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Another definitions of exceptional simple Lie groups of type $E_{7(-25)}$ and $E_{7(-133)}$

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We have proved in [3], [4] that

$$E_{7,1}=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ |\ \alpha\mathfrak{M}=\mathfrak{M},\ \{\alpha P,\ \alpha Q\}=\{P,\ Q\}\}$$

is a connected simple Lie group of type $E_{7(-25)}$ and

$$E_7=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C,\ \mathfrak{P}^C)\ |\ \alpha\mathfrak{M}^C=\mathfrak{M}^C,\ \{\alpha P,\ \alpha Q\}=\{P,\ Q\},\ <\alpha P,\ \alpha Q>=<P,\ Q>\}$$

is a simply connected compact simple Lie group of type E_7, where $\mathfrak{M}$, $\mathfrak{M}^C$ are the Freudenthal's manifolds and $\{P,\ Q\}$, $<P,\ Q>$ inner products in $\mathfrak{P}$, $\mathfrak{P}^C$ respectively. In this paper, we shall give another expressions of these groups. Our results are as follows. These groups $E_{7,1}$ and E_7 are also defined by

$$E_{7,1}=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ |\ \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q\ \},$$

$$E_7=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C,\ \mathfrak{P}^C)\ |\ \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q,\ <\alpha P,\ \alpha Q>=<P,\ Q>\}$$

respectively, where $\times : \mathfrak{P}\times\mathfrak{P}\longrightarrow\mathfrak{e}_{7,1}$, $\times : \mathfrak{P}^C\times\mathfrak{P}^C\longrightarrow\mathfrak{e}_7{}^C$ are mappings defined by Freudenthal in [2].

1. Jordan algebra $\mathfrak{J}$ and Lie algebra $\mathfrak{e}_{6,1}$.

Let $\mathfrak{C}$ denote the division algebra of Cayley numbers over the field of real numbers $\boldsymbol{R}$ and $\mathfrak{J}=\mathfrak{J}(3,\ \mathfrak{C})$ the Jordan algebra consisting of all 3×3 Hermitian matrices X with entries in $\mathfrak{C}$ with respect to the multiplication $X\circ Y=\dfrac{1}{2}(XY+YX)$. In $\mathfrak{J}$, the symmetric inner product $(X,\ Y)$, the crossed product $X\times Y$ and the cubic form $(X,\ Y,\ Z)$ are defined respectively by

$$(X,\ Y)=\mathrm{tr}\,(X\circ Y),$$

$$X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X,\ Y))E),$$

$$(X,\ Y,\ Z)=(X,\ Y\times Z)=(X\times Y,\ Z)$$

where E is the 3×3 unit matrix.

In [1], Freudenthal defined the exceptional simple Lie algebra $\mathfrak{e}_{6,1}$ of type E_6 explicitly by

$$\mathfrak{e}_{6,1}=\{\ \phi\in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})\ \mid\ (\phi X,\ X,\ X)=0\ \}$$

and, for $A,\ B\in\mathfrak{J}$, he constructed $A\vee B\in\mathfrak{e}_{6,1}$ by

$$(A\vee B)X=\frac{1}{2}(B,\ X)A+\frac{1}{6}(A,\ B)X-2B\times(A\times X) \qquad X\in\mathfrak{J}. \tag{i}$$

Finally, for $\phi\in\mathfrak{e}_{6,1}$, we denote the skew-transpose of ϕ by ϕ' with respect to the inner product $(X,\ Y)$:

$$(\phi X,\ Y)+(X,\ \phi' Y)=0,$$

then also $\phi'\in\mathfrak{e}_{6,1}$.

2. Lie algebra $\mathfrak{e}_{7,1}$.

Let $\mathfrak{P}$ be a 56-dimensional vector space defined by

$$\mathfrak{P}=\mathfrak{J}\oplus\mathfrak{J}\oplus\boldsymbol{R}\oplus\boldsymbol{R}.$$

For $\phi\in\mathfrak{e}_{6,1}$, $A,\ B\in\mathfrak{J}$ and $\rho\in\boldsymbol{R}$, we define a linear transformation $\Phi(\phi,\ A,\ B,\ \rho)$ of $\mathfrak{P}$ by

$$\Phi(\phi,\ A,\ B,\ \rho)\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}=\begin{pmatrix}\phi-\frac{1}{3}\rho 1 & 2B & 0 & A\\ 2A & \phi'+\frac{1}{3}\rho 1 & B & 0\\ 0 & A & \rho & 0\\ B & 0 & 0 & -\rho\end{pmatrix}\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}$$

$$=\begin{pmatrix}\phi X-\frac{1}{3}\rho X+2B\times Y+A\eta\\ 2A\times X+\phi' Y+\frac{1}{3}\rho Y+\xi B\\ (A,\ Y)+\rho\xi\\ (B,\ X)-\rho\eta\end{pmatrix}$$

Then Freudenthal showed in [2] that

$$\mathfrak{e}_{7,1}=\{\ \Phi=\Phi(\phi,\ A,\ B,\ \rho)\in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ \mid\ \phi\in\mathfrak{e}_{6,1},\ A,\ B\in\mathfrak{J},\ \rho\in\boldsymbol{R}\ \}$$

is a exceptional simple Lie algebra of type E_7. The Lie bracket $[\Phi_1,\ \Phi_2]$ in $\mathfrak{e}_{7,1}$ is given by

$$[\Phi(\phi_1,\ A_1,\ B_1,\ \rho_1),\ \Phi(\phi_2,\ A_2,\ B_2,\ \rho_2)]=\Phi(\phi,\ A,\ B,\ \rho)$$

where

$$\begin{cases}\phi=[\phi_1,\ \phi_2]+2A_1\vee B_2-2A_2\vee B_1,\\ A=(\phi_1+\dfrac{2}{3}\rho_1 1)A_2-(\phi_2+\dfrac{2}{3}\rho_2 1)A_1,\\ B=(\phi_1{}'-\dfrac{2}{3}\rho_1 1)B_2-(\phi_2{}'-\dfrac{2}{3}\rho_2 1)B_1,\\ \rho=(A_1,\ B_2)-(B_1,A_2).\end{cases}$$

For $P=(X,\ Y,\ \xi,\ \eta)$, $Q=(Z,\ W,\ \zeta,\ \omega)\in\mathfrak{P}$, he constructed $P\times Q\in\mathfrak{e}_{7,1}$ by

$$P\times Q=\Phi(\phi,\ A,\ B,\ \rho),\qquad \begin{cases}\phi=-\dfrac{1}{2}(X\vee W+Z\vee Y)\\ A=-\dfrac{1}{4}(2Y\times W-\xi Z-\zeta X),\\ B=\dfrac{1}{4}(2X\times Z-\eta W-\omega Y),\\ \rho=\dfrac{1}{8}((X,\ W)+(Z,\ Y)-3(\xi\omega+\zeta\eta)),\end{cases}$$

and showed the following formula

$$[\Phi,\ P\times Q]=\Phi P\times Q+P\times\Phi Q\qquad \Phi\in\mathfrak{e}_{7,1},\ P,\ Q\in\mathfrak{P}.$$

In $\mathfrak{P}$, we define a skew-symmetric inner produt $\{P,\ Q\}$ by

$$\{P,\ Q\}=(X,\ W)-(Z,\ Y)+\xi\omega-\zeta\eta$$

for $P=(X,\ Y,\ \xi,\ \eta)$, $Q=(Z,\ W,\ \zeta,\ \omega)$.

Proposition 1. *For $P,\ Q\in\mathfrak{P}$, we have*

$$(P\times P)Q=(P\times Q)P+\frac{3}{8}\{P,\ Q\}P.$$

Proof is straight-forward calculations using (i).

Finally we define a manifold $\mathfrak{M}$ in $\mathfrak{P}$, called the Freudenthal's manifold by

$$\mathfrak{M}=\{\ P\in\mathfrak{P}\ \mid\ P\times P=0,\ P\neq 0\ \}.$$

3. Lie group $E_{7,1}$.

Theorem 2. $E_{7,1}=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ \mid\ \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q\ \}$

is a connected simple Lie group of type E_7. *The polar decomposition of* $E_{7,1}$ *is given by*

$$E_{7,1}\simeq(U(1)\times E_6)/\boldsymbol{Z}_3\times\boldsymbol{R}^{54}$$

where $U(1)$ *is the unitary group and* E_6 *is a simply connected compact Lie group of type* E_6. *The center* $z(E_{7,1})$ *of* $E_{7,1}$ *is*

$$z(E_{7,1})=\{1,\ -1\}.$$

Proof, In [3], we showed that the group

$$E_{7(-25)}=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P},\ \mathfrak{P})\ \mid\ \alpha\mathfrak{M}=\mathfrak{M},\ \ \{\alpha P,\ \alpha Q\}=\{P,\ Q\}\}$$

is a connected simple Lie group of type E_7 which has the properties stated in Theorem 2. We shall prove that the group $E_{7,1}$ coincides with the group $E_{7(-25)}$. First we show that $E_{7,1}$ is a subgroup of $E_{7(-25)}$. In fact, for $\alpha\in E_{7,1}$, we have $\alpha\mathfrak{M}=\mathfrak{M}$, since $\alpha P\times\alpha P=\alpha(P\times P)\alpha^{-1}=0$ for $P\in\mathfrak{M}$. Act $\alpha\in E_{7,1}$ on $(P\times P)Q=(P\times Q)P+\frac{3}{8}\{P,\ Q\}P$ of Proposition 1, then $\alpha(P\times P)\alpha^{-1}\alpha Q=\alpha(P\times Q)\alpha^{-1}\alpha P+\frac{3}{8}\{P,\ Q\}\alpha P$, that is,

$$(\alpha P\times\alpha Q)\alpha Q=(\alpha P\times\alpha Q)\alpha P+\frac{3}{8}\{P,\ Q\}\alpha P.$$

On the other hand, from Proposition 1 again

$$(\alpha P\times\alpha P)\alpha Q=(\alpha P\times\alpha Q)\alpha P+\frac{3}{8}\{\alpha P,\ \alpha Q\}\alpha P.$$

Hence we have

$$\{\alpha P,\ \alpha Q\}=\{P,\ Q\}.$$

Therefore $E_{7,1}\subset E_{7(-25)}$. Conversely, we know that the Lie algebra of the group $E_{7(-25)}$ is $\mathfrak{e}_{7,1}$ [3] and any element of $\mathfrak{e}_{7,1}$ satisfies

$$[\Phi,\ P\times Q]=\Phi P\times Q+P\times\Phi Q.$$

This shows that the Lie algebras of the groups $E_{7(-25)}$ and $E_{7,1}$ coincide. Hence from the connectedness of $E_{7(-25)}$, we see that $E_{7(-25)}\subset E_{7,1}$. Thus we obtain $E_{7,1}=E_{7(-25)}$.

4. Lie group E_7.

Let $\mathfrak{C}^C$ and $\mathfrak{J}^C=\mathfrak{J}(3, \mathfrak{C}^C)$ be the complexifications of $\mathfrak{C}$ and $\mathfrak{J}$ respectively over the field of complex numbers C and put $\mathfrak{P}^C=\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus C\oplus C$. Then the statements in the preceding sections are also valid in the complex case. For example,

$$\mathfrak{e}_6{}^C=\{ \phi\in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi X, X, X)=0 \}$$

$$\mathfrak{e}_7{}^C=\{ \Phi(\phi, A, B, \rho) \mid \phi\in \mathfrak{e}_6{}^C, A, B\in\mathfrak{J}^C, \rho\in C\}$$

are the simple Lie algebras over C of type E_6 and E_7 respectively, and the Freudenthal's manifold $\mathfrak{M}^C$ is defined by

$$\mathfrak{M}^C=\{ P\in\mathfrak{P}^C \mid P\times P=0, P\neq 0 \}.$$

We define positive definite Hermitian inner products $\langle X, Y\rangle$ in $\mathfrak{J}^C$ and $\langle P, Q\rangle$ in $\mathfrak{P}^C$ respectively by

$$\langle X, Y\rangle=(\overline{X}, Y),$$

$$\langle P, Q\rangle=\langle X, Z\rangle+\langle Y, W\rangle+\overline{\xi}\zeta+\overline{\eta}\omega$$

where $\overline{X}$ is the conjugate of X with respect to the field C and $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega)\in\mathfrak{P}^C$.

In [4], we showed that the group

$$E_7=\{ \alpha\in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha\mathfrak{M}^C= \mathfrak{M}^C, \{\alpha P, \alpha Q\}=\{P, Q\}, \langle\alpha P, \alpha Q\rangle=\langle P, Q\rangle\}$$

is a simply connected compact simple Lie group of type E_7. By the preceding arguments, we see that conditions

$$\alpha\mathfrak{M}^C= \mathfrak{M}^C, \qquad \{\alpha P, \alpha Q\}=\{P, Q\}$$

is equivalent to

$$\alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q.$$

Hence we have

Theorem 3. $E_7=\{ \alpha\in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q, \langle\alpha P, \alpha Q\rangle=\langle P, Q\rangle\}$ *is a simply connected simple Lie group of type* E_7.

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