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On Two Theorems of U. Albrecht

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In [1] and [2] U. Albrecht has dealt with finite Galois extensions of division rings. There he has obtained a sufficient condition for a finite Galois extension of division rings to have no finite Galois groups [1, Satz 4] and a sufficient condition for a division ring of finite degree over its center to have no finite Galois groups whose orders coincide with the extension degree over the center [2, Satz 1].

First, by making use of two results in [3], we reprove more briefly [1, Satz 4].

Theorem 1. *Let K/L be a finite Galois extension of division rings of degree $[K:L]=q^r$ $(r\geqq 1)$, q a prime. If the centralizer $V_K(L)$ of L in K is not commutaive and if there is no prime $p\geqq q$ such that $V_K(L)$ contains a primitive p-th root of 1, then there is no finite Galois group of K/L.*

Proof. Let $V=V_K(L)$, $Z_0=V_L(L)$ and $C_0=V_V(V)$. Suppose that there exists a finite Galois group G^* of K/L. Then the restriction G of G^* to V is a finite Galois group of V/Z_0, and it is easily seen that the group G_0 consisting of all the inner automorphisms contained in G is a Galois group of V/C_0 and $[V:C_0]=q^s$ $(s\geqq 1)$. Now let p be an arbitrary prime factor of $|G_0|$, and choose a subgroup H of G_0 with $|H|=p$. If U is the fixed subring of H, then $p\geqq[V:U]>1$ and $[V:U]|q^s$, so that $p\geqq q$. By [3, Theorem 13.2 (a)] we have then $p=[V:U]$, i. e., $p=q$. Hence $|G_0|$ is a power of q. But this contradicts [3, Lemma 10.3].

According to Albrecht [2], a positive integer n is said to satisfy the condition P if for each group G of order n there exists an arrangement, say $p_1{}^{r_1}, \cdots, p_m{}^{r_m}$, of the prime powers which appear in the prime factorization of n such that $G=G_1$ has a normal subgroup H_1 of order $p_1{}^{r_1}$, $G_2=G_1/H_1$ has a normal subgroup H_2 of order $p_2{}^{r_2}$, and so on.

Let K be a division ring of finite degree over its center Z, and let $p_1, \cdots, p_m$ be the distinct prime factors of $[K:Z]$. Albrecht [2, Satz 1] has proved the following: If $[K:Z]$ satisfies the condition P and Z contains no primitive p_i-th roots of 1 for $i=1, \cdots, m$, then there is no finite Galois group of K/Z whose order coincides with $[K:Z]$.

Now, we are in a position to prove the following theorem that yields at once [2, Satz 1].

Theorem 2. *Let K be a division ring of finite degree over its center Z, and G a finite Galois group of K/Z. Let p be a prime factor of $|G|$, and $|G|=p^e n$, $(p, n)=1$. If $|G|$ coincides with $[K:Z]$ and if Z contains no primitive p-th roots of 1, then G cannot contain a normal subgroup of order n.*

Proof. Suppose G contains a normal subgroup H of order n, and set $L=K^H$, the fixed subring of H. Then by [3, Lemma 10.2], $[L:Z]=(G:H)=p^e$ and G/H is naturally isomorphic to a Galois group of L/Z. Since Z contains no primitive p-th roots of 1, by [3, Lemma 10.4] we see that $L=V_L(Z)=V_L(L)\cdot Z$ is a commutative field. Now let M be a maximal subfield of K including L. Then, $p^e n=[K:Z]=[M:Z]^2=p^{2e}[M:L]^2$. But this is impossible.

References

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[3] H. TOMINAGA and T. NAGAHARA : Galois Theory of Simple Rings, Okayama Math. Lectures, 1970.