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Characterizations of Division Rings. II

By MASAYUKI ÔHORI

Department of Mathematics, Faculty of Science,
Shinshu University

and HISAO TOMINAGA

Department of Mathematics, Faculty of Science,
Okayama University

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The present objective is to reprove more briefly the main theorem of F. Radó [2] and to improve considerably a proposition of M. A. Quadri [1], by means of the following explicit theorem.

Theorem. *A ring R $(\neq 0)$ is a division ring if (and only if) for every non-zero $x \in R$ there holds that $xR=R$ or $Rx=R$.*

Proof. It is obvious that R is a reduced ring. Let x be an arbitrary non-zero element of R. Then, by hypothesis, there exists $x' \in R$ such that $x^2x'=x$ or $x'x^2=x$. By a brief computation, we obtain $(xx'x-x)^2=0$, whence it follows $xx'x=x$. Since the non-zero idempotents xx' and $x'x$ are central, we obtain $xx'R=Rxx'=R$ and $x'xR=Rx'x=R$, and therefore $xx'=1=x'x$.

Remark 1. As is well known every indecomposable strongly regular ring is a division ring. Moreover, a careful examination of the above proof shows that every indecomposable reduced ring whose elements are right or left π-regular is a division ring.

Corollary 1 ([2, Theorem]). *A distributive near ring R $(\neq 0)$ is a division ring if (and only if) R contains a right regular element a ($ab=0$ implies $b=0$) and for every non-zero $x \in R$ there exists $y \in R$ such that $xy=a$ or $yx=a$.*

Proof. First, we claim that R is a ring. As is easily seen, $(-b)c=-bc=b(-c)$ for any b, $c \in R$. Hence, $a\{(b+c)-(c+b)\}=a(b+c)+(-a)(c+b)=0$, whence it follows $b+c=c+b$. Next, we prove that a is left regular, too. Suppose $ra=0$ with some non-zero $r \in R$. Then, by hypothesis there exists $s \in R$ such that $rs=a$ or $sr=a$. If $a=sr$ then $a^2=sra=0$, a contradiction. Hence, $a=rs$ and $sr \neq 0$ by $rsr=ar \neq 0$. There exists $t \in R$ such that $a=srt$, since $a=tsr$ yields a contradiction $a^2=0$. Then, $art=rsrt=ra=0$, and hence $rt=0$. But this implies $a=srt=0$, contrary to assumption. We have thus seen that a is (right and left) regular. Now, let x be an arbitrary non-zero element of R. Then axa is non-zero and there exists $y \in R$ such that $axay=a$ or $yaxa=a$, so that $xR=R$ or $Rx=R$. Hence R is a division ring by Theorem.

Now, assume that the center C of a ring $R\,(\neq 0)$ contains a multiplicative semigroup S such that for every $a \in R$ there exists $s \in S$ with $as=a$. We define a relation $<$ on R as follows: $b<a$ if and only if $a=bs$ with some $s \in S$. It is immediate that the relation $<$ is reflexive and transitive, so we can define an equivalence relation $\equiv$ on R: $a \equiv b$ if and only if $a<b$ and $b<a$. For any $a \in R$, we denote by $[a]$ the equivalence class of a with respect to $\equiv$. Let $[R/S]$ be the totality of all such equivalence classes. We now define multiplication in $[R/S]$ by $[a]\cdot[b]=[ab]$. As is easily verified, this multiplication is well-defined and $[R/S]$ forms a semigroup with zero $[0]$. Assume further that for every non-zero $x \in R$ there holds that $[x]\cdot[R/S]=[R/S]$ or $[R/S]\cdot[x]=[R/S]$. If $[x]\cdot[R/S]=[R/S]$ (resp. $[R/S]\cdot[x]=[R/S]$), then for any $y \in R$ there exists $z \in R$ such that $[x]\cdot[z]=[y]$ (resp. $[z]\cdot[x]=[y]$), and therefore $xzs=y$ (resp. $szx=y$) with some $s \in S$. This implies $xR=R$ (resp. $Rx=R$). Hence R is a division ring by Theorem. Conversely, if R is a division ring then we can take C (or $C\setminus\{0\}$) as S and $[R/S]\setminus\{[0]\}$ is seen to be a group. We have thus improved [1, Proposition 2] as follows:

Corollary 2. *A ring $R(\neq 0)$ is a division ring if and only if the center of R contains a multiplicative semigroup S such that for every non-zero $x \in R$ 1) $xs=x$ with some $s \in S$ and 2) $[x]\cdot[R/S]=[R/S]$ or $[R/S]\cdot[x]=[R/S]$.*

Remark 2. As is well known, there does exist a semigroup M without identity such that $xM=M$ for all $x \in M$.

References

[1] M. A. QUADRI: A characterisation of a field, Aligarh Bull. Math. 1 (1971), 113–114.
[2] F. RADÓ: On the definition of skew-fields, Arch. Math. 32 (1979), 441–444.