_2)$. We can determine $c=-9$ putting $\Phi_1=\Phi_2=\Phi(0,\ 0,\ 0,\ \rho)\in\mathfrak{e}_7{}^C$. Thus $B(\Phi_1,\ \Phi_2)=-9(\Phi_1,\ \Phi_2)$.

§4. Hermitian symmetric pair.

Let G be a connected Lie group, K a subgroup of G and s an involutive automorphism of G. Let $\mathfrak{g}$ be the Lie algebra of G. We decompose $\mathfrak{g}$ as a vector space using the differential of s (which is also denoted by s) into

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{n}$$

where $\mathfrak{k}=\{X\in\mathfrak{g}\mid sX=X\}$ and $\mathfrak{n}=\{X\in\mathfrak{g}\mid sX=-X\}$. Let g be an inner product on $\mathfrak{n}$. Suppose that $\mathfrak{n}$ has a complex structure J.

Definition ([11]). The connected Lie group G has an Hermitian symmetric pair $(G,\ K;\ s,\ g,\ J)$ if and only if

(1) s is not identity.

(2) K is a closed subgroup of G such that $(G_s)_0\subset K\subset G_s$, where G_s is the set of fixed points of s and $(G_s)_0$ is the identity component of G_s.

(3) $\mathrm{Ad}K$ is a compact subgroup of $GL(\mathfrak{g})$ (where Ad is the adjoint representation of G).

(4) g is a positive definite inner product on $\mathfrak{n}$ satisfying

$$g(\mathrm{Ad}kX,\ \mathrm{Ad}kY)=g(X,\ Y),\qquad k\in K,\ X,\ Y\in\mathfrak{n},$$

$$g(JX,\ JY)=g(X,\ Y),\qquad X,\ Y\in\mathfrak{n},$$

$$J(\mathrm{Ad}|\mathfrak{n}\ k)=(\mathrm{Ad}|\mathfrak{n}\ k)J,\qquad k\in K.$$

Lemma 4 ([11] P. 117). *Let G be a connected Lie group and has an Hermitian symmetric pair $(G,\ K;\ s,\ g,\ J)$. Then the homogeneous space G/K has an Hermitian symmetric structure.*

We shall construct Hermitian symmetric pairs of the groups $E_{6,\sigma}$ and $E_{7,\iota}$ respectively later on. As the results, we see that the homogeneous spaces $E_{6,\sigma}/U(1)Spin(10)$ and $E_{7,\iota}/U(1)E_6$ have Hermitian symmetric structures.

II. Bounded symmetric domain of type E_6.

§5. Hermitian symmetric pair of $E_{6,\sigma}$.

We define an involutive automorphism $\boldsymbol{\sigma}$ of the group $E_{6,\sigma}$ (which is a Cartan involution) by

$$\boldsymbol{\sigma}\alpha=\sigma\alpha\sigma, \qquad \alpha\in E_{6,\sigma}.$$

The decomposition $\mathfrak{e}_{6,\sigma}=\mathfrak{k}\oplus\mathfrak{n}$ as in §4 with respect to $\boldsymbol{\sigma}$ is given by

$$\mathfrak{k}=\mathfrak{D}_0+\{\widetilde{A}_1(y)\,|\,y\in\mathfrak{C}\}+\{\sqrt{-1}(\xi_1E_1+\xi_2E_2+\xi_3E_3+F_1(x))^{\sim}\,|\,\xi_i\in\boldsymbol{R},\sum_{i=1}^{3}\xi_i=0,\ x\in\mathfrak{C}\},$$

$$\mathfrak{n}=\{\sqrt{-1}\widetilde{A}_2(y_2)+\sqrt{-1}\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x_3)\,|\,x_2,\ x_3,\ y_2,\ y_3\in\mathfrak{C}\}.$$

We define an inner product g on $\mathfrak{n}$ by

$$g(\sqrt{-1}\widetilde{A}_2(y_2)+\sqrt{-1}\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x_3),\ \sqrt{-1}\widetilde{A}_2(y_2')+\sqrt{-1}\widetilde{A}_3(y_3')$$
$$+2\widetilde{F}_2(x_2')+2\widetilde{F}_3(x_3'))=(y_2,\ y_2')+(y_3,\ y_3')+(x_2,\ x_2')+(x_3,\ x_3'),$$

and a linear transformation J of $\mathfrak{n}$ by

$$J=-\frac{2}{3}\sqrt{-1}\,\mathrm{ad}(2E_1-E_2-E_3)^{\sim}.$$

Hence for each $N=\sqrt{-1}\widetilde{A}_2(y_2)+\sqrt{-1}\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x_3)\in\mathfrak{n}$, we have

$$J(N)=\frac{2}{3}[2E_1-E_2-E_3,\ A_2(y_2)+A_3(y_3)]^{\sim}-\frac{1}{3}\sqrt{-1}[2E_1-E_2-E_3,\ F_2(x_2)+F_3(x_3)]^{\sim}$$
$$=\sqrt{-1}\widetilde{A}_2(x_2)-\sqrt{-1}\widetilde{A}_3(x_3)-2\widetilde{F}_2(y_2)+2\widetilde{F}_3(y_3),$$

so J is a complex structure on $\mathfrak{n}$.

Proposition 5. *$(E_{6,\sigma},\ U(1)Spin(10)\ ;\ \sigma,\ g,\ J)$ is an Hermitian symmetric pair of the group $E_{6,\sigma}$.*

Proof. We shall check the conditions of Definition in §4. In [10] Proposition 6, we have seen $\{\alpha\in E_{6,\sigma}\,|\,\sigma\alpha\sigma=\alpha\}=U(1)Spin(10)$. Now obviously conditions (1), (2) and (3) are satisfied. Instead of the first condition (4), it suffices to show that the inner product g is ad$\mathfrak{k}$-invariant :

$$g(\mathrm{ad}\,kX,\ Y)+g(X,\ \mathrm{ad}kY)=0, \qquad X,\ Y\in\mathfrak{n},\ k\in\mathfrak{k}.$$

For $N=\sqrt{-1}\widetilde{A}_2(y_2)+\sqrt{-1}\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x_3)$, $N'=\sqrt{-1}\widetilde{A}_2(y_2')+\sqrt{-1}\widetilde{A}_3(y_3')$ $+2\widetilde{F}_2(x_2')+2\widetilde{F}_3(x_3')\in\mathfrak{n}$, taking the inner product $(\phi_1,\ \phi_2)$ on $\mathfrak{e}_6{}^C$, we have $(N,\ N')=-(A(y_2)+A_3(y_3),\ A_2(y_2')+A_3(y_3'))-4(F_2(x_2)+F_3(x_3),\ F_2(x_2')+F_3(x_3'))$, and using $\widetilde{A}_2(y)=4[\widetilde{E}_3,\ \widetilde{F}_2(y)]$ and $\widetilde{A}_3(y)=-4[\widetilde{E}_2,\ \widetilde{F}_3(y)]$, we have

$$(N,\ N') = -4((\widetilde{A}_2(y_2)+\widetilde{A}_3(y_3))F_2(y_2'),\ E_3) + 4((\widetilde{A}_2(y_2)+\widetilde{A}_3\ (y_3))F_3(y_3'),\ E_2)$$
$$-8((x_2,\ x_2')+(x_3,\ x_3'))$$
$$= -8((y_2,\ y_2')+(y_3,\ y_3')+(x_2,\ x_2')+(x_3,\ x_3')).$$

So g is ad$\mathfrak{k}$-invariant, since the inner product $(\phi_1,\ \phi_2)$ on $\mathfrak{e}_6{}^C$ is ad$\mathfrak{e}_6{}^C$-invariant. And for the above $N,\ N' \in \mathfrak{n}$, we have

$$g(JN,\ JN') = g(\sqrt{-1}\widetilde{A}_2(x_2) - \sqrt{-1}\widetilde{A}_3(x_3) - 2\widetilde{F}_2(y_2) + 2\widetilde{F}_3(y_3),$$
$$\sqrt{-1}\widetilde{A}_2(x_2') - \sqrt{-1}\widetilde{A}_3(x_3') - 2\widetilde{F}_2(y_2') + 2\widetilde{F}_3(y_3'))$$
$$= (x_2,\ x_2') + (x_3,\ x_3') + (y_2,\ y_2') + (y_3,\ y_3') = g(N,\ N'),$$

and for $k \in \mathfrak{k}$, $N \in \mathfrak{n}$

$$J\mathrm{ad}k\ N = -\frac{2}{3}\sqrt{-1}[(2E_1-E_2-E_3)^{\sim},\ [k,\ N]]$$
$$= -\frac{2}{3}\sqrt{-1}[k,\ [(2E_1-E_2-E_3)^{\sim}\ N] - \frac{2}{3}\sqrt{-1}[[(2E_1-E_2-E_3)^{\sim},\ k],\ N]$$
$$= \mathrm{ad}k\ JN, \quad ([(2E_1-E_2-E_3)^{\sim},\ k]=0).$$

Hence the condition (4) is satisfied. Thus the proof is completed.

From Lemma 4 and Proposition 5, we see that the homogeneous space $E_{6,\sigma}/U(1)$ $Spin(10)$ has a structure of an Hermitian symmetric space.

§6. Realization of the symmetric space $E_{6,\sigma}/U(1)Spin(10)$.

The space $[\mathfrak{J}_1]$ has a differentiable structure induced by that of the manifold $\mathfrak{J}_1$, because on the manifold $\mathfrak{J}_1$ the group U(1) acts freely.

Proposition 6. *The homogeneous space* $E_6/U(1)Spin(10)$ *is diffeomorphic to the manifold* $[\mathfrak{J}_1] = [\{X \in \mathfrak{J}^C \mid X \times X = 0,\ \langle X,\ X\rangle = 1\}]$.

Proof. The group E_6 acts on $\mathfrak{J}_1$, since for $\alpha \in E_6$, $X \in \mathfrak{J}_1$ we have

$$\alpha X \times \alpha X = \tau\alpha\tau(X \times X) = 0, \qquad \langle \alpha X,\ \alpha X\rangle = \langle X,\ X\rangle = 1.$$

From [12] Proposition 5, for each $X \in \mathfrak{J}_1$ there exists $\alpha \in E_6$ such that $\alpha X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3$, $\xi_i \in \boldsymbol{C}$. $\alpha X \in \mathfrak{J}_1$ implies $\alpha X = \xi_i E_i$, $|\xi_i| = 1$ for some $i = 1,\ 2,\ 3$. If $i=1$, then $\phi(\xi_1{}^{-\frac{1}{4}})\alpha X = E_1$, and if $i = 2$ or 3, then $\phi(\xi_i{}^{\frac{1}{2}})\alpha X = E_i$. From [13] Theorem 5.53, E_2 and E_3 can be transformed into E_1 using the elements of the group F_4. Hence for $X \in \mathfrak{J}_1$, we have $\alpha X = E_1$ for some $\alpha \in E_6$. Therefore E_6 acts transitively on $\mathfrak{J}_1$ and $[\mathfrak{J}_1]$. Let $\alpha \in E_6$ fix the point $[E_1] \in [\mathfrak{J}_1]$. Then $\alpha E_1 = \theta E_1$ for some $\theta \in U(1)$. So $\phi(\theta^{-\frac{1}{4}})\alpha E_1 = E_1$, that is, $\phi(\theta^{-\frac{1}{4}})\alpha \in Spin(10)$. Therefore $\alpha \in U(1)Spin(10)$. Conversely, let α be an arbitrary element of $U(1)Spin(10)$. Then $\alpha[E_1] = [E_1]$. Thus the homogeneous space $E_6/U(1)Spin(10)$ is diffeomorphic to the manifold $[\mathfrak{J}_1]$.

Lemma 7. *The group $E_6{}^C$ acts on the space $[\mathfrak{J}_\times]$ transitively. Let U be the isotropy subgroup of $E_6{}^C$ at $[E_1] \in [\mathfrak{J}_\times]$. Then the homogeneous space E_6/U is homeomorphic to the space $[\mathfrak{J}_\times]$.*

Proof is similar to that of Proposition 6.

From now on, we identify E_6/U with $[\mathfrak{J}_\times]$ and introduce the differentiable and complex structure of E_6/U into $[\mathfrak{J}_\times]$.

Now, we shall realize the symmetric space $E_{6,\sigma}/U(1)Spin(10)$. Any element α of the group $E_{6,\sigma}$ leaves the inner product $\langle X, Y\rangle_\sigma$ invariant and satisfies $\alpha X \times \alpha X = \tau\sigma\alpha\sigma\tau(X \times X)$. Hence $E_{6,\sigma}$ acts on the space $\mathfrak{J}_\sigma$ and $[\mathfrak{J}_\sigma]$. Since the isotropy subgroup of $E_{6,\sigma}$ at $[E_1] \in [\mathfrak{J}_\sigma]$ is $U(1)Spin(10)$ (this follows from the equivalence relation in $\mathfrak{J}_\sigma$ and the definition of the groups $U(1)$ and $Spin(10)$), we shall consider $E_{6,\sigma}/U(1)Spin(10)$ as the orbit space $E_{6,\sigma}[E_1]$ in $[\mathfrak{J}_\sigma]$. To describe the orbit space $E_{6,\sigma}[E_1]$ explicitely, we need the following arguments.

Let $X = X(\xi, x) \in \mathfrak{J}_\sigma$, then $X \times X = 0$ and $\langle X, X\rangle_\sigma = 1$, so we have

$$(5)\quad \begin{cases} \xi_2\xi_3 = (x_1, x_1), & \xi_3\xi_1 = (x_2, x_2), \quad \xi_2\xi_1 = (x_3, x_3), \\ \xi_1 x_1 = x_2 x_3, & \xi_2 x_2 = x_3 x_1, \quad \xi_3 x_3 = x_1 x_2, \\ |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + 2\langle x_1, x_1\rangle - 2\langle x_2, x_2\rangle - 2\langle x_3, x_3\rangle = 1. \end{cases}$$

Lemma 8 ([9]-I, P. 161, Corollary 1). *Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$, $\boldsymbol{\sigma}$ a Cartan involution of $\mathfrak{g}$, $\boldsymbol{\sigma}'$ an involutive automorphism of $\mathfrak{g}$ such that $\boldsymbol{\sigma}\boldsymbol{\sigma}' = \boldsymbol{\sigma}'\boldsymbol{\sigma}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ the Cartan decompositon. Let $\mathfrak{n}_\pm$ be the (± 1)-eigen spaces of $\boldsymbol{\sigma}'$ in $\mathfrak{n}$, and K the subgroup corresponding to $\mathfrak{k}$. Then the map $(X, Y, k) \to (\exp X)(\exp Y)k$ is a diffeomorphism of $\mathfrak{n}_+ \times \mathfrak{n}_- \times K$ onto G.*

We define a mapping $\sigma' : \mathfrak{J}^C \to \mathfrak{J}^C$ by

$$\sigma' \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & x_3 & -\overline{x}_2 \\ \overline{x}_3 & \xi_2 & -x_1 \\ -x_2 & -\overline{x}_1 & \xi_3 \end{pmatrix}$$

and an involutive automorphism $\boldsymbol{\sigma}'$ of $\mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C)$ by

$$\boldsymbol{\sigma}'\phi = \sigma'\phi\sigma', \qquad \phi \in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C),$$

Lemma 9. *The mapping $\boldsymbol{\sigma}'$ is an involutive automorphism of the Lie algebra $\mathfrak{e}_{6,\sigma}$ and commute with the Cartan involution $\boldsymbol{\sigma}$ of $\mathfrak{e}_{6,\sigma}$.*

Proof. Let ϕ be an arbitrary element of $\mathfrak{e}_{6,\sigma}$. For $X \in \mathfrak{J}^C$, we have $(\boldsymbol{\sigma}'\phi X, X, X) = (\sigma'\phi\sigma' X, X \times X) = (\phi\sigma' X, \sigma'(X \times X)) = (\phi\sigma' X, \sigma' X, \sigma' X) = 0$, and for $X, Y \in \mathfrak{J}^C$

$$\langle \boldsymbol{\sigma}'\phi X, Y\rangle_\sigma + \langle X, \boldsymbol{\sigma}'\phi Y\rangle_\sigma = \langle \sigma'\phi\sigma' X, Y\rangle_\sigma + \langle X, \sigma'\phi\sigma' Y\rangle_\sigma$$

$$= \langle \sigma\sigma'\phi\sigma' X, Y\rangle + \langle X, \sigma\sigma'\phi\sigma' Y\rangle = \langle \sigma'\sigma\phi\sigma' X, Y\rangle + \langle X, \sigma'\sigma\phi\sigma' Y\rangle$$

$$= \langle \sigma\phi(\sigma' X), \sigma' Y\rangle + \langle \sigma' X, \sigma\phi(\sigma' Y)\rangle = \langle \phi(\sigma' X), \sigma' Y\rangle_\sigma + \langle \sigma' X, \phi(\sigma' Y)\rangle_\sigma = 0.$$

Hence $\boldsymbol{\sigma}' \in \mathfrak{e}_{6,\sigma}$. And the commutativity $\boldsymbol{\sigma}\boldsymbol{\sigma}' = \boldsymbol{\sigma}'\boldsymbol{\sigma}$ is clear. Thus the proof is completed.

For $\mathfrak{n} = \{\sqrt{-1}\tilde{A}_2(y_2) + \sqrt{-1}\tilde{A}_3(y_3) + 2\tilde{F}_2(x_2) + 2\tilde{F}_3(x_3) \mid x_2, x_3, y_2, y_3 \in \mathfrak{C}\}$,

the decomposition $\mathfrak{n} = \mathfrak{n}_+ \oplus \mathfrak{n}_-$ as in Lemma 8 with respect to $\boldsymbol{\sigma}'$ is given by

$$\mathfrak{n}_+ = \{\sqrt{-1}\tilde{A}_3(y_3) + 2\tilde{F}_3(x_3) \mid x_3, y_3 \in \mathfrak{C}\},$$

$$\mathfrak{n}_- = \{\sqrt{-1}\tilde{A}_2(y_2) + 2\tilde{F}_2(x_2) \mid x_2, y_2 \in \mathfrak{C}\}.$$

Therefore any $\alpha \in E_{6,\sigma}$ is represented by the form :

$$\alpha = \exp(\sqrt{-1}\tilde{A}_3(y_3) + 2\tilde{F}_3(x_3))\exp(\sqrt{-1}A_2(y_2) + 2\tilde{F}_2(x_2))k, \qquad k \in U(1)Spin(10).$$

Now we shall calculate $\exp(\sqrt{-1}\tilde{A}_3(y_3) + 2\tilde{F}_3(x_3))\exp(\sqrt{-1}\tilde{A}_2(y_2) + 2\tilde{F}_2(x_2))E_1$.

First of all, $\exp(\sqrt{-1}\tilde{A}_2(y) + 2\tilde{F}_2(x))E_1$ is calculated as follows.

$$\exp(\sqrt{-1}\tilde{A}_2(y) + 2\tilde{F}_2(x))E_1$$

$$= E_1 + F_2(z) + \frac{2}{2!}(\langle z, z\rangle E_1 + (z, z)E_3) + \frac{2}{3!}F_2(\langle z, z\rangle z + (z, z)\tilde{z})$$

$$+ \frac{4}{4!}(\langle z, z\rangle^2 E_1 + 2(z, z)\langle z, z\rangle E_3 + (z, z)(\tilde{z}, \tilde{z})E_1)$$

$$+ \frac{4}{5!}F_2(\langle z, z\rangle^2 z + 2(z, z)\langle z, z\rangle\tilde{z} + (z, z)(\tilde{z}, \tilde{z})z)$$

$$+ \frac{8}{6!}(\langle z, z\rangle^3 E_1 + 3\langle z, z\rangle^2(z, z)E_3 + 3\langle z, z\rangle(z, z)(\tilde{z}, \tilde{z})E_1 + (z, z)^2(\tilde{z}, \tilde{z})E_3)$$

$$+ \cdots.$$

where $z = x + \sqrt{-1}y$. Let $[\frac{n}{2}]$ be the maximal integer not greater than $\frac{n}{2}$,

$[\frac{n}{2}]' = [\frac{n}{2}]$ for $n \geqq 2$ and $[\frac{n}{2}]' = 1$ for $n = 0,\ 1$. Then we have

$$\exp(\sqrt{-1}\widetilde{A}_2(y_2) + 2\widetilde{F}_2(x_2))E_1 = \xi(z_2)E_1 + \eta(z_2)E_3 + F_2(u(z_2))$$

where
$$\begin{cases} \xi(z_2) = \sum_{n=0}^{\infty} \frac{2^n}{(2n)\,!} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} (z_2,\ z_2)^k (\tilde{z}_2,\ \tilde{z}_2)^k \langle z_2,\ z_2 \rangle^{n-2k}, \\ \eta(z_2) = \sum_{n=0}^{\infty} \frac{2^n}{(2n)\,!} \sum_{k=0}^{[\frac{n}{2}]'-1} \binom{n}{2k+1} (z_2,\ z_2)^{k+1} (\tilde{z}_2,\ \tilde{z}_2)^k \langle z_2,\ z_2 \rangle^{n-2k-1}, \\ u(z_2) = \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!} \Big(\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} (z_2,\ z_2)^k (\tilde{z}_2,\ \tilde{z}_2)^k \langle z_2,\ z_2 \rangle^{n-2k} z_2 + \\ \quad + \sum_{k=0}^{[\frac{n}{2}]'-1} \binom{n}{2k+1} (z_2,\ z_2)^{k+1} (\tilde{z}_2,\ \tilde{z}_2)^k \langle z_2,\ z_2 \rangle^{n-2k-1} \tilde{z}_2 \Big), \quad (z_2 = x_2 + \sqrt{-1}y_2). \end{cases}$$

Next, we calculate $\exp(\sqrt{-1}\widetilde{A}_3(y_3) + 2\widetilde{F}_3(x_3))\exp(\sqrt{-1}\widetilde{A}_2(y_2) + 2\widetilde{F}_2(x_2))E_1$.

$$\begin{aligned} &\exp(\sqrt{-1}\widetilde{A}_3(y_3) + 2\widetilde{F}_3(x_3))\exp(\sqrt{-1}\widetilde{A}_2(y_2) + 2\widetilde{F}_2(x_2))E_1 \\ &= \exp(\sqrt{-1}\widetilde{A}_3(y_3) + 2\widetilde{F}_3(x_3))(\xi(z_2)E_1 + \eta(z_2)E_3 + F_2(u(z_2)) \\ &= \xi(z_2)(\xi(\tilde{z}_3)E_1 + \eta(\tilde{z}_3)E_2 + F_3(u(\tilde{z}_3)) + \eta(z_2)E_3 + F_2(v(u(z_2),\ z_3)) + F_1(v'(z_2,\ z_3)) \end{aligned}$$

where $z_3 = x_3 + \sqrt{-1}y_3$, $v(u(z_2),\ z_3) = \sum_{n=0}^{\infty} \frac{1}{(2n)\,!} (\overbrace{\ldots (u(z_2)\tilde{z}_3)\tilde{z}_3)\tilde{z}_3 \ldots)\tilde{z}_3}^{n})$

and $v'(z_2,\ z_3) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \tilde{z}_3 (\overbrace{\ldots (\tilde{z}_3(z_3}^{n}(\tilde{z}_3 u(z_2)) \ldots)$. Therefore from (5) we have

$$(6)\quad \begin{cases} (u(z_2),\ u(z_2)) = \xi(z_2)\eta(z_2),\ \ \xi(z_2)^2 + |\eta(z_2)|^2 - 2\langle u(z_2),\ u(z_2) \rangle = 1, \\ (u(\tilde{z}_3),\ u(\tilde{z}_3)) = \xi(\tilde{z}_3)\eta(\tilde{z}_3),\ \ \xi(\tilde{z}_3)^2 + |\eta(\tilde{z}_3|^2 - 2\langle u(\tilde{z}_3),\ u(\tilde{z}_3) \rangle = 1, \\ (v(u(z_2),\ z_3),\ v(u(z_2),\ z_3)) = \xi(z_2)\xi(\tilde{z}_3)\eta(z_2), \\ v(u(z_2),\ z_3)u(\tilde{z}_3) = \xi(\tilde{z}_3)v'(z_2,\ z_3), \\ -\langle u(z_2),\ u(z_2) \rangle = -\langle v(u(z_2),\ z_3),\ v(u(z_2),\ z_3) \rangle + \langle v'(z_2,\ z_3),\ v'(z_2,\ z_3) \rangle. \end{cases}$$

We define mappings u: $\mathfrak{J}^C \longrightarrow \mathfrak{J}^C$ and $v(\ \ ,\ z_0)$: $\mathfrak{J}^C \longrightarrow \mathfrak{J}^C$ respectively by

$$z \longmapsto u(z), \qquad z \longmapsto v(z,\ z_0).$$

We shall show that the mappings u and $v(\ \ ,\ z_0)$ are both surjections. To do this, we prepare the following elements $\exp(\sqrt{-1}\widetilde{A}_i(a))$, $\exp(\widetilde{F}_i(a))$ of the group $E_{6,\sigma}$ $(i=2,\ 3,\ a\in\mathfrak{C})$.

(i) $\exp(\sqrt{-1}\widetilde{A}_i(a))X(\xi,\ x)=Y(\eta,\ y)$

$$\text{where}\ \begin{cases} \eta_{i-1}=\dfrac{\xi_{i-1}+\xi_{i+1}}{2}+\dfrac{\xi_{i-1}-\xi_{i+1}}{2}\cosh 2|a|-\sqrt{-1}\dfrac{(a,\ x_i)}{|a|}\sinh 2|a|, \\ \eta_i=\xi_i, \\ \eta_{i+1}=\dfrac{\xi_{i-1}+\xi_{i+1}}{2}-\dfrac{\xi_{i-1}-\xi_{i+1}}{2}\cosh 2|a|+\sqrt{-1}\dfrac{(a,\ x_i)}{|a|}\sinh 2|a|, \end{cases}$$

$$\begin{cases} y_{i-1}=x_{i-1}\ \cosh|a|+\sqrt{-1}\dfrac{\overline{ax_{i+1}}}{|a|}\sinh|a|, \\ y_i=x_i-\dfrac{2(a,\ x_i)a}{|a|^2}\sinh^2|a|+\sqrt{-1}\dfrac{(\xi_{i-1}-\xi_{i+1})a}{2|a|}\sinh 2|a|, \\ y_{i+1}=x_{i+1}\ \cosh|a|-\sqrt{-1}\dfrac{\overline{x_{i-1}a}}{|a|}\sinh|a|, \end{cases}$$

(ii) $\exp(F_i(a))X(\xi,\ x)=Y(\eta,\ y)$

$$\text{where}\ \begin{cases} \eta_{i-1}=\dfrac{\xi_{i-1}-\xi_{i+1}}{2}+\dfrac{\xi_{i-1}+\xi_{i+1}}{2}\ \cosh|a|+\dfrac{(a,\ x_i)}{|a|}\sinh|a|, \\ \eta_i=\xi_i, \\ \eta_{i+1}=-\dfrac{\xi_{i-1}-\xi_{i+1}}{2}+\dfrac{\xi_{i-1}+\xi_{i+1}}{2}\ \cosh|a|+\dfrac{(a,\ x_i)}{|a|}\sinh|a|, \end{cases}$$

$$\begin{cases} y_{i-1}=x_{i-1}\ \cosh\left|\dfrac{a}{2}\right|+\dfrac{\overline{ax_{i+1}}}{|a|}\sinh\left|\dfrac{a}{2}\right|, \\ y_i=x_i+\dfrac{2(a,\ x_i)a}{|a|^2}\sinh^2\left|\dfrac{a}{2}\right|+\dfrac{(\xi_{i-1}+\xi_{i+1})a}{2|a|}\sinh|a|, \\ y_{i+1}=x_{i+1}\ \cosh\left|\dfrac{a}{2}\right|+\dfrac{\overline{x_{i-1}a}}{|a|}\sinh\left|\dfrac{a}{2}\right|, \end{cases}$$

(the indices are considered as mod. 3, and if $a=0$, then $a\dfrac{\sinh|a|}{|a|}$ means 0).

Lemma 10. *The mapping u is onto.*

Proof. The Lie subalgebra $\mathfrak{h}$ of $\mathfrak{e}_{6,\sigma}$ generated by $\{\sqrt{-1}\widetilde{A}_2(y)+2\widetilde{F}_2(x)\,|\,x,\ y\in\mathfrak{C}\}$ is $\{\sqrt{-1}\widetilde{A}_2(y)+2\widetilde{F}_2(x)+\sqrt{-1}r(E_1-E_3)^{\sim}\,|\,x,\ y\in\mathfrak{C},\ r\in\boldsymbol{R}\}$. Let H be the connected subgroup of $E_{6,\sigma}$ corresponding to $\mathfrak{h}$. Then from [1] (6. 4. 6), we have {the F_2-component of $h[E_1]\,|\,h\in H\}=\{$the F_2-component of $\exp(\sqrt{-1}\widetilde{A}_2(y)+2\widetilde{F}_2(x))[E_1]\,|$

$x,\ y\in\mathfrak{C}\}$. By formal computation we have

$$\exp(\sqrt{-1}\tilde{A}_2(y))\exp(2\tilde{F}_2(x))E_1=\xi_1E_1+\xi_3E_3+F_2(x_2)$$

where
$$\begin{cases}\xi_1=\dfrac{1}{2}(\cosh 2|x|+\cosh 2|y|-\sqrt{-1}\dfrac{(y,\ x)}{|y|\cdot|x|}\sinh 2|x|\sinh 2|y|),\\ \xi_3=\dfrac{1}{2}(\cosh 2|x|-\cosh 2|y|+\sqrt{-1}\dfrac{(y,\ x)}{|y|\cdot|x|}\sinh 2|x|\sinh 2|y|),\\ x_2=\dfrac{1}{2}\sinh 2|x|(\dfrac{x}{|x|}-\dfrac{2(y,\ x)y}{|y|^2\cdot|x|}\sinh^2|y|)+\sqrt{-1}\dfrac{y}{2|y|}\sinh 2|y|.\end{cases}$$

We put $a=\frac{1}{2}\sinh 2|x|\ (\frac{x}{|x|}-\frac{2(y,\ x)y}{|y|^2\cdot|x|}\sinh^2|y|)$ and $b=\frac{y}{2|y|}\sinh 2|y|$. If $a\neq rb$ for all $r\in\boldsymbol{R}^*=\boldsymbol{R}-\{0\}$, then there doesn't exist $s\in\boldsymbol{R}^*$ such that $x=sy$, and then we have $\frac{x}{|x|}-\frac{2(y,\ x)y}{|y|^2\cdot|x|}\sinh^2|y|\neq 0$. Therefore for any $b\in\mathfrak{C}$, when we move x for all points of $\mathfrak{C}$, the point a ranges over all points of $\mathfrak{C}-\{rb\,|\,r\in\boldsymbol{R}^*\}$. If $a=rb$ for some $r\in\boldsymbol{R}^*$, then there exists $s\in\boldsymbol{R}^*$ such that $x=sy$, and then we have $\frac{x}{|x|}-\frac{2(y,\ x)y}{|y|^2\cdot|x|}\sinh^2|y|=\frac{s\cdot y}{|s|\cdot|y|}(1-2\sinh^2|y|)$. Let $\sinh^2|y|=\frac{1}{2}$. Then there exists $\zeta\in\boldsymbol{R}^*$ such that $|b|=\zeta$. Therefore when we move x and y for all points of $\mathfrak{C}$, the point x_2 doesn't range at most over $\{rb+\sqrt{-1}b\,|\,r\in\boldsymbol{R}^*,\ b\in\mathfrak{C},\ |b|=\zeta\}$. For $x=sy$ and $w=ty$ ($y\in\mathfrak{C},\ s,\ t\in\boldsymbol{R}^*$), the F_2-component y_2 of $\exp(2\tilde{F}_2(w))\exp(\sqrt{-1}\tilde{A}_2(y))\exp(2\tilde{F}_2(x))E_1$ is given by

$$y_2=\frac{3sy}{2|sy|}(1-2\sinh^2|y|)\sinh^2|sy|+\frac{ty}{2|ty|}\cosh 2|sy|\sinh|ty|$$
$$+\sqrt{-1}\frac{y}{2|y|}\sinh 2|y|(1+2\sinh^2|ty|).$$

Therefore when we move $y\in\mathfrak{C},\ s,\ t\in\boldsymbol{R}^*$, the point y_2 ranges over all points of $\{rb+\sqrt{-1}b\,|\,r\in\boldsymbol{R}^*,\ b\in\mathfrak{C},\ |b|=\zeta\}$. Thus we have {the F_2-component of $hE_1\,|\,h\in H\}=\mathfrak{C}^{\boldsymbol{C}}$. Similarly we have {the E_1-component of $hE_1\,|\,h\in H\}=\boldsymbol{C}$. Therefore these imply {the F_2-component of $\exp(\sqrt{-1}\tilde{A}_2(y)+2\tilde{F}_2(x))E_1\,|\,x,y\in\mathfrak{C}\}=\mathfrak{C}^{\boldsymbol{C}}$. Thus the mapping u is onto.

Let $z_0=x_0+\sqrt{-1}y_0$ ($x_0,\ y_0\in\mathfrak{C}$) be an arbitrary point of $\mathfrak{C}^{\boldsymbol{C}}$ and fixed.

Lemma 11. *The mapping* $v(\ \ ,z_0)$ *is onto.*

Proof. The Lie subalgebra of $\mathfrak{e}_{6,\sigma}$ generated by $\{\sqrt{-1}\tilde{A}_3(y_0)+2\tilde{F}_3(x_0)\}$ is $\{\sqrt{-1}\tilde{A}_3(ty_0)+2\tilde{F}_3(sx_0)+\sqrt{-1}r(\widetilde{E_1-E_2})\,|\,r,\ s,\ t\in\boldsymbol{R}^*\}$. If x_0 and y_0 are both small enough, there exist $r,\ s,\ t\in\boldsymbol{R}^*$ such that

$$\exp(\sqrt{-1}\tilde{A}_3(y_0)+2\tilde{F}_3(x_0))=\exp(\sqrt{-1}\tilde{A}_3(ty_0))\exp(2\tilde{F}_3(sx_0))\exp(\sqrt{-1}r(\widetilde{E_1-E_2})).$$

By formal computation for $a \in \mathfrak{C}^C$ we have

$$\exp(\sqrt{-1}\widetilde{A}_3(ty_0)) \exp(2\widetilde{F}_3(sx_0))F_2(a)$$

$$= F_1\Big(\frac{sax_0}{|sx_0|}\sinh|sx_0|\cosh|ty_0| - \sqrt{-1}\frac{tay_0}{|ty_0|}\cosh|sx_0|\sinh|ty_0|\Big)$$

$$+ F_2\Big(a\cosh|sx_0|\cosh|ty_0| + \sqrt{-1}\frac{st(ax_0)\overline{y}_0}{|sx_0|\cdot|ty_0|}\sinh|sx_0|\sinh|ty_0|\Big).$$

If we put $a = a_1 + \sqrt{-1}a_2$ $(a_1, a_2 \in \mathfrak{C})$ and the above F_2–component $= b_1 + \sqrt{-1}b_2$ $(b_1, b_2 \in \mathfrak{C})$, we have

$$\begin{cases} b_1 = a_1 \cosh|sx_0|\cosh|ty_0| - \dfrac{st(a_2x_0)\overline{y}_0}{|sx_0|\cdot|ty_0|}\sinh|sx_0|\sinh|ty_0|, \\[2ex] b_2 = a_2 \cosh|sx_0|\cosh|ty_0| + \dfrac{st(a_1x_0)\overline{y}_0}{|sx_0|\cdot|ty_0|}\sinh|sx_0|\sinh|ty_0|. \end{cases}$$

Therefore these points $b_1 + \sqrt{-1}b_2$ range over all points of $\mathfrak{C}^C$ independent of x_0 and y_0, when points a move all over $\mathfrak{C}^C$. On the other hand, it holds that $\exp(\sqrt{-1}r(E_1-E_2)^{\sim})F_2(a) = e^{\frac{r}{2}\sqrt{-1}}F_2(a)$. Therefore points $v(a, z_0)$ range over all points of $\mathfrak{C}^C$. For not small x_0, $y_0 \in \mathfrak{C}$, there exist a large integer n and small numbers r, s, $t \in \boldsymbol{R}^*$ such that

$$\exp(\sqrt{-1}\widetilde{A}_3(y_0) + 2\widetilde{F}_3(x_0)) = (\exp(\sqrt{-1}\widetilde{A}_3(ty_0))\exp(2\widetilde{F}_3(sx_0))\exp(\sqrt{-1}r(E_1 - E_2)^{\sim}))^n.$$

Similarly as the above argument, points $v(a, z_0)$ range over all points of $\mathfrak{C}^C$ independent of z_0. Thus the mapping $v(\ \ , z_0)$ is onto.

For $z = x + \sqrt{-1}y$ $(x, y \in \mathfrak{C})$, $\xi(z)$ is a positive real number and satisfies $\xi(z) > |\eta(z)|$. Using the condition (6) we can put $\exp(\sqrt{-1}\widetilde{A}_3(y_3) + 2\widetilde{F}_3(x_3)) \cdot$ $\exp(\sqrt{-1}\widetilde{A}_2(y_2)+2\widetilde{F}_2(x_2))E_1$ by $\begin{pmatrix} \xi\eta & \eta x & \overline{y} \\ \eta\overline{x} & \eta\xi' & \frac{1}{\xi}\overline{yx} \\ y & \frac{1}{\xi}yx & \eta' \end{pmatrix}$. Moreover from (6), we have

$$\begin{cases} (x, x) = \xi\xi', & \xi^2 + |\xi'|^2 - 2\langle x, x\rangle = 1, \\[1ex] (y, y) = \xi\eta\eta', & \eta^2 + |\eta'|^2 - 2\langle y, y\rangle + \dfrac{2}{\xi^2}\langle yx, yx\rangle = 1. \end{cases}$$

These imply that ξ^2 is a solution of the quadratic equation :

$$X^2-(1+2\langle x,\ x\rangle)X+|(x,\ x)|^2=0,$$

and from $\xi>0$ and $\xi^2\geqq|\xi'|^2$ we have

$$(7)\quad \xi=\frac{1}{\sqrt{2}}\sqrt{1+2\langle x,\ x\rangle+\sqrt{(1+2\langle x,\ x\rangle^2-4|(x,\ x)|^2}},\ \left(\xi'=\frac{1}{\xi}(x,\ x)\right).$$

Similarly η^2 is a solution of the quadratic equation :

$$X^2-\left(1+2\langle y,\ y\rangle-\frac{2}{\xi^2}\langle yx,\ yx\rangle\right)X+\frac{1}{\xi^2}|(y,\ y)|^2=0,$$

and from $\eta>0$ and $\eta^2\geqq|\eta'|^2$ we have

$$(8)\quad \eta=\frac{1}{\sqrt{2}}\sqrt{1+2\langle y,\ y\rangle-\frac{2}{\xi^2}\langle yx,\ yx\rangle+\sqrt{(1+2\langle y,\ y\rangle-\frac{2}{\xi^2}\langle yx,\ yx\rangle)^2-\frac{4}{\xi^2}|(y,\ y)|^2}},$$

$$\left(\eta'=\frac{1}{\xi\eta}(y,\ y)\right).$$

Thus we have

Proposition 12. *The homogeneous space $E_{6,\sigma}/U(1)Spin(10)$ is homeomorphic to the space D :*

$$D=\left\{\begin{bmatrix}\xi\eta & \eta x & \bar{y}\\ \eta\bar{x} & \frac{\eta}{\xi}(x,\ x) & \frac{1}{\xi}\overline{xy}\\ y & \frac{1}{\xi}yx & \frac{1}{\xi\eta}(y,\ y)\end{bmatrix}\in[\mathfrak{J}_\sigma]\ \middle|\ x,\ y\in\mathfrak{C}^C,\ \xi \text{ and } \eta \text{ are given by (7), (8)}\right\}$$

Proof. From the Preceding arguments (Lemma 10 and 11), the group $E_{6,\sigma}$ acts on the space D transitively. The isotropy subgroup of $E_{6,\sigma}$ at $[E_1]\in D$ is $U(1)Spin(10)$. Thus $E_{6,\sigma}/U(1)Spin(10)$ is homeomorphic to D.

From now on, we identify $E_{6,\sigma}/U(1)Spin(10)$ with D, and introduce the differentiable and complex structure of $E_{6,\sigma}/U(1)Spin(10)$ into D.

§7. Harish-Chandra imbedding.

Let $\mathfrak{n}^C$ be the complexification of $\mathfrak{n}$. We shall decompose $\mathfrak{n}^C$ into the $(\pm\sqrt{-1})$-eigen spaces $\mathfrak{n}^{\pm}$ with respect to the complex structure J on $\mathfrak{n}$. Since this J is $-\frac{2}{3}\sqrt{-1}\mathrm{ad}(\widetilde{2E_1-E_2-E_3})$, for $\widetilde{A}_2(y_2)+\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x)\in\mathfrak{n}^C$ we have

$$J(\widetilde{A}_2(y_2)+\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x_3))=\sqrt{-1}(\widetilde{A}_2(x_2)-\widetilde{A}_3(x_3)+2\widetilde{F}_2(y_2)-2\widetilde{F}_3(y_3)).$$

This implies

$$\mathfrak{n}^+ = \{\widetilde{A}_2(y) + \widetilde{A}_3(x) + 2\widetilde{F}_2(y) - 2\widetilde{F}_3(x) \,|\, x,\ y \in \mathfrak{C}^C\},$$

$$\mathfrak{n}^- = \{\widetilde{A}_2(y) + \widetilde{A}_3(x) - 2\widetilde{F}_2(y) + 2\widetilde{F}_3(x) \,|\, x,\ y \in \mathfrak{C}^C\}.$$

We define a mapping $f: \mathfrak{n}^+ \longrightarrow [\mathfrak{J}_\times]$ by

$$f(N) = (\exp N)[E_1] = \begin{bmatrix} 1 & x & \bar{y} \\ \bar{x} & (x,\ x) & \bar{x}\bar{y} \\ y & yx & (y,\ y) \end{bmatrix}$$

where $N = \widetilde{A}_2(y) + \widetilde{A}_3(x) + 2\widetilde{F}_2(y) - 2\widetilde{F}_3(x) \in \mathfrak{n}^+$. Therefore f is an injection. Let ϕ be the natural mapping of $D = E_{6,\sigma}/U(1)Spin(10)$ into $[\mathfrak{J}_\times] = E_6{}^C/U$.
Then we have

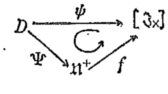
Lemma 13. $\phi(D) \subset f(\mathfrak{n}^+)$.

Proof. Let $X = \begin{bmatrix} \xi\eta & \eta x & \bar{y} \\ \eta\bar{x} & \dfrac{\eta}{\xi}(x,\ x) & \dfrac{1}{\xi}\bar{y}\bar{x} \\ y & \dfrac{1}{\xi}yx & \dfrac{1}{\xi\eta}(y,\ y) \end{bmatrix}$ be an arbitrary point of D.

Then we have $\phi(x) = \begin{bmatrix} 1 & \dfrac{1}{\xi}x & \dfrac{1}{\xi\eta}\bar{y} \\ \dfrac{1}{\xi}\bar{x} & \dfrac{1}{\xi^2}(x,\ x) & \dfrac{1}{\xi^2\eta}\bar{y}\bar{x} \\ \dfrac{1}{\xi\eta}y & \dfrac{1}{\xi^2\eta}yx & \dfrac{1}{\xi^2\eta^2}(y,\ y) \end{bmatrix} \in [\mathfrak{J}_\times]$. On the other hand, we have $f\Big(\widetilde{A}_2\Big(\dfrac{1}{\xi\eta}y\Big) + \widetilde{A}_3\Big(\dfrac{1}{\xi}x\Big) + 2\widetilde{F}_2\Big(\dfrac{1}{\xi\eta}y\Big) - 2\widetilde{F}_3\Big(\dfrac{1}{\xi}x\Big)\Big) = \phi(x)$. Thus $\phi(D) \subset f(\mathfrak{n}^+)$.

From the above Lemma, we can define a holomorphic imbedding $\Psi: D \longrightarrow \mathfrak{n}^+$ by

$$\phi(X) = f(\Psi(X))$$



for each $X \in D$ [2]. This imbedding Ψ is called a Harish-Chandra imbedding.

Lemma 14. *The imbedding Ψ is given by*

$$\Psi\begin{bmatrix} \xi\eta & \eta x & \bar{y} \\ \eta\bar{x} & \dfrac{\eta}{\xi}(x,\ x) & \dfrac{1}{\xi}\bar{y}\bar{x} \\ y & \dfrac{1}{\xi}yx & \dfrac{1}{\xi\eta}(y, y) \end{bmatrix} = \widetilde{A}_2\Big(\frac{1}{\xi\eta}y\Big) + \widetilde{A}_3\Big(\frac{1}{\xi}x\Big) + 2\widetilde{F}_2\Big(\frac{1}{\xi\eta}y\Big) - 2\widetilde{F}_3\Big(\frac{1}{\xi}x\Big).$$

Proof is similar to that of Lemma 13.

Let π be a natural mapping of $\mathfrak{n}^+$ onto $\mathfrak{C}^C\times\mathfrak{C}^C$ defined by

$$\pi(\tilde{A}_2(y)+\tilde{A}_3(x)+2\tilde{F}_2(y)-2\tilde{F}_3(x))=(x,\ y),$$

and denote the mapping $\pi\circ\Psi$ also by Ψ.

Theorem 15. *The imbedding Ψ maps D onto $D(V)$:*

$$D(V)=\left\{\left(\frac{x}{\xi},\ \frac{y}{\eta}\right)\in\mathfrak{C}^C\times\mathfrak{C}^C\,|\,x,\ y\in\mathfrak{C}^C,\right.$$

$$\xi=\frac{1}{\sqrt{2}}\sqrt{1+2\langle x,\ x\rangle+\sqrt{(1+2\langle x,\ x\rangle)^2-4|(x,\ x)|^2}},$$

$$\left.\eta=\frac{1}{\sqrt{2}}\sqrt{\xi^2+2\xi^2\langle y,\ y\rangle-2\langle yx,\ yx\rangle+\sqrt{(\xi^2+2\xi^2\langle y,\ y\rangle-2\langle yx,\ yx\rangle)^2-4\xi^2|(y,y)|^2}}\right\}$$

Moreover $D(V)$ is a bounded domain of $\mathfrak{C}^C\times\mathfrak{C}^C$, since the imbedding Ψ is holomorphic.

Proof. Let $X=\begin{bmatrix}\xi\eta & \eta x & \bar{y}\\ \eta\bar{x} & \frac{\eta}{\xi}(x,\ x) & \frac{1}{\xi}\overline{yx}\\ y & \frac{1}{\xi}yx & \frac{1}{\xi\eta}(y,y)\end{bmatrix}\in D$. From Lemma 14 we have

$\Psi(X)=\left(\frac{x}{\xi},\ \frac{y}{\xi\eta}\right)$. Now we denote $\xi\eta$ by η, so $\Psi(X)=\left(\frac{x}{\xi},\ \frac{y}{\eta}\right)\in D(V)$. Conversely let $(x,\ y)\in D(V)$. If we put $\lambda=(1+|(x,\ x)|^2+|(y,\ y)|^2+2\langle yx,\ yx\rangle-2\langle x,\ x\rangle-2\langle y,\ y\rangle)^{-\frac{1}{2}}$, then we have

$$\begin{bmatrix}\lambda & \lambda x & \lambda\bar{y}\\ \lambda\bar{x} & \lambda(x,\ x) & \lambda\overline{yx}\\ \lambda y & \lambda yx & \lambda(y,y)\end{bmatrix}\in D \text{ and } \Psi\begin{bmatrix}\lambda & \lambda x & \lambda\bar{y}\\ \lambda\bar{x} & \lambda(x,\ x) & \lambda\overline{yx}\\ \lambda y & \lambda yx & \lambda(y,y)\end{bmatrix}=(x,\ y).$$

Therefore $\Psi(D)=D(V)$.

§8. Symmetric structure of D and $D(V)$.

Any point $X\in D$ is represented by $(\exp N)[E_1]$ for some $N\in\mathfrak{n}$. For $N=\sqrt{-1}\tilde{A}_2(y_2)+\sqrt{-1}\tilde{A}_3(y_3)+2\tilde{F}_2(x_2)+2\tilde{F}_3(x_3)\in\mathfrak{n}$, we have

$$\lim_{t\to 0}\frac{1}{t}((\exp tN)E_1-E_1)=NE_1=F_2(x_2+\sqrt{-1}y_2)+F_3(x_3-\sqrt{-1}y_3).$$

Hence we can regard the space $\{F_2(x)+F_3(y) \mid x,\ y\in\mathfrak{C}^C\}$ as the tangent space D_1 of D at $[E_1]$. Therefore the mapping :

$$\mathfrak{n}\ni\sqrt{-1}\widetilde{A}_2(y_2)+\sqrt{-1}\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x_3)\longrightarrow F_2(x_2+\sqrt{-1}y_2)$$
$$+F_3(x_3-\sqrt{-1}y_3)\in D_1$$

gives a linear isomorphism of $\mathfrak{n}$ to D_1.

We define an inner product g_1 on D_1 by

$$g_1(X,\ Y)=6(\langle X,\ Y\rangle+\langle Y,\ X\rangle),\qquad X,\ Y\in D_1\subset\mathfrak{J}^C.$$

For $X=F_2(x_2+\sqrt{-1}y_2)+F_3(x_3-\sqrt{-1}y_3)$, $Y=F_2(x_2'+\sqrt{-1}y_2')+F_3(x_3'-\sqrt{-1}y_3')$ $\in D_1$ we have

$$g_1(X,\ Y)=48((x_2,\ x_2')+(x_3,\ x_3')+(y_2,\ y_2')+(y_3,\ y_3')),$$

hence using this g_1 we can define an Hermitian metric $\bar{g}$ on D (Lemma 4).

Let X' be a representative element of the class $X\in D$. We define a transformation $s_1 : D\longrightarrow D$ by $s_1(X)=[\sigma X']$. For any $X=(\exp N)[E_1]\in D$ $(N\in\mathfrak{n})$, we have

$$s_1((\exp N)[E_1])=[\sigma(\exp N)E_1]=\sigma(\exp N)\sigma[E_1]=\boldsymbol{\sigma}(\exp N)[E_1].$$

Therefore s_1 is a symmetry at the point $[E_1]$ (Lemma 4). For any $X=(\exp N_0)[E_1]$ $\in D$ $(N_0\in\mathfrak{n})$, we define a transformation s_X of D by

$$s_X((\exp N)[E_1])=(\exp 2N_0)(\exp(-N))[E_1],$$

then s_X is a symmetry at the point X. In fact, for $(\exp N)[E_1]\in D$ we have

$$(\exp N_0)s_1(\exp(-N_0))(\exp N)[E_1]=(\exp N_0)\sigma(\exp(-N_0))\sigma\sigma(\exp N)\sigma[E_1]$$
$$=(\exp N_0)(\exp N_0)(\exp(-N))[E_1]=s_X((\exp N)[E_1]),$$

so s_X is a symmetry at X (Lemma 4).

Thus we have following

Theorem 16. *$(D,\ \bar{g})$ is a non-compact Hermitian symmetric space of type E_6.*

Remark. The compact dual space of D is $[\mathfrak{J}_1]=E_6/U(1)Spin(10)$.

From the symmetric structure of $(D,\ \bar{g})$ we can induce a symmetric structure of $D(V)$ using the imbedding Ψ.

Now we shall consider the symmetric structure only at the origin of $D(V)$. For $N=\sqrt{-1}\widetilde{A}_2(y_2)+\sqrt{-1}\widetilde{A}_3(y_3)+2\widetilde{F}_2(x_2)+2\widetilde{F}_3(x_3)\in\mathfrak{n}$, we have

$$\lim_{t\to 0}\frac{1}{t}(\Psi((\exp tN)[E_1])-\Psi([E_1]))=(x_3-\sqrt{-1}y_3,\ x_2+\sqrt{-1}y_2).$$

Hence we can regard the space $\{(x,\ y)\in\mathfrak{C}^C\times\mathfrak{C}^C \mid x,\ y\in\mathfrak{C}^C\}$ as the tangent space $D(V)_0$ of $D(V)$ at 0. Therefore the mapping :

$\mathfrak{n} \ni \sqrt{-1}\tilde{A}_2(y_2) + \sqrt{-1}\tilde{A}_3(y_3) + 2\tilde{F}_2(x_2) + 2\tilde{F}_3(x_3) \longrightarrow (x_3 - \sqrt{-1}y_3,\ x_2 + \sqrt{-1}y_2) \in D(V)_0$

gives a linear isomorphism of $\mathfrak{n}$ to $D(V)_0$.

Let $\tilde{g}$ be the Bergman metric on $D(V)$ and $\tilde{g}_0$ the restriction of $\tilde{g}$ on $D(V)_0$. Let B be the Killing form of the Lie algebra $\mathfrak{e}_{6,\sigma}$. Then from [3] P. 397 we have $\tilde{g}_0 = \frac{1}{2} B|\mathfrak{n}$. On the other hand, from Proposition 2 $B|\mathfrak{n}$ is given by

$$B(N_1,\ N_2) = 96\left(\left(y_2^1,\ y_2^2\right) + \left(y_3^1,\ y_3^2\right) + \left(x_2^1,\ x_2^2\right) + \left(x_3^1,\ x_3^2\right)\right)$$

where $N_i = \sqrt{-1}\tilde{A}_2\left(y_2^i\right) + \sqrt{-1}\tilde{A}_3\left(y_3^i\right) + 2\tilde{F}_2\left(x_2^i\right) + 2\tilde{F}_3\left(x_3^i\right) \in \mathfrak{n}\ \ (i=1,\ 2)$.

Therefore for $(x_i,\ y_i) \in D(V)_0\ (i=1,\ 2)$ $\tilde{g}_0$ is given by

$$g_0((x_1,\ y_1),\ (x_2,\ y_2)) = 12(\langle x_1,\ x_2\rangle + \langle x_2,\ x_1\rangle + \langle y_1,\ y_2\rangle + \langle y_2,\ y_1\rangle).$$

This implies that the metric induced by $\tilde{g}$ using the imbedding Ψ coinside with $\tilde{g}$.

Let s_0 be the symmetry of $D(V)$ at 0 induced by $(D,\ \tilde{g})$ using the imbedding Ψ. For any point $(x,\ y) \in D(V)$ there exists $X \in D$ such that $\Psi(X) = (x,\ y)$ (Theorem 15). Therefore we have

$$s_0(x,\ y) = \Psi(s_1(X)) = \Psi([\sigma X']) = (-x,\ -y).$$

Thus we have following

Theorem 17. $D(V) = \left\{\left(\frac{x}{\xi},\ \frac{y}{\eta}\right) \in \mathfrak{C}^C \times \mathfrak{C}^C \,|\, x,\ y \in \mathfrak{C}^C,\right.$

$$\xi = \frac{1}{\sqrt{2}}\sqrt{1 + 2\langle 2x,\ x\rangle + \sqrt{(1+2\langle x,\ x\rangle)^2 - 4|(x,\ x)|^2}},$$

$$\left.\eta = \frac{1}{\sqrt{2}}\sqrt{\xi^2 + 2\xi^2\langle y,\ y\rangle - 2\langle yx,\ yx\rangle + \sqrt{(\xi^2 + 2\xi^2\langle y,\ y\rangle - 2\langle yx,\ yx\rangle)^2 - 4\xi^2|(y,\ y)|^2}}\right\}$$

is an irreducible bounded symmetric domain of type E_6. In particular, the restriction $\tilde{g}_0 = \tilde{g}|D(V)_0$ of the Bergman metric $\tilde{g}$ on $D(V)$ and the symmetry s_0 of $D(V)$ at $0 \in D(V)$ are given respectively by

$$g_0((x_1,\ y_1),\ (x_2,\ y_2)) = 12(\langle x_1,\ x_2\rangle + \langle x_2,\ x_1\rangle + \langle y_1,\ y_2\rangle + \langle y_2,\ y_1\rangle),\quad (x_i,\ y_i) \in D(V)_0,$$

$$s_0(x,\ y) = (-x,\ -y), \qquad (x,\ y) \in D(V).$$

III. Bounded symmetric domain of type E_7.

§9. Hermitian symmetric pair of $E_{7,\iota}$.

We define a linear transformation ι of $\mathfrak{P}^C$ by

$$\iota(X, Y, \xi, \eta)=(X, -Y, \xi, -\eta),$$

and define an involutive automorphism $\boldsymbol{\iota}$ of the group $E_{7,\iota}$ (which is a Cartan involution) by

$$\boldsymbol{\iota}\alpha=\iota\alpha\iota, \qquad \alpha\in E_{7,\iota}.$$

The decomposition $\mathfrak{e}_{7,\iota}=\mathfrak{k}\oplus\mathfrak{n}$ as in §4 with respect to $\boldsymbol{\iota}$ is given by

$$\mathfrak{k}=\{\Phi(\phi, 0, 0, \rho)\in\mathfrak{e}_{7,\iota} \mid \phi\in\mathfrak{e}_6, \ \rho\in\boldsymbol{C}, \ \rho+\bar{\rho}=0\},$$

$$\mathfrak{n}=\{\Phi(0, A, \bar{A}, 0)\in\mathfrak{e}_{7,\iota} \mid A\in\mathfrak{J}^C\}.$$

We denote the element $\Phi(0, A, \bar{A}, 0)\in\mathfrak{e}_{7,\iota}$ by $\Phi(A)$ briefly. We define an inner product g on $\mathfrak{n}$ by

$$g(\Phi(A), \Phi(B))=\langle A, B\rangle+\langle B, A\rangle,$$

and a linear transformation J of $\mathfrak{n}$ by

$$J=\mathrm{ad}\,\Phi(0, 0, 0, -\frac{3}{2}\sqrt{-1}).$$

Therefore for each $\Phi(A)\in\mathfrak{n}$ we have

$$J(\Phi(A))=[\Phi(0, 0, 0, -\frac{3}{2}\sqrt{-1}), \Phi(0, A, \bar{A}, 0)]=-\sqrt{-1}\Phi(A),$$

so J is a complex structure on $\mathfrak{n}$.

Proposition 18. *$(E_{7,\iota}, U(1)E_6; \boldsymbol{\iota}, g, J)$ is an Hermitian symmetric pair of the group $E_{7,\iota}$.*

Proof. We shall check the conditions of Definition in §4. In [5] Proposition 12, we have seen $\{\alpha\in E_{7,\iota} \mid \iota\alpha\iota=\alpha\}=U(1)E_6$. Now obviously conditions (1), (2) and (3) are satisfied. Instead of the first condition (4), it suffices to show that the inner product g is ad$\mathfrak{k}$-invariant. For $\Phi(A)$, $\Phi(B)\in\mathfrak{n}$ and $\Phi(\phi, 0, 0, \rho)\in\mathfrak{k}$ we have

$$g([\Phi(\phi, 0, 0, \rho), \Phi(A)], \Phi(B))+g(\Phi(A), [\Phi(\phi, 0, 0, \rho), \Phi(B)])$$

$$=g(\Phi(\phi A+\frac{2}{3}\rho A), \Phi(B))+g(\Phi(A), \Phi(\phi B+\frac{2}{3}\rho B))$$

$$=\langle\phi A+\frac{2}{3}\rho A, B\rangle+\langle B, \phi A+\frac{2}{3}\rho A\rangle+\langle A, \phi B+\frac{2}{3}\rho B\rangle+\langle\phi B+\frac{2}{3}\rho B, A\rangle$$

$$=\langle\phi A, B\rangle+\langle A, \phi B\rangle+\langle B, \phi A\rangle+\langle\phi B, A\rangle=0,$$

so g is ad$\mathfrak{k}$-invariant. And for $\Phi(A)$, $\Phi(B)\in\mathfrak{n}$, we have

$$g(J\Phi(A),\ J\Phi(B))=g(-\sqrt{-1}\Phi(A),\ -\sqrt{-1}\Phi(B))$$
$$=\langle -\sqrt{-1}A,\ -\sqrt{-1}B\rangle+\langle -\sqrt{-1}B,\ -\sqrt{-1}A\rangle$$
$$=\langle A,\ B\rangle+\langle B,\ A\rangle=g(\Phi(A),\ \Phi(B)),$$

and for $\Phi(\phi,\ 0,\ 0,\ \rho)\in\mathfrak{k}$ and $\Phi(A)\in\mathfrak{n}$

$$J\operatorname{ad}\Phi(\phi,\ 0,\ 0,\ \rho)\Phi(A)=J\Phi(\phi A+\frac{2}{3}\rho A)=\Phi(-\sqrt{-1}(\phi A+\frac{2}{3}\rho A))$$
$$=\operatorname{ad}\Phi(\phi,\ 0,\ 0,\ \rho)J\Phi(A).$$

Hence the condition (4) is satisfied. Thus the proof is completed.

From Lemma 4 and Proposition 18, we see that the homogeneous space $E_{7,\iota}/U(1)E_6$ has a structure of an Hermitian symmetric space.

§10. Realization of the symmetric space $E_{7,\iota}/U(1)E_6$.

The space $[\mathfrak{M}_1]$ has a differentiable structure induced by that of the manifold $\mathfrak{M}_1$, because on the manifold $\mathfrak{M}_1$ the group $U(1)$ acts freely.

Proposition 19. *The homogeneous space $E_7/U(1)E_6$ is diffeomorphic to the manifold $[\mathfrak{M}_1]$.*

Proof. From [4] Theorem 9, the group E_7 acts on the manifold $\mathfrak{M}_1$ transitively (and differentiably). On the other hand, the isotropy subgroup of E_7 at $[1]\in[\mathfrak{M}_1]$ is $U(1)E_6$. Thus $E_7/U(1)E_6$ is diffeomorphic to $[\mathfrak{M}_1]$.

Lemma 20. *The group $E_7{}^C$ acts on the space $[\mathfrak{M}^C]$ transitively. Let U be the isotropy subgroup of $E_7{}^C$ at $[1]\in[\mathfrak{M}^C]$. Then the homogeneous space $E_7{}^C/U$ is homeomorphic to the space $[\mathfrak{M}^C]$.*

Proof is similar to that of [5] Theorem 7.

From now on, we identify $E_7{}^C/U$ with $[\mathfrak{M}^C]$ and introduce the differentiable and complex structure of $E_7{}^C/U$ into $[\mathfrak{M}^C]$.

Now, we shall realize the symmetric space $E_{7,\iota}/U(1)E_6$. Any element of the group $E_{7,\iota}$ leaves the manifold $\mathfrak{M}^C$ and the inner product $\langle P,\ Q\rangle_\iota$ invariant. Therefore $E_{7,\iota}$ acts on the space $\mathfrak{M}_\iota$ and $[\mathfrak{M}_\iota]$ (however not transitively).

For $a\in C$, we define an element $\alpha_1(a)$ of $E_{7,\iota}$ by

$$\alpha_1(a)=\begin{pmatrix} 1+(\cosh|a|-1)p_1 & 2\bar{a}\dfrac{\sinh|a|}{|a|}E_1 & 0 & a\dfrac{\sinh|a|}{|a|}E_1 \\ 2a\dfrac{\sinh|a|}{|a|}E_1 & 1+(\cosh|a|-1)p_1 & \bar{a}\dfrac{\sinh|a|}{|a|}E_1 & 0 \\ 0 & a\dfrac{\sinh|a|}{|a|}E_1 & \cosh|a| & 0 \\ \bar{a}\dfrac{\sinh|a|}{|a|}E_1 & 0 & 0 & \cosh|a| \end{pmatrix}$$

$= \exp \Phi(aE_1)$

where the mapping $p_1 : \mathfrak{J}^C \longrightarrow \mathfrak{J}^C$ is defined by

$$p_1 \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}$$

and the action of $\alpha_1(a)$ on $\mathfrak{P}^C$ is defined as similar to that of $\Phi(aE_1)$. Similarly we can define elements $\alpha_2(a)$, $\alpha_3(a)$ of $E_{7,\iota}$ [5].

In order to find a realization of $E_{7,\iota}/U(1)E_6$, we prepare a few Lemmas.

Lemma 21. *The isotropy subgroup of the group $E_{7,\iota}$ at $[1] \in [\mathfrak{M}_\iota]$ is $U(1)E_6$.*

Proof. From [5] Theorem 5, we have $E_{7,\iota} = U(1)E_6 \exp(\mathfrak{n})$, i. e., any $\alpha \in E_{7,\iota}$ has the form

$$\alpha = \theta\beta \exp \Phi(A), \qquad \theta \in U(1),\ \beta \in E_6,\ A \in \mathfrak{J}^C.$$

Since $A \in \mathfrak{J}^C$ can be transformed in a diagonal form by a certain element $\beta' \in E_6$:

$$\beta' A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad a_i \in \boldsymbol{C}, \text{ we have } \alpha = \theta\beta\beta'^{-1}\alpha_1(a_1)\alpha_2(a_2)\alpha_3(a_3)\beta'.$$

Therefore we have

$$\alpha[1] = \theta\beta\beta'^{-1}\alpha_1(a_1)\alpha_2(a_2)\alpha_3(a_3)\beta'[1] = \theta\beta\beta'^{-1}\alpha_1(a_1)\alpha_2(a_2)\alpha_3(a_3)[1]$$

$$= \Bigg[\theta^{-1}\beta\beta'^{-1} \begin{pmatrix} \cosh|a_1|\bar{a}_2\frac{\sinh|a_2|}{|a_2|}\bar{a}_3\frac{\sinh|a_3|}{|a_3|} & 0 & 0 \\ 0 & \bar{a}_1\frac{\sinh|a_1|}{|a_1|}\cosh|a_2|\bar{a}_3\frac{\sinh|a_3|}{|a_3|} & 0 \\ 0 & 0 & \bar{a}_1\frac{\sinh|a_1|}{|a_1|}\bar{a}_2\frac{\sinh|a_2|}{|a_2|}\cosh|a_3| \end{pmatrix}$$

$$+ \theta\tau\beta\beta'^{-1} \begin{pmatrix} a_1\frac{\sinh|a_1|}{|a_1|}\cosh|a_2|\cosh|a_3| & 0 & 0 \\ 0 & \cosh|a_1|a_2\frac{\sinh|a_2|}{|a_2|}\cosh|a_3| & 0 \\ 0 & 0 & \cosh|a_1|\cosh|a_2|a_3\frac{\sinh|a_3|}{|a_3|} \end{pmatrix}^{\bullet}$$

$$+ \theta^3\cosh|a_1|\cosh|a_2|\cosh|a_3| + \left(\theta^{-3}\bar{a}_1\frac{\sinh|a_1|}{|a_1|}\bar{a}_2\frac{\sinh|a_2|}{|a_2|}\bar{a}_3\frac{\sinh|a_3|}{|a_3|}\right)^{\bullet}\Bigg].$$

If $\alpha[1] = [1]$, then we have $a_1 = a_2 = a_3 = 0$. Hence $\alpha = \theta\beta \in U(1)E_6$. Conversely let $\alpha \in U(1)E_6$, then we have $\alpha[1] = [1]$.

Lemma 22. *The group $E_{7,\iota}$ acts transitively on D*:

$$D=\{[X,\ Y,\ \xi,\ \eta]\in[\mathfrak{M}_\iota]\mid |\langle Y,\ V\rangle|<|\xi| \text{ for all } V\in\mathfrak{J}_1\}.$$

Proof. Let $P=[X,\ Y,\ \xi,\ \eta]\in D$. From the definition of D, we have $\xi\neq 0$, hence $P=[\frac{1}{\xi}Y\times Y,\ Y,\ \xi,\ \frac{1}{\xi^2}\det Y]$. Transforming Y in a diagonal form $\eta_1E_1+\eta_2E_2+\eta_3E_3$ ($\eta_i\in \boldsymbol{C}$) by a certain element $\tau\beta\tau\in E_6$, we have

$$\beta P=\left[\ \frac{1}{\xi}\begin{pmatrix}\eta_2\eta_3 & 0 & 0\\ 0 & \eta_3\eta_1 & 0\\ 0 & 0 & \eta_1\eta_2\end{pmatrix}+\begin{pmatrix}\eta_1 & 0 & 0\\ 0 & \eta_2 & 0\\ 0 & 0 & \eta_3\end{pmatrix}^{\circ}+\xi+\left(\frac{1}{\xi^2}\eta_1\eta_2\eta_3\right)^{\circ}\ \right].$$

Therefore $\beta P\in[\mathfrak{M}_\iota]$ implies

$$\left(1-\frac{|\eta_1|^2}{|\xi|^2}\right)\left(1-\frac{|\eta_2|^2}{|\xi|^2}\right)\left(1-\frac{|\eta_3|^2}{|\xi|^2}\right)=\frac{1}{|\xi|^2}. \qquad \text{(i)}$$

On the other hand, Y and ξ satisfies the condition $|\langle Y,\ V\rangle|<|\xi|$ for all $V\in\mathfrak{J}_1$. Hence we have $|\langle\eta_1E_1+\eta_2E_2+\eta_3E_3,\ V\rangle|<|\xi|$ for all $V\in\mathfrak{J}_1$ (Proposition 6), especially $|\langle\eta_1E_1+\eta_2E_2+\eta_3E_3,\ E_i\rangle|=|\eta_i|<|\xi|$ for $i=1,\ 2,\ 3$. Now we can put $\frac{\eta_i}{\xi}=\frac{\bar a_i}{|a_i|}\tanh|a_i|$ for some $a_i\in\boldsymbol{C}$, $i=1,\ 2,\ 3$. This and (i) imply $|\xi|=\cosh|a_1|\cosh|a_2|\cosh|a_3|$. Therefore we have

$$P=\left[\ \beta^{-1}\begin{pmatrix}\cosh|a_1|\bar a_2\frac{\sinh|a_2|}{|a_2|}\bar a_3\frac{\sinh|a_3|}{|a_3|} & 0 & 0\\ 0 & \bar a_1\frac{\sinh|a_1|}{|a_1|}\cosh|a_2|\bar a_3\frac{\sinh|a_3|}{|a_3|} & 0\\ 0 & 0 & \bar a_1\frac{\sinh|a_1|}{|a_1|}\bar a_2\frac{\sinh|a_2|}{|a_2|}\cosh|a_3|\end{pmatrix}\right.$$

$$+\tau\beta^{-1}\begin{pmatrix}a_1\frac{\sinh|a_1|}{|a_1|}\cosh|a_2|\cosh|a_3| & 0 & 0\\ 0 & \cosh|a_1|a_2\frac{\sinh|a_2|}{|a_2|}\cosh|a_3| & 0\\ 0 & 0 & \cosh|a_1|\cosh|a_2|a_3\frac{\sinh|a_3|}{|a_3|}\end{pmatrix}^{\circ}$$

$$\left.+\cosh|a_1|\cosh|a_2|\cosh|a_3|+\left(\bar a_1\frac{\sinh|a_1|}{|a_1|}\bar a^2\frac{\sinh|a_2|}{|a_2|}\bar a_3\frac{\sinh|a_3|}{|v_3|}\right)^{\circ}\ \right]$$

$$=\beta^{-1}\alpha_1(a_1)\alpha_2(a_2)\alpha_3(a_3)[1].$$

Conversely let $\alpha\in E_{7,\iota}$. $\alpha[1]$ has a form appeared in the proof of Lemma 21 and we denote it by $[\theta^{-1}\beta\beta'^{-1}X,\ \theta\tau\beta\beta'^{-1}Y,\ \xi,\ \eta]$ briefly. Hence this implies $|\langle\theta\tau\beta\beta'^{-1}Y, V\rangle| = |\langle Y,\ \beta'\beta^{-1}V\rangle| \leq \max\ (\sinh|a_1|\cosh|a_2|\cosh|a_3|,\ \cosh|a_1|\sinh|a_2|\cosh|a_3|,\ \cosh|a_1|\cosh|a_2|\sinh|a_3|) < \cosh|a_1|\cosh|a_2|\cosh|a_3| = |\xi|$ for all $V \in \mathfrak{J}_1$. Therefore $\alpha[1] \in D$. Thus Lemma 22 is proved.

Thus we have

Proposition 23. *The homogeneous space $E_{7,\iota}/U(1)E_6$ is homeomorphic to the space $D = \{[X,\ Y,\ \xi,\ \eta] \in [\mathfrak{M}_\iota] \mid |\langle Y,\ V\rangle| < |\xi| \text{ for all } V \in \mathfrak{J}_1\}$.*

Proof. The group $E_{7,\iota}$ acts transitively on D (lemma 22) and its isotropy subgroup of $E_{7,\iota}$ at $[1] \in D$ is $U(1)E_6$ (Lemma 21). Therefore the homogeneous space $E_{7,\iota}/U(1)E_6$ is homeomorphic to D.

From now on, we identify $E_{7,\iota}/U(1)E_6$ with D and introduce the differentiable and complex structure of $E_{7,\iota}/U(1)E_6$ into D.

§11. Harish-Chandra imbedding.

Let $\mathfrak{n}^C$ be the complexification of $\mathfrak{n}$. We shall decompose $\mathfrak{n}^C$ into the $(\pm\sqrt{-1})$-eigen spaces $\mathfrak{n}^\pm$ with respect to the complex structure J on $\mathfrak{n}$. Since this J is $\operatorname{ad}\Phi(0,\ 0,\ 0,\ -\frac{3}{2}\sqrt{-1})$, for $\Phi(0,\ A,\ B,\ 0) \in \mathfrak{n}^C$ we have

$$J\Phi(0,\ A,\ B,\ 0) = \Phi(0,\ -\sqrt{-1}A,\ \sqrt{-1}B,\ 0).$$

This implies $\mathfrak{n}^+ = \{\Phi(0,\ 0,\ B,\ 0) \in \mathfrak{e}_7{}^C \mid B \in \mathfrak{J}^C\}$ and $\mathfrak{n}^- = \{\Phi(0,\ A,\ 0,\ 0) \in \mathfrak{e}_7{}^C \mid A \in \mathfrak{J}^C\}$.

We define a mapping $f:\ \mathfrak{n}^+ \longrightarrow [\mathfrak{M}^C]$ by

$$f(\Phi(0,\ 0,\ B,\ 0)) = (\exp\Phi(0,\ 0,\ B,\ 0))[1] = [B\times B,\ B,\ 1,\ \det B].$$

Hence f is an injection. Let ϕ be the natural mapping of $D = E_{7,\iota}/U(1)E_6$ into $[\mathfrak{M}^C] = E_7{}^C/U$. Then we have following

Lemma 24. $\phi(D) \subset f(\mathfrak{n}^+)$.

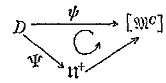
Proof. For any $P = [X,\ Y,\ \xi,\ \eta] = [\frac{1}{\xi}Y\times Y,\ Y,\ \xi,\ \frac{1}{\xi^2}\det Y] \in D$, we have

$$\phi(P) = [\frac{1}{\xi}Y\times Y,\ Y,\ \xi,\ \frac{1}{\xi^2}\det Y] = [\frac{1}{\xi^2}Y\times Y,\ \frac{1}{\xi}Y,\ 1,\ \frac{1}{\xi^3}\det Y] = f(\Phi(0,\ 0,\ \frac{1}{\xi}Y,\ 0)).$$

Thus $\phi(D) \subset f(\mathfrak{n}^+)$.

From the above Lemma, we can define a holomorphic imbedding $\Psi:\ D \longrightarrow \mathfrak{n}^+$ by

$$\phi(P) = f(\Psi(P))$$



for each $P \in D$ [2]. This imbedding Ψ is called a Harish-Chandra imbedding.

Lemma 25. *The imbedding Ψ is given by*

$$\Psi([X,\ Y,\ \xi,\ \eta]) = \Phi(0,\ 0,\ \frac{1}{\xi}Y,\ 0).$$

Proof is similar to that of Lemma 24.

Let π be a natural mapping of $\mathfrak{n}^+$ onto $\mathfrak{J}^C$ defined by $\pi(\Phi(0,\ 0,\ B,\ 0)) = B$, and denote the mapping $\pi\circ\Psi$ also by Ψ.

Theorem 26. *The imbedding Ψ maps D onto $D(VI)$:*

$$D(VI) = \{Z \in \mathfrak{J}^C |\ |\langle Z,\ V\rangle| < 1 \text{ for all } V \in \mathfrak{J}_1\}.$$

Moreover $D(VI)$ is a bounded domain of $\mathfrak{J}^C$, since the imbedding Ψ is holomorphic.

Proof. Let $P = [\frac{1}{\xi} Y \times Y,\ Y,\ \xi,\ \frac{1}{\xi^2}\det Y] \in D$. Then it holds

$$|\langle Y,\ V\rangle| < |\xi| \qquad \text{for all } V \in \mathfrak{J}_1.$$

This implies

$$\Psi(P) = \frac{1}{\xi}Y, \qquad |\langle \frac{1}{\xi}Y,\ V\rangle| < 1 \text{ for all } V \in \mathfrak{J}_1.$$

Therefore $\Psi(P) \in D(VI)$. Conversely let $Z \in D(VI)$. Transforming Z in a diagonal form $\beta Z = \zeta_1 E_1 + \zeta_2 E_2 + \zeta_3 E_3$ ($\zeta_i \in \boldsymbol{C}$) by a certain element $\beta \in E_6$, we have

$$\begin{aligned}
&\langle Z \times Z,\ Z \times Z\rangle - \langle Z,\ Z\rangle + 1 - |\det Z|^2 \\
&= \langle \beta Z \times \beta Z,\ \beta Z \times \beta Z\rangle - \langle \beta Z,\ \beta Z\rangle + 1 - |\det \beta Z|^2 \\
&= (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)(1 - |\zeta_3|^2).
\end{aligned}$$

From Proposition 6, $Z \in D(VI)$ implies $|\zeta_i| < 1$ for $i = 1,\ 2,\ 3$. Therefore we have

$$0 < \langle Z \times Z,\ Z \times Z\rangle - \langle Z,\ Z\rangle + 1 - |\det Z|^2 \leqq 1$$

If we put $\xi = \left(\langle Z \times Z,\ Z \times Z\rangle - \langle Z,\ Z\rangle + 1 - |\det Z|^2\right)^{-\frac{1}{2}}$ and $P = [\xi Z \times Z,\ \xi Z,\ \xi,\ \xi \det Z]$, then we have $P \in D$ and $\Psi(P) = Z$. Therefore $\Psi(D) = D(VI)$.

§12. Symmetric structure of D and $D(VI)$.

Any point $P \in D$ is represented by $(\exp \Phi(A))[1]$ for some $A \in \mathfrak{J}^C$. For $\Phi(A) \in \mathfrak{n}$, we have

$$\lim_{t \to 0} \frac{1}{t}((\exp t\Phi(A))1 - 1) = \Phi(A)1 = (0,\ \bar{A},\ 0,\ 0).$$

Hence we can regard the space $\{(0,\ X,\ 0,\ 0) \in \mathfrak{P}^C | X \in \mathfrak{J}^C\}$ as the tangent space D_1 of D at $[1]$. Therefore the mapping :

$$\mathfrak{n} \ni \Phi(A) \longrightarrow (0,\ \bar{A},\ 0,\ 0) \in D_1$$

gives a linear isomorphism of $\mathfrak{n}$ to D_1.

We define an inner product g_1 on D_1 by

$$g_1((0,\ X,\ 0,\ 0),\ (0,\ Y,\ 0,\ 0)) = 18(\langle X,\ Y\rangle + \langle Y,\ X\rangle).$$

Using this g_1 we can define an Hermitian metric $\bar{g}$ on D (Lemma 4).

Let P' be a representative element of the class $P \in D$. We define a transformation $s_1 : D \longrightarrow D$ by $s_1(P) = [\iota P']$. For any $P = (\exp\Phi(A))[1] \in D$ ($A \in \mathfrak{J}^C$), we have

$$s_1((\exp\Phi(A))[1]) = [\iota(\exp(A))1] = \iota(\exp\Phi(A))\iota[1] = \iota(\exp\Phi(A))[1].$$

Therefore s_1 is a symmetry at the point $[1]$ (Lemma 4). For any $P = (\exp\Phi(A))[1] \in D$, we define a transformation s_P of D by

$$s_P((\exp\Phi(B))[1]) = (\exp\Phi(2A))(\exp\Phi(-B))[1],$$

then s_P is a symmetry at $P \in D$. In fact, for $(\exp\Phi(B))[1] \in D$ we have

$$(\exp\Phi(A))s_1(\exp\Phi(-A))(\exp\Phi(B))[1] = (\exp\Phi(A))\iota(\exp\Phi(-A))\iota\iota(\exp\Phi(B))\iota[1]$$
$$= (\exp\Phi(2A))(\exp\Phi(-B))[1] = s_P((\exp\Phi(B))[1]),$$

so s_P is a symmetry at P (Lemma 4).

Thus we have following

Theorem 27. *$(D,\ \bar{g})$ is a non-compact Hermitian symmetric space of type E_7.*

Remark. The compact dual space of D is $[\mathfrak{M}_1] = E_7/U(1)E_6$.

From the symmetric structure of $(D,\ \bar{g})$ we can induce a symmetric structure of $D(VI)$ using the imbedding Ψ.

Now we shall consider the symmetric structure only at the origin of $D(VI)$. For $A \in \mathfrak{J}^C$, A is transformed in a diagonal form $\beta A = a_1E_1 + a_2E_2 + a_3E_3$, $\beta \in E_6$ ($a_i \in \boldsymbol{C}$). Hence we have for $t \in \boldsymbol{R}$

$$\Psi((\exp t\Phi(A))[1]) = \tau\beta^{-1}\Big(\frac{a_1}{|a_1|}\tanh t|a_1|E_1 + \frac{a_2}{|a_2|}\tanh t|a_2|E_2 + \frac{a_3}{|a_3|}\tanh t|a_3|E_3\Big).$$

Therefore this implies

$$\lim_{t\to 0}\frac{1}{t}(\Psi((\exp t\Phi(A))[1]) - \Psi([1])) = \bar{A},$$

and we can regard the space $\mathfrak{J}^C$ as the tangent space $D(VI)_0$ of $D(VI)$ at 0. Hence the mapping :

$$\mathfrak{n} \in \Phi(A) \longrightarrow \bar{A} \in D(VI)_0$$

gives a linear isomorphism of $\mathfrak{n}$ to $D(VI)_0$.

Let $\tilde{g}$ be the Bergman metric on $D(VI)$ and $\tilde{g}_0$ the restriction of $\tilde{g}$ on $D(VI)_0$. Let B be the Killing form of the Lie algebra $\mathfrak{e}_{7,\iota}$. Then from [3] P. 397 we have $\tilde{g}_0 = \frac{1}{2} B|\mathfrak{n}$. On the other hand, from Proposition 3, $B|\mathfrak{n}$ is given by

$$B(\Phi(A),\ \Phi(B)) = 36(\langle A,\ B\rangle + \langle B,\ A\rangle).$$

Therefore for $X,\ Y \in D(VI)_0$ g_0 is given by

$$g_0(X,\ Y) = 18(\langle X,\ Y\rangle + \langle Y,\ X\rangle).$$

This implies that the metric induced by $\bar{g}$ using the imbedding Ψ coinside with $\tilde{g}$.

Let $\tilde{s}_0$ be the symmetry of $D(VI)$ at 0 induced by $(D,\ \bar{g})$ using the imbedding Ψ. For any point $Z \in D(VI)$, there exists $P \in D$ such that $\Psi(P) = Z$ (Theorem 26). Hence we have

$$s_0(Z) = \Psi(s_1(P)) = \Psi([\iota P']) = -Z.$$

Thus we have following

Theorem 28. $D(VI) = \{Z \in \mathfrak{J}^C \mid |\langle Z,\ V\rangle| < 1 \text{ for all } V \in \mathfrak{J}_1\}$ *is an irreducible bounded symmetric domain of type* E_7. *In particular, the restriction* $\tilde{g}_0 = \tilde{g}|D(VI)_0$ *of the Bergman metric* $\tilde{g}$ *on* $D(VI)$ *and the symmetry* $\tilde{s}_0$ *of* $D(VI)$ *at* $0 \in D(VI)$ *are given respectively by*

$$\tilde{g}_0(X,\ Y) = 18(\langle X,\ Y\rangle + \langle Y,\ X\rangle), \qquad X,\ Y \in D(VI)_0,$$
$$\tilde{s}_0(Z) = -Z, \qquad Z \in D(VI).$$

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