

Erratum to the Paper :

Trace of the Fundamental Solution of $\Delta + r^k D \dots$,

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There is a misunderstanding of the author on the quoted Wey's result in the above titled paper which appeared in this journal vol. 14, 47–85. The notion of spectre difference is nonsense in general. Therefore §6 of the above paper has no meanings.

Trace of the Fundamental Solution of $\Delta+r^k D$ and Spectre Difference

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Introduction

It is known that by a compact perturbation G , the spectre set of a Selfadjoint operator H remains invariant except finite set of point spectres and such finite set can be taken arbitrarily with appropriate choice of G ([5], [18]). The spectre difference $d=d(H, G)$ of H and $H+G$ is defined to be d_1-d_2 , where d_1 is the number of point spectres of $H+G$ which are not in the spectre set of H and d_2 is the number of point spectres of H which are not in the spectre set of $H+G$. The positive spectre difference $d_+(H, G)$ and negative spectre difference $d_-(H, G)$ are defined similarly for the sets of positive spectres and negative spectres of H and $H+G$. If ζ -function $\zeta_H(s)$ and η -function $\eta_H(s)$ of H etc., are defined and continued analytically at $s=0$, we have although they have poles at $s=0$

$$d(H, G)=d_+(H, G)+d_-(H, G)=\lim_{s \rightarrow 0} \zeta_{(D+H)^2}(s)-\zeta_{D^2}(s),$$

$$d_+(H, G)-d_-(H, G)=\lim_{s \rightarrow 0} \eta_{D+H}(s)-\eta_D(s).$$

Let M be a compact smooth manifold, E a smooth vector bundle over M , $D:C^\infty(M, E) \rightarrow C^\infty(M, E)$ an elliptic operator, θ its lower order perturbation such that D and θ both allow selfadjoint L^2 -extensions, then $\zeta_D(s)$ and $\eta_D(s)$ etc., are defined and continued analytically at $s=0$ ([10]). The positive and negative spectre differences are unitary invariants and since a connection of D with respect to F , a vector bundle over M , is a lower order perturbation of $\{D_U \otimes 1_F\}$, $D=\{D_U\}$ ([2]), these gives invariants of connection under the bundle automorphisms (cf. [2], [4], [13]).

The purpose of this paper is to give formulas to compute positive and negative spectre differences of D and $D+\theta$ which at least does not need to compute the spectre set of $D+\theta$.

For this purpose, we consider the operator $\Delta+r^k D$ on $\mathbb{R}^n \times M$, $n \geq 2$, where Δ

is the Laplacian on \mathbf{R}^n and r is the euclidean norm function on \mathbf{R}^n . Although degenerate parabolic operator (cf. [12], [15]) was used in the study of η -invariant ([1], cf. [3]), our operator is degenerate elliptic if D is negative (cf. [14]) and degenerate ultrahyperbolic (cf. [11]) if D is positive. But under suitable boundary condition ((4) in §1 and (4)' in §3), we can treat this operator without assumption of positivity or negativity of D . If D is negative, our boundary condition can be replaced by 0-boundary condition, but for positive D , our boundary condition can not be replaced 0-boundary condition in general.

To express spectre difference by using the trace of the fundamental solution of $\Delta + r^k D$, first we consider this operator in $L^2(\mathbf{R}^+, r^{n-1} dr) \otimes \mathcal{H}^{p,n} \otimes E_\lambda$, where $\mathcal{H}^{p,n}$ is the space of harmonic polynomials of homogeneous degree p and E_λ is the λ -proper space of D . By virtue of a formula of Lommel ([8]), we get explicit fundamental solution of our operator in this space and its trace is also computed using formulas of Bessel functions ([9]) (§1). Next we construct fundamental solution of $\Delta + r^k D$ on $L^2([0, a], r^{n-1} dr) \otimes \mathcal{H}^{p,n} \otimes L^2(M, E)$ and on $L^2(\mathbf{R}^+, r^{n-1} dr) \otimes \mathcal{H}^{p,n} \otimes L^2(M, E)$ and compute its trace. The trace is expressed as a formula containning $\zeta_{D,\pm}(s)$ and $\zeta_{D,-}(s)$. Here $\zeta_{D,\pm}(s)$ are defined to use positive (or negative) proper values of D . (§§3-4).

To summalize the fundamental solution of $\Delta + r^k D$ on $L^2(\mathbf{R}^+, r^{n-1} dr) \otimes \mathcal{H}^{p,n} \otimes L^2(M, E)$ in p , we use a family of operators $T_{w,\alpha}$ with complex parameters w and α , $\operatorname{Re} \alpha > 0$, such that $T_{w,\alpha}$ is analytic in w and α and $\lim_{w \rightarrow 0} T_{w,\alpha} = I$, the identity map and set $\Delta_{w,\alpha} = T_{w,\alpha} \Delta$ (§3). Then for $\operatorname{Re} w > n-2$, we can construct the fundamental solution of $\Delta_{w,\alpha} + r^k D$ on $L^2(\mathbf{R}^n \times M, \pi^*(E))$ and on $L^2(B_a \times M, \pi^*(E))$. Here π is the projection onto M and $B_a = \{x \in \mathbf{R}^n \mid ||x|| < a\}$. The fundamental solution of $\Delta + r^k D$ is obtained by analytic continuation in w . We denote these fundamental solutions by $G_{k,(w,\alpha)}$ and $G_{k,a,(w,\alpha)}$. Their traces are computed as the formulas containning $\zeta_{D,\pm}(s)$, $\zeta_n(w, \alpha)$ and $\zeta_{n,k}(w, \alpha)$. $\lim_{k \rightarrow \infty} \operatorname{tr} G_{k,a,(w,\alpha)}$ is also computed if $\zeta_{D,\pm}(0)$ exist as a formula containning $\zeta_n(w, \alpha)$ and $\zeta_{\infty,n}(w, \alpha)$. Here $\zeta_n(w, \alpha)$, $\zeta_{n,k}(w, \alpha)$ and $\zeta_{\infty,n}(w, \alpha)$ are defined in §4 and their analytic continuations and the values or residues at $w=0$ are computed in §5.

Using these results, under the assumptions that $\zeta_{D,\pm}(0)$ exist, we have

$$\lim_{\alpha \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w \operatorname{tr} G_{k,a,(w,\alpha)} = -\frac{1}{n} \zeta_D(0), \quad n=2, 4,$$

$$\lim_{\alpha \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w \operatorname{tr} G_{k,a,(w,\alpha)}$$

$$= \frac{\sin(n\pi/2)}{8\pi} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2}-2\right) \right\}^2 \zeta_D(0), \quad n \text{ is odd},$$

$$\lim_{\alpha \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{\partial^{n-2}}{\partial \alpha^{n-2}} \operatorname{tr} G_{k, \alpha, (0, \alpha)} + \frac{a^2}{2(n-2)} \dim \ker D = \frac{1}{2(n-2)} \zeta_D(0),$$

n is even, $n \geq 6$.

We also denote the fundamental solutions of $A_{w, \alpha} + r^k D$ and $A_{w, \alpha} - r^k D = A_{w, \alpha} + r^k (-D)$ by $G^+_{k, (w, \alpha)}$ and $G^-_{k, (w, \alpha)}$. Then we get

$$\lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 \{ \operatorname{tr} G^+_{k, (w, \alpha)} - \operatorname{tr} G^-_{k, (w, \alpha)} \} = \frac{\pi^2}{2n} \eta_D(0), \quad n=2, 4,$$

$$\begin{aligned} & \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 \{ \operatorname{tr} G^+_{k, (w, \alpha)} - \operatorname{tr} G^-_{k, (w, \alpha)} \} \\ &= -\frac{\sin(n\pi/2)}{16} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2}-2\right) \right\}^2 \pi \eta_D(0), \quad n \text{ is odd}, \end{aligned}$$

$$\lim_{k \rightarrow \infty} k^2 \frac{\partial^{n-2}}{\partial \alpha^{n-2}} \{ \operatorname{tr} G^+_{k, (0, \alpha)} - \operatorname{tr} G^-_{k, (0, \alpha)} \} = \frac{\pi^2}{2(n-2)} \eta_D(0), \quad n \text{ is even, } n \geq 6.$$

By these formulas, we can express the spectre differences of D and $D+\theta$ as follows

$$\lim_{\alpha \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w [\operatorname{tr} \{ G_{k, \alpha, (w, \alpha)} r^k \theta G_{k, \alpha, (w, \alpha)} (I + r^k \theta G_{k, \alpha, (w, \alpha)})^{-1} \}]$$

$$= \frac{1}{n} (d_+ + d_-), \quad n=2, 4,$$

$$\lim_{\alpha \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w [\operatorname{tr} \{ G_{k, \alpha, (w, \alpha)} r^k \theta G_{k, \alpha, (w, \alpha)} (I + r^k \theta G_{k, \alpha, (w, \alpha)})^{-1} \}]$$

$$= -\frac{\sin(n\pi/2)}{8\pi} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2}-2\right) \right\}^2 (d_+ + d_-), \quad n \text{ is odd},$$

$$\lim_{\alpha \rightarrow \infty} \lim_{k \rightarrow \infty} \left[\frac{\partial^{n-2}}{\partial \alpha^{n-2}} [\operatorname{tr} \{ G_{k, \alpha, (0, \alpha)} r^k \theta G_{k, \alpha, (0, \alpha)} (I + r^k \theta G_{k, \alpha, (0, \alpha)})^{-1} \}] \right]$$

$$+ \frac{a^2}{2(n-2)} \{ \dim \ker(D+\theta) - \dim \ker D \}$$

$$= -\frac{1}{2(n-2)} (d_+ + d_-), \quad n \text{ is even, } n \geq 6,$$

$$\lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 [\operatorname{tr} \{ G^+_{k, (w, \alpha)} r^k \theta G^+_{k, (w, \alpha)} (I + r^k \theta G^+_{k, (w, \alpha)})^{-1} \}]$$

$$+ \operatorname{tr} \{ G^-_{k, (w, \alpha)} r^k \theta G^-_{k, (w, \alpha)} (I - r^k \theta G^-_{k, (w, \alpha)})^{-1} \}]$$

$$= -\frac{\pi^2}{2n} (d_+ - d_-), \quad n=2, 4,$$

$$\begin{aligned}
& \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 [\operatorname{tr}\{G^+_{k,(w,\alpha)} r^k \theta G^+_{k,(w,\alpha)} (I + r^k \theta G^+_{k,(w,\alpha)})^{-1}\} \\
& + \operatorname{tr}\{G^-_{k,(w,\alpha)} r^k \theta G^-_{k,(w,\alpha)} (I - r^k \theta G^-_{k,(w,\alpha)})^{-1}\}] \\
& = \frac{\sin(n\pi/2)}{16} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2} - 2\right) \right\}^2 \pi(d_+ - d_-), \quad n \text{ is odd}, \\
& \lim_{k \rightarrow \infty} k^2 \frac{\partial^{n-2}}{\partial \alpha^{n-2}} [\operatorname{tr}\{G^+_{k,(0,\alpha)} r^k \theta G^+_{k,(0,\alpha)} (I + r^k \theta G^+_{k,(0,\alpha)})^{-1}\}] \\
& + \operatorname{tr}\{G^-_{k,(0,\alpha)} r^k \theta G^-_{k,(0,\alpha)} (I - r^k \theta G^-_{k,(0,\alpha)})^{-1}\}] \\
& = \frac{1}{2(n-2)} (d_+ - d_-), \quad n \text{ is even}, \quad n \geq 6.
\end{aligned}$$

The applications of these results will be given in forthcoming papers.

§1. Trace of the fundamental solution of $L_{n,p} + r^k$

1. We set $\beta = \beta(n, a) = \sqrt{(1-n/2)^2 - a}$, $\beta(c) = \beta(n, a+c)$ and $\mu(a) = 2\beta(n, a)/(k+2)$, $\mu(a, c) = \mu(a+c)$, $k \neq -2$. Then by a formula of Lommel ([8], [9], [17]), we have

Lemma 1. *The fundamental system of solutions of the equation*

$$\frac{d^2 f}{dr^2} + \frac{n-1}{r} \frac{df}{dr} + \left\{ \frac{a}{r^2} + cr^k \right\} f = 0,$$

is given $b\{r^{1-n/2} J_{\mu(a)}((2\sqrt{c}/(k+2))r^{(k+2)/2})\}$ if $k \neq -2$, $\{r^{1-n/2} Y_{\mu(a)}((2\sqrt{c}/(k+2))r^{(k+2)/2})\}$ if $k = -2$, $\{r^{1-n/2 \pm \beta(c)}\}$ if $k = -2$, $a+c \neq (1-n/2)^2$ and $\{r^{1-n/2}, r^{1-n/2} \log r\}$ if $k = -2$, $a+c = (1-n/2)^2$.

Definition. For $n \geq 2$, $p \geq 0$ (both integers), we set

$$(1) \quad L_{n,p} = \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} + \frac{1}{r^2} p(2-n-p),$$

$$\alpha = \alpha(\lambda) = \sqrt{(p + \frac{n}{2} - 1)^2 - \lambda}, \quad \nu(p) = \nu(p, k) = \frac{2p+n-2}{k+2}, \quad k \neq -2.$$

By lemma 1, if λ is a real number, a fundamental system of solutions of $(L_{n,p} + \lambda r^k)y = 0$ is given by the following $\{y_{+, \lambda}, y_{-, \lambda}\}$.

$$(2)_+ \quad y_{+, \lambda}(r) = r^{1-\frac{n}{2}} J_{\nu(p)}\left(\frac{2\sqrt{-\lambda}}{k+2} r^{\frac{k+2}{2}}\right), \quad y_{-, \lambda}(r) = r^{1-\frac{n}{2}} Y_{\nu(p)}\left(\frac{2\sqrt{-\lambda}}{k+2} r^{\frac{k+2}{2}}\right),$$

$$\lambda > 0, \quad k \neq -2,$$

$$(2)_- \quad y_{+, \lambda}(r) = r^{1-\frac{n}{2}} I_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right), \quad y_{-, \lambda}(r) = r^{1-\frac{n}{2}} K_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right),$$

$$\lambda < 0, \quad k \neq -2,$$

$$(2)_0 \quad y_{+, \lambda}(r) = r^{1-\frac{n}{2}+\alpha}, \quad y_{-, \lambda}(r) = r^{1-\frac{n}{2}-\alpha},$$

$$k=-2, \quad \lambda \neq \left(p + \frac{n}{2} - 1\right)^2 \text{ or } \lambda=0, \quad (n, p) \neq (2, 0),$$

$$(2)_{0,0} \quad y_{+, \lambda}(r) = r^{1-\frac{n}{2}}, \quad y_{-, \lambda}(r) = r^{1-\frac{n}{2}} \log r,$$

$$k=-2, \quad \lambda = \left(p + \frac{n}{2} - 1\right)^2 \text{ or } \lambda=0, \quad (n, p)=(2, 0).$$

The Wronskians $W(y_{+, \lambda}, y_{-, \lambda})$ of these $\{y_{+, \lambda}, y_{-, \lambda}\}$ are given by

$$(3) \quad W(y_{+, \lambda}, y_{-, \lambda}) = \frac{k+2}{\pi} r^{1-n}, \quad \lambda > 0, \quad k \neq -2,$$

$$W(y_{+, \lambda}, y_{-, \lambda}) = -\frac{k+2}{2} r^{1-n}, \quad \lambda < 0, \quad k \neq -2,$$

$$(3)_0 \quad W(y_{+, \lambda}, y_{-, \lambda}) = -2\alpha(\lambda)r^{1-n}, \quad \lambda \neq \left(p + \frac{n}{2} - 1\right)^2, \quad k=-2,$$

$$W(y_{+, \lambda}, y_{-, \lambda}) = r^{1-n}, \quad \lambda = \left(p + \frac{n}{2} - 1\right)^2, \quad k=-2 \text{ or } \lambda=0, \quad (n, p)=(2, 0).$$

We set

$$G_\lambda(r, \rho) = G_{\lambda, k, a}(r, \rho) = W(y_{+, \lambda}, y_{-, \lambda})(\rho)^{-1} y_{+, \lambda}(\rho) y_{-, \lambda}(r), \quad a \geq r \geq \rho > 0,$$

$$= W(y_{+, \lambda}, y_{-, \lambda})(\rho)^{-1} y_{+, \lambda}(r) y_{-, \lambda}(\rho), \quad a \geq \rho \geq r > 0,$$

$$G_{\lambda, k, a} f = \int_0^a G_\lambda(r, \rho) f(\rho) d\rho, \quad G_{\lambda, k, \infty} f = \int_0^\infty G_\lambda(r, \rho) f(\rho) d\rho.$$

Then G_λ is the fundamental solution of $L_{n, p} + \lambda r^k$ with the selfadjoint boundary condition

$$(4) \quad u(0)=0, \quad \frac{u(a)}{u'(a)} = \frac{y_{-, \lambda}(a)}{y_{-, \lambda}'(a)}, \quad y_{-, \lambda}'(a) \neq 0,$$

$$u(0)=0, \quad u'(a)=0, \quad y_{-, \lambda}'(a)=0.$$

In the rest of this §, we assume k to be a real number. Then, by the asymptotic formulas of Bessel functions ([6], [9], [17]), we have for $k > -2$

$$y_{+, \lambda}(r) = r^{1-\frac{n}{2}} \left\{ \sqrt{\frac{k+2}{\pi \sqrt{\lambda}}} r^{-\frac{k+2}{4}} \cos \left(\frac{2\sqrt{\lambda}}{k+2} r^{\frac{k+2}{2}} - \frac{\nu(p)}{2} \pi - \frac{\pi}{4} \right) (1 + O(\lambda^{-\frac{3}{4}} r^{-\frac{3}{4}(k+2)})) \right. \\ \left. + O\left(\lambda^{-\frac{3}{4}} r^{-\frac{3}{4}(k+2)}\right) \right\},$$

$$y_{-, \lambda}(r) = r^{1-\frac{n}{2}} \left\{ \sqrt{\frac{k+2}{\pi \sqrt{|\lambda|}}} r^{-\frac{k+2}{4}} \sin \left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}} - \frac{\nu(p)}{2} \pi - \frac{\pi}{4} \right) (1 + O(\lambda^{-\frac{3}{4}} r^{-\frac{3}{4}(k+2)})) \right.$$

$$\left. + O\left(\lambda^{-\frac{3}{4}} r^{-\frac{3}{4}(k+2)}\right)\right\},$$

$\lambda > 0, r \rightarrow \infty$ or $\lambda \rightarrow \infty$.

$$y_{+, \lambda}(r) = r^{1-\frac{n}{2}} \left[\frac{1}{2} \sqrt{\frac{k+2}{\pi \sqrt{|\lambda|}}} r^{-\frac{k+2}{4}} \left\{ \exp \left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}} \right) (1 + O(|\lambda|^{-\frac{1}{2}} r^{-\frac{k+2}{2}})) \right. \right. \\ \left. \left. + \exp \left(-\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}} - (\nu(p) + \frac{1}{2}) \pi \sqrt{-1} \right) (1 + O(|\lambda|^{-\frac{1}{2}} r^{-\frac{k+2}{2}})) \right\} \right],$$

$$y_{-, \lambda}(r) = r^{1-\frac{n}{2}} \frac{1}{2} \sqrt{\frac{(k+2)\pi}{\sqrt{|\lambda|}}} r^{-\frac{k+2}{4}} \exp \left(-\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}} (1 + O(|\lambda|^{-\frac{1}{2}} r^{-\frac{k+2}{2}})) \right),$$

$\lambda < 0, r \rightarrow \infty$ or $\lambda \rightarrow -\infty$.

Hence to set $G_\lambda(r, \rho) = \rho^{n/2} r^{1-n/2} H_\lambda(r, \rho)$, we have $H_\lambda(r, \rho)|_{\rho \geq r} = H_\lambda(\rho, r)|_{r \geq \rho}$ and for $k > -2$

$$(5)_0 \quad H_\rho(r, \rho) = O\left(\left(\frac{\rho}{r}\right)^{\frac{n}{2}+p-1}\right), \quad r \geq \rho, \quad r \rightarrow 0,$$

$$(5)_+ \quad H_{\rho, k}(r, \rho) \sim \frac{1}{\sqrt{|\lambda|}} \left(\frac{1}{\rho r} \right)^{\frac{k+2}{4}} \left\{ \sin \left(\frac{2\sqrt{|\lambda|}}{k+2} \rho^{\frac{k+2}{2}} - \frac{\nu(p)}{2} \pi - \frac{\pi}{4} \right) \right. \\ \left. \cos \left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}} - \frac{\nu(p)}{2} \pi - \frac{\pi}{4} \right) \right\}, \\ \sim \frac{1}{2\sqrt{|\lambda|}} \left(\frac{1}{\rho r} \right)^{\frac{k+2}{4}} \left[\sin \left(\frac{2\sqrt{|\lambda|}}{k+2} |\rho^{\frac{k+2}{2}} - r^{\frac{k+2}{2}}| \right) \right. \\ \left. - \cos \left(\frac{2\sqrt{|\lambda|}}{k+2} \left(\rho^{\frac{k+2}{2}} + r^{\frac{k+2}{2}} \right) - \nu(p) \pi \right) \right],$$

$\rho \geq r, r \rightarrow \infty, \lambda > 0$ or $\lambda \rightarrow \infty$,

$$(5)_- \quad H_{\lambda, k}(r, \rho) \sim \frac{1}{2\sqrt{|\lambda|}} \left(\frac{1}{\rho r} \right)^{\frac{k+2}{4}} \left[\exp \left(-\frac{4\sqrt{|\lambda|}}{k+2} |\rho^{\frac{k+2}{2}} - r^{\frac{k+2}{2}}| \right) \right. \\ \left. + \exp \left(-\frac{2\sqrt{|\lambda|}}{k+2} \left(\rho^{\frac{k+2}{2}} - r^{\frac{k+2}{2}} \right) - \left(\nu(p) + \frac{1}{2} \right) \pi \sqrt{-1} \right) \right],$$

$r \rightarrow \infty, \lambda > 0$ or $\lambda \rightarrow -\infty$.

By (5), we obtain

Lemma 2. (i). $G_{\lambda, k, a}$ is defined on $L^2([0, a], r^{n-1} dr)$ and maps it into $L^2([0, a],$

$r^{n-1}dr)$ unless $k=-2$, $\lambda=(p+n/2-1)^2$.

(ii). If $\lambda>0$, $G_{\lambda,k}$ is defined on $L^2(\mathbb{R}^+, r^{n-1}dr)$ and maps it into $L^2(\mathbb{R}^+, r^{n-1}dr)$ if $k>2$.

(iii). If $\lambda<0$, $G_{\lambda,k}$ is defined on $L^2(\mathbb{R}^+, r^{n-1}dr)$ if $k>-2$ and maps it into $L^2(\mathbb{R}^+, r^{n-1}dr)$ if $k>2$.

(iv). $\operatorname{tr} G_{\lambda,k}$ exists if $k>2$.

2. **Lemma 3.** (i). As a function of ρ , $H_{\lambda,k}(r, \rho)$ belongs in $L^2([0, a], d\rho)$ for all r , $0 < r \leq a$, and at $r \rightarrow 0$, we have

$$\| \rho^{\frac{1}{2}} H_{\lambda,k}(r, \rho) \|_{L^2([0, a], d\rho)} = O(r), \quad k \neq -2, \quad n+2p \neq 4,$$

or $k=-2$, $\lambda \leq (p+n/2-1)^2 - 1$,

$$\| \rho^{\frac{1}{2}} H_{\lambda,k}(r, \rho) \|_{L^2([0, a], d\rho)} = O(r |\log r|^{\frac{1}{2}}), \quad k \neq -2, \quad n+2p=4,$$

$$\| \rho^{\frac{1}{2}} H_{\lambda,-2}(r, \rho) \|_{L^2([0, a], d\rho)} = O(r^{\sqrt{(p+n/2-1)^2 - \lambda}}),$$

$$(p+n/2-1)^2 - 1 < \lambda < (p+n/2-1)^2,$$

$$\| \rho^{\frac{1}{2}} H_{\lambda,-2}(r, \rho) \|_{L^2([0, a], d\rho)} = O(1), \quad \lambda > (p+n/2-1)^2.$$

(ii). if $G_{\lambda,k,a}$ is defined on $L^2([0, a], r^{n-1}dr)$, then

$$(6) \quad \| G_{\lambda,k,a} \| = \left(\int_0^a r \| \rho^{\frac{1}{2}} H_{\lambda,k}(r, \rho) \|_{L^2([0, a], d\rho)}^2 dr \right)^{\frac{1}{2}}.$$

Proof. Since we have

$$(7) \quad r^{-2\alpha} \int_0^r \rho^{1+2\alpha} d\rho + r^2 \int_r^a \rho^{1-2\alpha} d\rho = \frac{a^{2-2\alpha}}{2-2\alpha} r^{2\alpha} - \frac{4\alpha^2 r^2}{4-4\alpha},$$

we get (i) by (5)_0. By (i), for $f \in L^2([0, a], r^{n-1}dr)$, $\|f\|=1$, we get by Schwarz inequality

$$\begin{aligned} \| G_\lambda f \| &= \int_0^a r \left| \int_0^a \rho^{\frac{n}{2}} H_\lambda(r, \rho) f(\rho) d\rho \right|^2 dr \leq \\ &\leq \int_0^a r \int_0^a \left| \rho^{\frac{1}{2}} H_\lambda(r, \rho) \right|^2 d\rho \int_0^a |f(\rho)|^2 \rho^{n-1} d\rho dr \\ &= \int_0^a r \| \rho^{\frac{1}{2}} H_\lambda(r, \rho) \|_{L^2([0, a], d\rho)}^2 dr. \end{aligned}$$

Hence we have (ii).

Proposition 1. At $|\lambda| \rightarrow \infty$, we have

$$(8) \quad \| G_{\lambda,k,a} \| = O(|\lambda|^{-\frac{1}{2}}), \quad -2 \leq k < 2,$$

$$\|G_{\lambda, 2, \alpha}\| = O(|\lambda|^{-\frac{1}{2}} \log |\lambda|),$$

$$\|G_{\lambda, k, \alpha}\| = O(|\lambda|^{-\frac{2}{k+2}}), \quad k > 2,$$

$$\|G_{\lambda, k}\| = O(|\lambda|^{-\frac{2}{k+2}}), \quad k > 2.$$

Proof. To set $\alpha = \sqrt{(p+n/2-1)^2 - \lambda}$, we have by (7)

$$\left(\int_0^a r \| \rho^{\frac{1}{2}} H_{\lambda, -2}(r, \rho) \|_{L^2([0, a], d\rho)}^2 dr \right)^{\frac{1}{2}} = \frac{a^2}{4|\alpha|},$$

$$\lambda \neq (p+n/2-1)^2, \quad (p+n/2-1)^2 - 1,$$

because $\sqrt{(1/4\alpha^2)\alpha^4(1-\alpha^2)/(4-4\alpha^2)} = a^2/4|\alpha|$. Then, since $|\alpha| = O(|\lambda|^{1/2})$, we get

(8) for $k = -2$.

If $k > -2$, set $\sigma = (2\sqrt{|\lambda|}/(k+2))\rho^{(k+2)/2}$, we have for $0 \leq \rho \leq a$ (or $0 \leq \sigma$),

$$(9) \quad \begin{aligned} \rho &= \left(\frac{k+2}{2\sqrt{|\lambda|}} \right)^{\frac{2}{k+2}} \sigma^{\frac{2}{k+2}}, \quad 0 \leq \sigma \leq \frac{2\sqrt{|\lambda|}}{k+2} a^{\frac{k+2}{2}} \text{ or } 0 \leq \sigma, \\ d\rho &= \left(\frac{k+2}{2\sqrt{|\lambda|}} \right)^{\frac{2}{k+2}} \frac{2}{k+2} \sigma^{-\frac{k}{k+2}} d\sigma. \end{aligned}$$

Hence we get the last formula. Other parts of proposition follows from (5) and lemma 3.

Note. Set $G_{\lambda, (0)}(r, \rho) = G_{\lambda, k}(r, \rho) - (y_{-, \lambda}(a)/y_{+, \lambda}(a)) W(y_{+, \lambda}, y_{-, \lambda})(\rho)^{-1} \cdot y_{+, \lambda}(r) y_{-, \lambda}(\rho)$, if $y_{+, \lambda}(a) \neq 0$, and

$$\begin{aligned} G_{\lambda, a, (0)} f &= \int_0^a G_{\lambda, (0)}(r, \rho) f(\rho) d\rho, \quad y_{+, \lambda}(a) \neq 0, \\ G_{\lambda, a, (0)} f &= \int_0^a G_{\lambda}(r, \rho) f(\rho) d\rho, \quad f \in (y_{+, \lambda})^\perp \quad \text{in } L^2(0, a, r^{n-1} dr), \\ y_{+, \lambda}(a) &= 0, \end{aligned}$$

$G_{\lambda, a, (0)}$ is the fundamental solution of $L_{n, p} + \lambda r^k$ with the boundary condition

$$(4)_0 \quad u(0) = u(a) = 0.$$

Since $y_{+, \lambda}(a) \neq 0$ for $k > -2$ if $\lambda < 0$ by a theorem of Hurwitz ([17]) and

$$\frac{y_{-, \lambda}(a)}{y_{+, \lambda}(a)} \sim \pi \exp \left(-\frac{4\sqrt{|\lambda|}}{k+2} a^{\frac{k+2}{2}} - \frac{\nu(p)}{2} - \frac{\pi}{4} \right) (1 + O(|\lambda|^{-\frac{1}{2}} a^{-\frac{k+2}{2}})), \quad \lambda \rightarrow -\infty,$$

$G_{\lambda, k, a, (0)}$ and $G_{\lambda, a, (0)}$ are defined on $L^2([0, a], r^{n-1} dr)$ and on $L^2(\mathbb{R}^+, r^{n-1} dr)$ and they satisfy same estimates as (8) for $\lambda \rightarrow -\infty$. But if $\lambda > 0$, $G_{\lambda, k, a, (0)}$ may not be

defined on $L^2([0, a], r^{n-1}dr)$ and since

$$\begin{aligned} \frac{y_{-, \lambda}(a)}{y_{+, \lambda}(a)} &\sim \tan\left(\frac{2\sqrt{\lambda}}{k+2}a^{\frac{k+2}{2}} - \frac{\nu(p)}{2} - \frac{\pi}{4}\right)\left(1 + O\left(\frac{k+2}{\sqrt{\lambda}}a^{-\frac{k+2}{2}}\right)\right) \\ &+ O\left(\frac{k+2}{\sqrt{\lambda}}a^{-\frac{k+2}{2}}\right), \quad \lambda \rightarrow \infty, \end{aligned}$$

we have no such estimate for $\lambda \rightarrow \infty$.

3. Lemma 4. (i). If $a < \infty$, we have

$$(10) \quad \operatorname{tr} G_{\lambda, -2, a} = -\frac{a^2}{4} \left\{ \left(p + \frac{n}{2} - 1\right)^2 - \lambda \right\}^{-\frac{1}{2}}, \quad \lambda \neq \left(p + \frac{n}{2} - 1\right)^2,$$

$$\operatorname{tr} G_{(p+n/2-1)^2, -2, a} = -\frac{a^2}{2} \left(\log a - \frac{1}{2} \right).$$

(ii). If $k > 2$, we have

$$(11) \quad \begin{aligned} \operatorname{tr} G_{\lambda, k} &= -\left(\frac{1}{k+2}\right)^{\frac{2k}{k+2}} B\left(\frac{k-2}{k+2}, -\frac{2}{k+2}\right) \frac{\Gamma(\nu(p) + \frac{2}{k+2})}{\Gamma(\nu(p) + \frac{k}{k+2})} |\lambda|^{-\frac{2}{k+2}}, \quad \lambda < 0, \\ \operatorname{tr} G_{\lambda, k} &= \sin \frac{(k-2)\pi}{2(k+2)} \operatorname{tr} G_{-\lambda, k}, \quad \lambda > 0. \end{aligned}$$

Proof. Since $\operatorname{tr} G_{\lambda, -2, a} = -(1/2)\{(\nu(p) + n/2 - 1)^2 - \lambda\}^{-1/2} \int_0^a r dr$, $\lambda \neq (\nu(p) + n/2 - 1)^2$, $\operatorname{tr} G_{(\nu(p) + n/2 - 1)^2, -2, a} = \int_0^a r \log r dr$, we get (i).

To show (ii), we set $s = (2\sqrt{|\lambda|}/(k+2)) r^{(k+2)/2}$. Then by (9), we have

$$\begin{aligned} &\int_0^\infty r J_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2}r^{\frac{k+2}{2}}\right) Y_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2}r^{\frac{k+2}{2}}\right) dr \\ &= \lambda^{-\frac{2}{k+2}} \left(\frac{2}{k+2}\right)^{\frac{k-2}{k+2}} \int_0^\infty s^{-\frac{k-2}{k+2}} J_{\nu(p)}(s) Y_{\nu(p)}(s) ds, \quad \lambda > 0, \\ &\int_0^\infty r I_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2}r^{\frac{k+2}{2}}\right) K_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2}r^{\frac{k+2}{2}}\right) dr \\ &= |\lambda|^{-\frac{2}{k+2}} \left(\frac{2}{k+2}\right)^{\frac{k-2}{k+2}} \int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds, \quad \lambda < 0. \end{aligned}$$

We know that, if $a > 1$, $\operatorname{Re} k > 2$,

$$\int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(as) ds$$

$$= \left(\frac{a}{2}\right)^{-\frac{4}{k+2}} \left(\frac{1}{a}\right)^{\nu(p)} \frac{\Gamma\left(\nu(p) + \frac{2}{k+2}\right) \Gamma\left(\frac{2}{k+2}\right)}{4\Gamma(\nu(p)+1)} \\ F\left(\nu(p) + \frac{2}{k+2}, -\frac{2}{k+2}, \nu(p)+1, -\frac{1}{a^2}\right),$$

([9]). But since for sufficiently large M , $|K_{\nu(p)}(as)| < 2|K_{\nu(p)}(s)|$, $s \geq M$, $a \geq 1$, and since $|s^{-(k-2)/(k+2)} I_{\nu(p)}(s) K_{\nu(p)}(as)|$ is integrable on $[0, \infty)$ if $\operatorname{Re} k > 2$, we obtain

$$\lim_{a \rightarrow 1+0} \int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(as) ds = \int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds,$$

by Lebesgue's theorem. Hence we have

$$\int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds \\ = \lim_{a \rightarrow 1+0} \left(\frac{a}{2}\right)^{-\frac{4}{k+2}} \left(\frac{1}{a}\right)^{\nu(p)} \frac{\Gamma\left(\nu(p) + \frac{2}{k+2}\right) \Gamma\left(\frac{2}{k+2}\right)}{4\Gamma(\nu(p)+1)} \\ F\left(\nu(p) + \frac{2}{k+2}, -\frac{2}{k+2}, \nu(p)+1, -\frac{1}{a^2}\right).$$

On the other hand, since $\operatorname{Re} (\nu(p)+1) - (\nu(p)+2/(k+2)) > 0$, if $\operatorname{Re} k > 2$, we get

$$\lim_{a \rightarrow 1+0} F\left(\nu(p) + \frac{2}{k+2}, -\frac{2}{k+2}, \nu(p)+1, -\frac{1}{a^2}\right) \\ = F\left(\nu(p) + \frac{2}{k+2}, -\frac{2}{k+2}, \nu(p)+1, 1\right) = \frac{\Gamma(\nu(p)+1) \Gamma\left(\frac{k-2}{k+2}\right)}{\Gamma\left(\nu(p) + \frac{k}{k+2}\right) \Gamma\left(\frac{k}{k+2}\right)}.$$

Therefore we obtain

$$(12) \quad \int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds = 2^{-\frac{2k}{k+2}} \frac{\Gamma\left(\nu(p) + \frac{2}{k+2}\right) \Gamma\left(\frac{k-2}{k+2}\right) \Gamma\left(\frac{2}{k+2}\right)}{\Gamma\left(\nu(p) + \frac{k}{k+2}\right) \Gamma\left(\frac{k}{k+2}\right)} \\ = 2^{-\frac{2k}{k+2}} B\left(\frac{k-2}{k+2}, -\frac{2}{k+2}\right) \frac{\Gamma\left(\nu(p) + \frac{2}{k+2}\right)}{\Gamma\left(\nu(p) + \frac{k}{k+2}\right)},$$

if $\operatorname{Re} k > 2$. Then, since $\int_0^\infty s^{-(k-2)/(k+2)} J_{\nu(p)}(s) Y_{\nu(p)}(as) ds = -(2/\pi) \sin((k-2)\pi/2(k+2)) \int_0^\infty s^{-(k-2)/(k+2)} I_{\nu(p)}(s) K_{\nu(p)}(as) ds$, $a > 1$ ([9]), and since $|Y_{\nu(p)}(as)| \leq 1/\sqrt{s}$, $s \geq M$, $a \geq 1$ for large M and $|s^{-(k-2)/(k+2)-1/2} J_{\nu(p)}(s)|$ is integrable

on $[M, \infty)$, we obtain

$$\begin{aligned}
(12)' & \int_0^\infty s^{-\frac{k-2}{k+2}} J_{\nu(p)}(s) Y_{\nu(p)}(s) ds \\
& = \lim_{a \rightarrow 1+0} -\frac{2}{\pi} \sin\left(\frac{(k-2)\pi}{2(k+2)}\right) \int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(as) ds \\
& = -\frac{1}{\pi} 2^{-\frac{k-2}{k+2}} \sin\left(\frac{(k-2)\pi}{2(k+2)}\right) B\left(\frac{k-2}{k+2}, \frac{2}{k+2}\right) \frac{\Gamma\left(\nu(p)+\frac{2}{k+2}\right)}{\Gamma\left(\nu(p)+\frac{k}{k+2}\right)}.
\end{aligned}$$

By (12) and (12)', we obtain (ii). Because we have

$$\begin{aligned}
\operatorname{tr} G_{\lambda, k} &= \frac{\pi}{k+2} \int_0^\infty r J_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) Y_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) dr \\
&= \frac{\pi}{k+2} \lambda^{-\frac{k}{k+2}} \left(\frac{2}{k+2}\right)^{\frac{k-2}{k+2}} \int_0^\infty s^{-\frac{k-2}{k+2}} J_{\nu(p)}(s) Y_{\nu(p)}(s) ds, \quad \lambda > 0, \\
\operatorname{tr} G_{\lambda, k} &= -\frac{\pi}{k+2} \int_0^\infty r I_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) K_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) dr \\
&= -\frac{2}{k+2} |\lambda|^{-\frac{2}{k+2}} \left(\frac{2}{k+2}\right)^{\frac{k-2}{k+2}} \int_0^\infty s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds, \quad \lambda < 0.
\end{aligned}$$

Since we know ([6], I)

$$\Gamma\left(\frac{k-2}{k+2}\right) = \Gamma\left(1 - \frac{4}{k+2}\right) = \frac{\pi}{\Gamma\left(\frac{4}{k+2}\right) \sin\left(\frac{4\pi}{k+2}\right)},$$

$$\Gamma\left(\frac{k}{k+2}\right) = \frac{\pi}{\Gamma\left(\frac{2}{k+2}\right) \sin\left(\frac{2\pi}{k+2}\right)},$$

$$\Gamma\left(\frac{4}{k+2}\right) = 2^{\frac{4}{k+2}-1} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{2}{k+2}\right) \Gamma\left(\frac{2}{k+2} + \frac{1}{2}\right),$$

we get

$$(13) \quad B\left(\frac{k-2}{k+2}, \frac{2}{k+2}\right) = 2^{-\frac{4}{k+2}} \sqrt{\pi} \frac{\Gamma\left(\frac{2}{k+2}\right)}{\Gamma\left(\frac{2}{k+2} + \frac{1}{2}\right) \cos\left(\frac{2\pi}{k+2}\right)}.$$

On the other hand, since $\sin((k-2)\pi/2(k+2)) = \sin(\pi/2 - 2\pi/(k+2)) = \cos(2\pi/(k+2))$, to use $\nu(p) = (2p+n-2)/(k+2)$, we rewrite (11) as follows

$$(14) \quad \text{tr } G_{\lambda, k} = -2^{-\frac{4}{k+2}} \sqrt{\pi} \left(\frac{1}{k+2}\right)^{\frac{2k}{k+2}} \frac{\Gamma\left(\frac{2}{k+2}\right)}{\Gamma\left(\frac{2}{k+2} + \frac{1}{2}\right)} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \lambda^{-\frac{2}{k+2}}, \lambda > 0,$$

$$\text{tr } G_{\lambda, k} = \frac{1}{\cos\left(\frac{2\pi}{k+2}\right)} \text{tr } G_{|\lambda|, k}, \lambda < 0.$$

In the rest, we set (k may be a complex number)

$$(15) \quad C_{n, k, +} = -\frac{2^{-\frac{4}{k+2}} \pi \left(\frac{1}{k+2}\right)^{\frac{2k}{k+2}} \Gamma\left(\frac{2}{k+2}\right)}{(n-2)! \Gamma\left(\frac{2}{k+2} + \frac{1}{2}\right)}, \quad C_{n, k, -} = \frac{C_{n, k, +}}{\cos\left(\frac{2\pi}{k+2}\right)}.$$

§2. Analytic continuations in k and λ .

4. Since $r > 0$, for complex k , we determin $r^{(k+2)/2}$ to be

$$(16) \quad r^{\frac{k+2}{2}} = e^{\frac{(k+2)}{2} \log r}, \log r \in \mathbb{R}.$$

Then to set $\xi = \operatorname{Re}(k+2)/2 = 1 + (1/2) \operatorname{Re} k$, $\eta = \operatorname{Im}(k+2)/2 = (1/2) \operatorname{Im} k$, $\theta = \arg(k+2)/2 = \tan^{-1}(\xi/\eta)$, $-\pi < \theta \leq \pi$, we get

$$|\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}| = \frac{2\sqrt{|\lambda|}}{|k+2|} r^\xi, \arg \frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}} = \log r\eta - \theta.$$

We set $r_n = e^{(2n\pi+\theta)/n}$. By definition, $(2\sqrt{|\lambda|}/(k+2)) r_n^{(k+2)/2}$ is a positive real number. Hence to set $s = (2\sqrt{|\lambda|}/(k+2)) r^{(k+2)/2}$, $s_n = (2\sqrt{|\lambda|}/(k+2)) r_n^{(k+2)/2}$. $\arg s_n^{-(k-2)/(k+2)}$ does not depend on n because $s^{-(k-2)/(k+2)} = (2\sqrt{|\lambda|}/(k+2)) r^{2-(k+2)/2}$.

Since $G_{\lambda, k, a}$ given in n°1 is the fundamental solution of $L_{n, p} + r^k$ with the boundary condition (4) although k is a complex number, we have

$$\begin{aligned} \text{tr } G_{\lambda, k, a} &= \frac{\pi}{k+2} \int_0^a r J_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) Y_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) dr, \lambda > 0, \\ &= -\frac{2}{k+2} \int_0^a r I_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) K_{\nu(p)}\left(\frac{2\sqrt{|\lambda|}}{k+2} r^{\frac{k+2}{2}}\right) dr, \lambda < 0. \end{aligned}$$

Hence to set $s = (2\sqrt{|\lambda|}/(k+2)) r^{(k+2)/2}$, $\gamma_{c, d} = \{(2\sqrt{|\lambda|}/(k+2)) r^{(k+2)/2} | c \leq r \leq d\}$, we have

$$\begin{aligned} \text{tr } G_{\lambda, k, a} &= \frac{\pi}{k+2} |\lambda|^{-\frac{2}{k+2}} \left(\frac{2}{k+2}\right)^{\frac{k-2}{k+2}} \int_{r_0, a} s^{-\frac{k-2}{k+2}} J_{\nu(p)}(s) Y_{\nu(p)}(s) ds, \lambda > 0, \\ &= -\frac{2}{k+2} |\lambda|^{-\frac{2}{k+2}} \left(\frac{2}{k+2}\right)^{\frac{k-2}{k+2}} \int_{r_0, a} s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds, \lambda < 0. \end{aligned}$$

Lemma 5. To set

$$I_{n,+} = \int_{s_n}^{s_{n+1}} r^{-\frac{k-2}{k+2}} J_{\nu(p)}(r) Y_{\nu(p)}(r) dr,$$

$$I_{n,-} = \int_{s_n}^{s_{n+1}} r^{-\frac{k-2}{k+2}} I_{\nu(p)}(r) K_{\nu(p)}(r) dr,$$

we have

$$(16) \quad \int_{\gamma r_n, r_{n+1}} s^{-\frac{k-2}{k+2}} J_{\nu(p)}(s) Y_{\nu(p)}(s) ds = I_{n,+},$$

$$\int_{\gamma r_n, r_{n+1}} s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds = I_{n,-}.$$

Proof. Since $J_{\nu(p)}(s) Y_{\nu(p)}(s)$ (respectively $I_{\nu(p)}(s) K_{\nu(p)}(s)$) is a 1-valued function and $s_n^{-(k-2)/(k+2)}$ and $s_{n+1}^{-(k-2)/(k+2)}$ lie on same branch of $s^{-(k-2)/(k+2)}$, the starting point and ending point of $\gamma r_n, r_{n+1}$ lie on same branch of $s^{-(k-2)/(k+2)}$. $J_{\nu(p)}(s) Y_{\nu(p)}(s)$ (respectively $s^{-(k-2)/(k+2)} I_{\nu(p)}(s) K_{\nu(p)}(s)$). Hence we have the lemma.

Lemma 6. If $\operatorname{Re} k > 2$ and $\lambda < 0$, we have

$$(14)_- \quad \operatorname{tr} G_{\lambda, k} = (n-2)! C_{n, k, -} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} |\lambda|^{-\frac{2}{k+2}}.$$

Proof. By lemma 5, we have

$$\int_{\gamma_0, r_n} s^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds = \int_0^{s_n} r^{-\frac{k-2}{k+2}} I_{\nu(p)}(s) K_{\nu(p)}(s) ds.$$

On the other hand, by the asymptotic formulas of $I_{\nu(p)}(s)$ and $K_{\nu(p)}(s)$, we get

$$(17) \quad I_{\nu(p)}(s) K_{\nu(p)}(s) = \frac{1}{2s} \left(1 + O\left(\frac{1}{s}\right)\right), \quad |s| \rightarrow \infty.$$

Hence for $r_n < \rho < r_{n+1}$, we get $\lim_{n \rightarrow \infty} \int_{\gamma r_n, \rho} s^{-(k-2)/(k+2)} I_{\nu(p)}(s) K_{\nu(p)}(s) ds = 0$, because $\operatorname{Re} k > 2$. Therefore we obtain the lemma by (12).

Definition. Let $g(s)$ be a locally integrable function on \mathbb{C} such that continuous on $\mathbb{C} - \mathbb{R}^+$, \mathbb{R}^+ is the negative real axis, holomorphic on $\mathbb{C} - \mathbb{R}$ and $|g(s)| \leq C e^{-|Im s|}$, $Im s \neq 0$, for some constant C . Then we set

$$B_{\lambda, k}(\mathbb{R}^+) = \left\{ f \mid f(r) = g\left(\frac{2\sqrt{\lambda}}{k+2} r^{\frac{k+2}{2}}\right) \text{ for some } g, \quad f(r) \in L^2(\mathbb{R}^+, r^{n-1} dr) \right\}.$$

The closure of $B_{\lambda, k}(\mathbb{R}^+)$ in $L^2(\mathbb{R}^+, r^{n-1} dr)$ is denoted by $\overline{B_{\lambda, k}(\mathbb{R}^+)}$.

Lemma 6. If $\lambda > 0$ and $\operatorname{Re} k > 2$, $G_{\lambda, k}$ is defined on $\overline{B_{\lambda, k}}(\mathbb{R}^+)$ and as an operator on $\overline{B_{\lambda, k}}(\mathbb{R}^+)$, we have

$$(14)_{+}' \quad \operatorname{tr} G_{\lambda, k} = (n-2)! C_{n, k, +} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \lambda^{-\frac{2}{k+2}}.$$

Proof. To set $\sigma = (2\sqrt{-\lambda}/(k+2))\rho^{(k+2)/2}$ and take $r_n < a$, we have for $f \in B_{\lambda, k}(\mathbb{R}^+)$,

$$\begin{aligned} \int_{r_n}^a G_{\lambda, k}(r, \rho) f(\rho) d\rho &= \left(\frac{k+2}{2\sqrt{-\lambda}}\right)^{\frac{2}{k+2}} \frac{2}{k+2} \\ &\int_{r_n}^a \sigma^{-\frac{2}{k+2}} G\left(r, \left(\frac{k+2}{2\sqrt{-\lambda}}\sigma\right)^{\frac{2}{k+2}}\right) g(\sigma) d\sigma. \end{aligned}$$

But since $G(r, ((k+2)/2\sqrt{-\lambda})\sigma)^{2/(k+2)} = O((1/\sqrt{\sigma}) e^{|Im \sigma|})$, $\sigma \rightarrow \infty$, by the asymptotic formulas of $J_{\nu(p)}(s)$ and $Y_{\nu(p)}(s)$, we get $\lim_{n \rightarrow \infty} \int_{r_n}^a G_{\lambda, k}(r, \rho) f(\rho) d\rho = 0$ by the definition of $B_{\lambda, k}(\mathbb{R}^+)$ if $\operatorname{Re} k > 2$. Hence $G_{\lambda, k}$ is defined on $\overline{B_{\lambda, k}}(\mathbb{R}^+)$. Then, since $\operatorname{tr} G_{\lambda, k} = \lim_{n \rightarrow \infty} \int_0^a G_{\lambda, k}(r, r) dr$, we obtain the lemma by lemma 5 and (12)'.

5. Let λ be a complex number with $\operatorname{Re} \lambda \neq 0$. Then we determine $-\pi/4 < \sqrt{-\lambda} < \pi/4$, $\operatorname{Re} \lambda > 0$, and set

$$\begin{aligned} (2)_{+}' \quad y_{+, \lambda}(r) &= r^{1-\frac{n}{2}} J_{\nu(p)}\left(\frac{2\sqrt{-\lambda}}{k+2} r^{\frac{k+2}{2}}\right), \\ y_{-, \lambda}(r) &= r^{1-\frac{n}{2}} Y_{\nu(p)}\left(\frac{2\sqrt{-\lambda}}{k+2} r^{\frac{k+2}{2}}\right), \quad \operatorname{Re} \lambda > 0, \\ (2)_{-}' \quad y_{+, \lambda}(r) &= r^{1-\frac{n}{2}} I_{\nu(p)}\left(\frac{2\sqrt{-\lambda}}{k+2} r^{\frac{k+2}{2}}\right), \\ y_{-, \lambda}(r) &= r^{1-\frac{n}{2}} K_{\nu(p)}\left(\frac{2\sqrt{-\lambda}}{k+2} r^{\frac{k+2}{2}}\right), \quad \operatorname{Re} \lambda < 0. \end{aligned}$$

To use these $y_{+, \lambda}$, $y_{-, \lambda}$, we define $G_{\lambda, k}$ similarly as in n°1.

Lemma 6'. (i). If $\operatorname{Re} \lambda < 0$, $\operatorname{Re} k > 2$, $G_{\lambda, k}$ is defined on $L^2(\mathbb{R}^+, r^{n-1} dr)$ and we have

$$(14)_{-}'' \quad \operatorname{tr} G_{\lambda, k} = (n-2)! C_{n, k, -} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} (-\lambda)^{-\frac{2}{k+2}}.$$

(ii) If $\operatorname{Re} \lambda > 0$, $\operatorname{Re} k > 2$, $G_{\lambda, k}$ is defined on $\overline{B_{\lambda, k}}(\mathbb{R}^+)$ and as an operator on $\overline{B_{\lambda, k}}(\mathbb{R}^+)$, we have

$$(14)_{\pm}'' \quad \text{tr } G_{\lambda, k} = (n-2)! C_{n, k, \pm} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \lambda^{-\frac{2}{k+2}}.$$

Proof. (i) follows from (17). Since $-\pi/2 < \arg \lambda < \pi/2$ if $\operatorname{Re} \lambda > 0$, $B_{\lambda, k}(\mathbf{R}^+)$ is defined if $\operatorname{Re} \lambda > 0$. Hence we obtain (ii).

We note that as the functions of k , the right hand sides of $(14)_{\pm}''$ are continued meromorphically on $C - [-2, -\infty]$ and if $\operatorname{Re} k > -2$, the right hand side of $(14)_+''$ is holomorphic and the right hand side of $(14)_-''$ has poles of order 1 at $k = -2(2m-1)/(2m+1)$, $m = 0, 1, 2, \dots$.

Definition. (i). If $\operatorname{Re} \lambda > 0$ and $\operatorname{Re} k > -2$, we define $\operatorname{tr} G_{\lambda, k}$ by the right hand side of $(14)_+''$.

(ii). If $\operatorname{Re} \lambda < 0$ and $\operatorname{Re} k > -2$, $k \neq -2(2m-1)/(2m+1)$, $m = 0, 1, 2, \dots$, we define $\operatorname{tr} G_{\lambda, k}$ by the right hand side of $(14)_-''$.

Lemma 7. In the domain $\{k | \operatorname{Re} k > -2\}$, we have

$$(18) \quad \lim_{k \rightarrow \infty} C_{n, k, \pm} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} = -\frac{1}{(n-2)!} \frac{1}{2(2p+n)}.$$

Proof. Since we have $(1/(k+2))^{2k/(k+2)} = (1/(k+2))^2 1/(k+2)^{-4/(k+2)}$, $\Gamma(2/(k+2)) = ((k+2)/2) \Gamma((2/(k+2))+1)$, $\Gamma((2p+n)/(k+2)) = ((k+2)/(2p+n)) \Gamma(((2p+n)/(k+2))+1)$, we get (18) for $C_{n, k, +}$ because $\Gamma(1/2) = \sqrt{\pi}$. Then, since $\lim_{k \rightarrow \infty} \cos(\pi/(k+2)) = 1$, we get (18) for $C_{n, k, -}$.

Corollary. In the domain $\{k | \operatorname{Re} k > -2\}$, we have

$$(19) \quad \lim_{k \rightarrow \infty} \operatorname{tr} G_{\lambda, k} = \frac{-1}{2(2p+n)}, \quad \operatorname{Re} \lambda \neq 0.$$

Note. Since $\lim_{k \rightarrow \infty} (k+2)^2 (1 - 1/\cos(\pi/(k+2))) = -\pi^2/2$, we obtain

$$(19)' \quad \lim_{k \rightarrow \infty} k^2 [\operatorname{tr} G_{\lambda, k} - \operatorname{tr} G_{-\lambda, k}] = \frac{\pi^2}{4(2p+n)}, \quad \operatorname{Re} \lambda > 0.$$

§3. The operator $\mathcal{A}_{w, \alpha} + r^k D$.

6. Let M be a Riemannian manifold, E an Hermitian vector bundle over M , D a (formally selfadjoint) elliptic operator on $C^\infty(M, E)$. We fix a selfadjoint L^2 -extension of D and assume it allows spectral decomposition without continuous spectre.

Let $n \geq 2$ and \mathcal{A} be the Laplacian on \mathbf{R}^n with the coordinate $x = (x_1, \dots, x_n)$. Let $r = \sqrt{x_i^2}$ be the norm function on \mathbf{R}^n and set $x_1 = r \sin \theta_{n-1} \cdots \sin \theta_2 \sin \theta_1$, $x_2 =$

$r \sin \theta_{n-1} \cdots \sin \theta_2 \cos \theta_1, \dots, x_{n-1} = r \sin \theta_{n-1} \cos \theta_{n-2}, x_n = r \cos \theta_{n-1}$. Then Δ is written as $\partial^2 / \partial r^2 + (n-1) / r \partial / \partial r + 1 / r^2 \Delta_{n-1, \theta}$, where $\Delta_{n-1, \theta}$ is the Laplacian on S^{n-1} .

On $\mathbb{R}^n \times M$, $\Delta + r^k D$ acts on $C^\infty(\mathbb{R}^n \times M, \pi^*(E))$, where π is the projection from $\mathbb{R}^n \times M$ onto M . We denote the λ -proper space of D in $L^2(M, E)$ by E_λ and the space of homogeneous harmonic polynomials of degree p restricted on S^{n-1} by $\mathcal{H}^{n,p}$. Their O.N. basis are denoted by $\phi_{\lambda, i}(y)$ and $A_{(i)}^{n,p}(\theta)$. It is known ([6], II, [16])

$$(20) \quad \dim \mathcal{H}^{n,p} = h(p, n) = \frac{(2p+n-2)(n+p-3)!}{p!(n-2)!}, \quad (n, p) \neq (2, 0), \quad h(0, 2) = 1.$$

With suitable completion, we get $L^2(\mathbb{R}^n \times M, \pi^*(E)) = L^2(\mathbb{R}^n) \otimes L^2(M, E)$, $L^2(M, E) = \sum_\lambda E_\lambda$, $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^+, r^{n-1} dr) \otimes L^2(S^{n-1})$ and $L^2(S^{n-1}) = \sum_p \mathcal{H}^{n,p}$. Then we have

$$(21) \quad (\Delta + r^k D) | L^2([0, a], r^{n-1} dr) \otimes \mathcal{H}^{n,p} \otimes E_\lambda \\ = (L_{n,p} + \lambda r^k) \otimes 1_{\mathcal{H}^{n,p}} \otimes 1_{E_\lambda}, \quad 0 < a \leq \infty,$$

where $1_{\mathcal{H}^{n,p}}$ and 1_{E_λ} are the identity operators on $\mathcal{H}^{n,p}$ and E_λ . Hence to set

$$G^p_{\lambda, k, a}(f)(r, \theta, y)$$

$$= \int_0^a \int_{S^{n-1} \times M} G_{\lambda, k}(r, \rho) \sum_i \sum_{(i)} A_{(i)}^{n,p}(\theta) \overline{A_{(i)}^{n,p}(\rho)} \phi_{\lambda, i}(y) \overline{\phi_{\lambda, i}(\eta)} f(\rho, \varphi, \eta) d\eta d\varphi d\rho,$$

$G^p_{\lambda, k, a}$ is a fundamental solution of $\Delta + r^k D$ on $L^2([0, a], r^{n-1} dr) \otimes \mathcal{H}^{n,p} \otimes E_\lambda$ with the boundary condition

$$\begin{aligned} u(0, \theta, y) &= 0, \quad \frac{u(a, \theta, y)}{u'(a, \theta, y)} = \frac{y_{-, i}(a)}{y_{-, i}'(a)}, \quad y_{-, i}'(a) \neq 0, \\ u(0, \theta, y) &= 0, \quad u'(a, \theta, y) = 0, \quad y_{-, i}'(a) = 0. \end{aligned}$$

We denote $G^p_{\lambda, k, \infty}$ by $G^p_{\lambda, k}$. In this case, the boundary condition is $u \in L^2(\mathbb{R}^+, r^{n-1} dr) \otimes \mathcal{H}^{n,p} \otimes E_\lambda$.

In the rest, we denote $L^2([0, a], r^{n-1} dr) \otimes \mathcal{H}^{n,p} \otimes L^2(M, E)$ by $L^2(\mathbb{R}^n \times M, E)_{p,a}$ and $L^2(\mathbb{R}^+, r^{n-1} dr) \otimes \mathcal{H}^{n,p} \otimes L^2(M, E)$ by $L^2(\mathbb{R}^n \times M, E)_p$.

Definition. In $L^2(\mathbb{R}^n \times M, E)_{p,a}$ (respectively, in $L^2(\mathbb{R}^n \times M, E)_p$), we set

$$G^p_{k,a} = \sum_\lambda G^p_{\lambda, k, a}, \quad G^p_k = \sum_\lambda G^p_{\lambda, k}.$$

By definition, $G^p_{k,a}$ is densely defined in $L^2(\mathbb{R}^n \times M, E)_{p,a}$ and if f belongs in $\mathcal{D}(G^p_{k,a})$, we have $(\Delta + r^k D) G^p_{k,a} f = f$ and

$$(4') \quad G^p_{k,a} f(0, \theta, y) = 0,$$

$$\langle (G^b_{k,a}f)_\lambda(a, \theta, y), (G^b_{k,a}f)'_\lambda(a, \theta, y) \rangle = (y_{-\lambda}(a), y_{-\lambda}'(a)) \in \mathbb{C}\mathbb{P}^1,$$

$$\lambda \in \text{Spec } D.$$

Here $(G^b_{k,a}f)_\lambda$ means λ -component of $G^b_{k,a}f$.

By definition and proposition 1, we obtain for real k

Proposition 2. (i). $G^b_{-2,a}$ is bounded. For large m , $(G^b_{-2,a})^m$ is of trace class.

(ii). $G^b_{k,a}$ is bounded and $(G^b_{k,a})^m$ is of trace class for large m if $k > -2$.

(iii). G^b_k is bounded and $(G^b_k)^m$ is of trace class for large m unless

$\ker D \neq \{0\}$, if $k > 2$.

Note 1. By (20), we have $\|G^b_{k,a}\| = O(h(p, n)) = O(p^{n-2})$, $p \rightarrow \infty$.

Note 2. To use $G_{\lambda,k,(0)}$ instead of $G_{\lambda,k}$, we can construct the fundamental solution $G^b_{k,a,(0)}$ of $\mathcal{A}+r^k D$ with the boundary condition $u(0, \theta, y) = u(a, \theta, y) = 0$. In this case, by the note of n°2, if D has only finite positive proper values, $G^b_{k,a,(0)}$ is bounded. But if D has infinite number of positive proper values, $G^b_{k,a,(0)}$ may not be bounded. Since $\mathcal{D}(G^b_{k,a,(0)}) = \mathcal{N}^b_{k,a}^\perp$, $\mathcal{N}^b_{k,a} = \{u | u \in L^2(\mathbb{R}^n \times M, E)_p, a, u(0, \theta, y) = u(a, \theta, y) = 0, (\mathcal{A}+r^k D)u = 0\}$, and since

$$(22) \quad \mathcal{N}^b_{k,a} = \sum_{\lambda > 0, J_p p((2\sqrt{-\lambda})/(k+2))ak^{2/2}=0} \mathcal{H}^{b,n} \otimes E_\lambda,$$

$G^b_{k,a,(0)}$ is defined on $L^2(\mathbb{R}^n \times M, E)_p, a$ if D has no positive proper value. (22) also shows that to set

$$\mathcal{N}_{n,p,k;D} = \{a | \dim \mathcal{N}^b_{k,a} = \infty\} \subset \mathbb{R}^+,$$

$\mathcal{N}_{n,p,k;D}$ is the empty set if D has only finite positive proper values and by a theorem of Weyl ([5], [18]), $\mathcal{N}_{n,p,k;D}$ does not depend on the self-adjoint lower order perturbation of D .

7. Definition. Let f be an element of $L^2(\mathbb{R}^n)$, w and α are complex numbers with $\operatorname{Re} \alpha > 0$. Then we set

$$(23) \quad (T_{w,\alpha}f)(r, \theta) = \sum_p (p + \alpha)^w f_p(r) \sum_{(i)} A_{(i)}^{n,p}(\theta),$$

$$f(r, \theta) = \sum_p f_p(r) \sum_{(i)} A_{(i)}^{n,p}(\theta).$$

By definition, we have $\mathcal{D}(T_{w_1,\alpha}) \subset \mathcal{D}(T_{w_2,\beta})$ if $\operatorname{Re} w_1 > \operatorname{Re} w_2$ and if $\operatorname{Re} w \leq 0$, $\mathcal{D}(T_{w,\alpha}) = L^2(\mathbb{R}^n)$.

For a fixed f , $T_{w,\alpha}f$ is defined on $\operatorname{Re} w < c$, $\operatorname{Re} \alpha > 0$, where c is a constant determined by f , and holomorphic in w and α on this domain. Since $c \geq 0$, we have as a holomorphic function in w

$$(24) \quad \lim_{w \rightarrow 0} T_{w,\alpha}f = f.$$

We set $x=(r, \theta)$, $\xi=(s, \varphi)$ and define

$$Ef(s, r, \theta) = \int_{S^{n-1}} E(s, \theta, \varphi) f(r, \varphi) d\varphi, \quad 0 < s \leq 1, \quad E(\xi, x) \text{ is the Poisson kernel of } A,$$

$$(I_1 g)(r, \theta) = g(1, r, \theta), \quad g(s, x) \in C^{(\infty, 0)}_b(\mathbb{I} \times \mathbb{R}^n), \quad \mathbb{I} = [0, 1].$$

Here $C^{(\infty, 0)}(\mathbb{I} \times \mathbb{R}^n) = \{f | f \text{ is bounded on } [0, 1] \times \mathbb{R}^n, \text{ continuous on } (0, 1] \times \mathbb{R}^n \text{ and } C^\infty\text{-class in } s\}$. Then we have the following commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^n) & \xrightarrow{T_{w, \alpha}} & L^2(\mathbb{R}^n) \\ \downarrow E & \left(s^{1-\alpha} \frac{\partial}{\partial s} s^\alpha \right)^w & \downarrow I_1 \\ C_b^{(\infty, 0)}(\mathbb{I} \times \mathbb{R}^n) & \xrightarrow{\quad} & C_b^{(\infty, 0)}(\mathbb{I} \times \mathbb{R}^n). \end{array}$$

Here, $(s^{1-\alpha} \frac{\partial}{\partial s} s^\alpha)^w f$ is defined by (cf. [7])

$$\left(s^{1-\alpha} \frac{\partial}{\partial s} s^\alpha \right)^w f(s) = \mathcal{M}^{-1}[(\alpha - \sigma)^w \mathcal{M}[f](\sigma)](s), \quad \mathcal{M}[f](\sigma) = \int_0^1 s^{\sigma-1} f(s) ds.$$

By definition, as a function of w , $(s^{1-\alpha} \frac{\partial}{\partial s} s^\alpha)^w f(s)$ is defined on $\operatorname{Re} w > -c$ if f is $O(s^c)$, $s \rightarrow 0$, and holomorphic in w .

Lemma 8. *Let w be a complex number and not a positive integer, α a complex number with $\operatorname{Re} \alpha > c$. Then to set for $f(s) = O(s^{-c})$, $s \rightarrow 0$,*

$$\begin{aligned} S_{w, \alpha}[f](s) &= \frac{\sqrt{-1}}{2\pi} (1 - e^{2\pi\sqrt{-1}w}) \Gamma(1-w) \int_0^s \frac{1}{t} \left(\frac{t}{s} \right)^\alpha \left\{ \log \left(\frac{t}{s} \right) \right\}^{w-1} f(t) dt, \end{aligned}$$

we have

$$(25) \quad \left(s^{1-\alpha} \frac{\partial}{\partial s} s^\alpha \right)^w S_{w, \alpha}[f] = f, \quad f \in C^{(\infty, 0)}(\mathbb{I} \times \mathbb{R}^n),$$

$$(24)' \quad \lim_{w \rightarrow 0} S_{w, \alpha}[f] = f, \quad f \in C^{(\infty, 0)}(\mathbb{I} \times \mathbb{R}^n), \quad f(0) = 0.$$

Proof. By the inversion formula of Mellin transformation and the definition of $(s^{1-\alpha} \frac{\partial}{\partial s} s^\alpha)^w$, to set

$$S_{w, \alpha}(s, t) = \frac{1}{2\pi\sqrt{-1}} \int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty} (\alpha - \rho)^{-w} t^{-1} \left(\frac{t}{s} \right)^\sigma d\rho,$$

and define $S_{w, \alpha}[f](s)$ by $\int_0^\infty S_{w, \alpha}(s, t) f(t) dt$, we obtain (25). But, since $S_{w, \alpha}(s, t) = (1/2\pi) \int_{-\infty}^\infty (\alpha - \sqrt{-1}\sigma)^{-w} t^{-1} e^{\sqrt{-1}\sigma} \log(t/s) d\sigma$ and when w is not a positive integer,

this integral is equal to the limit of the integral along the half circle in the upper half plane if $t>s$, $\operatorname{Re} w>1$, and equal to the integral along the path starts from $-\sqrt{-1}\infty$, rounds $-\sqrt{-1}$ in the negative direction and ends at $-\sqrt{-1}\infty$ if $t<s$, $\operatorname{Re} w>1$, we get

$$\begin{aligned} S_{w,\alpha}(s, t) &= -\frac{\sqrt{-1}}{2\pi} \frac{1}{t} \left(\frac{t}{s}\right)^{\alpha} \left\{ \log\left(\frac{t}{s}\right) \right\}^{w-1} (1-e^{2\pi\sqrt{-1}w}) \Gamma(1-w), \quad t < s, \\ &= 0, \quad t > s, \end{aligned}$$

if $\operatorname{Re} w>1$ and w is not a positive integer. Then, since $S_{w,\alpha}[f]$ and $(s^{1-\alpha} \partial/\partial s s^\alpha)^w S_{w,\alpha}[f]$ both analytic in w , we have (25) for other w by analytic continuation.

To show (24)', we note that $f(s)=\int_0^s f'(t)dt$ by assumption. Hence we get $S_{w,\alpha}[f](s)=\int_0^s S_{w,\alpha}(s, t) \int_0^t f'(u)dudt=\int_0^s \int_u^s S_{w,\alpha}(s, t) dt f'(u)du=(\sqrt{-1}/2\pi)(e^{2\pi\sqrt{-1}w}-1) \Gamma(1-w) \int_0^s s^{-\alpha} \int_u^s t^{\alpha-1} \{\log(t/s)\}^{w-1} dt f'(u)du$. Then, since $\int_u^s t^{\alpha-1} \{\log(t/s)\}^{w-1} dt=\int_{u/s}^1 \rho^{\alpha-1} (\log \rho)^{w-1} d\rho=-(-\alpha)^{-w} \int_0^{-\alpha \log(u/s)} e^{-\eta} \eta^{w-1} d\eta$, we have

$$S_{w,\alpha}[f](s)=\frac{\sqrt{-1}}{2\pi} (1-e^{2\pi\sqrt{-1}w}) \Gamma(1-w) (-\alpha)^{-w} \int_0^s \gamma(w, -\alpha \log(\frac{u}{s})) f'(u) du,$$

where $\gamma(w, a)=\int_0^a t^{w-1} e^{-t} dt$ is the incomplete gamma function. Since $\gamma(w, a)=(1/w)\gamma(w+1, a)+a^w e^{-a}$ ([6], II), and $\gamma(1, a)=1-e^{-a}$, we obtain

$$\lim_{w \rightarrow 0} \frac{-1}{2\pi} (1-e^{2\pi\sqrt{-1}w}) \Gamma(1-w) (-\alpha)^{-w} \gamma(w, -\alpha \log(\frac{u}{s})) = 1, \quad \alpha \neq 0.$$

Hence we have $\lim_{w \rightarrow 0} S_{w,\alpha}[f](s)=\int_0^s f'(u) du=f(s)$.

Note. (25) holds if f is continuous in s . But, for (24)', $f(s)$ should be absolute continuous in s .

Definition. We set $\mathcal{A}_{w,\alpha}=T_{w,\alpha}\mathcal{A}$.

By definition, $\mathcal{A}_{w,\alpha}$ is densely defined in $L^2(\mathbb{R}^n)$ for any w and α . If $\mathcal{A}_{w,\alpha}f$ is defined for $w=w_0$ and $\operatorname{Re} w_0=c$, $\mathcal{A}_{w,\alpha}f$ is defined on the domain $\operatorname{Re} w < c$ and holomorphic in w . If $c \geq 0$, we have

$$(26) \quad \lim_{w \rightarrow 0} \mathcal{A}_{w,\alpha}f=\mathcal{A}f.$$

8. Let λ be a real number, c a complex number with $-\pi/2 < \arg c < \pi/2$. Then $G_{c^{-1}\lambda, k, \alpha}$ and $G_{c^{-1}\lambda, k}$ are defined by n°5 and we have

$$(27) \quad (cL_n, p+\lambda r^k) \left(\frac{1}{c} G_{c^{-1}\lambda, k, \alpha} f \right) = f, \quad (cL_n, p+\lambda r^k) \left(\frac{1}{c} G_{c^{-1}\lambda, k} f \right) = f.$$

By (27), if $-\pi/2 < \operatorname{Im} w \log(p + \operatorname{Re} \alpha) - \operatorname{Re} w \operatorname{Im} \alpha < \pi/2$, $(p + \alpha)^{-w} G_{(p+\alpha)-w, k, \alpha}$ and $(p + \alpha)^{-w} G_{(p+\alpha)-w, k}$ are defined and they are fundamental solutions of $(p + \alpha)^w L_{n, p} + \lambda r^k$. By definition, $(p + \alpha)^{-w} G_{(p+\alpha)-w, k, \alpha}$ and $(p + \alpha)^{-w} G_{(p+\alpha)-w, k}$ are holomorphic in α and w and they are continued analytically on whole w -plane if $\operatorname{Re} \alpha > 0$, α is not a positive integer, because Bessel functions defined on whole plane.

Definition. The analytic continuations of $(p + \alpha)^{-w} G_{(p+\alpha)-w, k, \alpha}$ and $(p + \alpha)^{-w} G_{(p+\alpha)-w, k}$ are denoted by $G_{\lambda, k, \alpha, (w, \alpha)}$ and $G_{\lambda, k, (w, \alpha)}$.

Since $(p + \alpha)^w L_{n, p} + \lambda r^k$ is analytic in w and α , we obtain by (27)

$$(27)' \quad ((p + \alpha)^w L_{n, p} + \lambda r^k) G_{\lambda, k, \alpha, (w, \alpha)} f = f,$$

$$((p + \alpha)^w L_{n, p} + \lambda r^k) G_{\lambda, k, (w, \alpha)} f = f.$$

Definition. We define an operator $G^{p_k, a, (w, \lambda)}$ in $L^2(\mathbb{R}^n \times M, E)_p, a$ by

$$G^{p_k, a, (w, \alpha)} = \sum_{\lambda \in \operatorname{Spec} D} G_{\lambda, k, \alpha, (w, \alpha)},$$

$$G_{\lambda, k, \alpha, (w, \alpha)}(f)(r, \theta, y)$$

$$= \int_0^\alpha \int_{S^{n-1} \times M} G_{\lambda, k, (w, \alpha)} \sum_{(i)} \sum_j A_{(i)}^{n, p}(\theta) \overline{A_{(i)}^{n, p}(\varphi)} \phi_{\lambda, j}(y) \overline{\phi_{\lambda, j}(\eta)} f(\rho, \varphi, \eta) d\eta d\varphi d\rho,$$

where $\{\phi_{\lambda, 1}, \dots, \phi_{\lambda, m}\}$ is the O.N. basis of E_λ . The operator $G^{p_k, (w, \alpha)}$ in $L^2(\mathbb{R}^n \times M, E)_p$ is similarly defined.

By (17), lemma 6₊ and Abel's continuity theorem, we obtain

Proposition 2'. (i). If $-\pi/2 < \arg p^{-w} < \pi/2$ and D has only finite number of positive proper values, $G^{p_k, (w, \alpha)}$ and $G^{p_k, a, (w, \alpha)}$ are defined on $\mathcal{D}(G^{p_k})$ and on $\mathcal{D}(G^{p_k, a})$.

(ii). If $-\pi/2 < \arg p^{-w} < \pi/2$ and D has infinite number of positive proper values, $\mathcal{D}(G^{p_k, (w, \alpha)})$ contains $\sum_{\lambda \in \operatorname{Spec} D, \lambda \neq 0} B_{\lambda, k}(\mathbb{R}^+) \otimes \mathcal{H}^{p, n} \otimes E_\lambda$.

(iii). If $G^{p_k, (w, \alpha)} f$ (or $G^{p_k, a, (w, \alpha)} f$) is defined on $\operatorname{Re} w > 0$ and as a holomorphic function of w , $\lim_{w \rightarrow 0} G^{p_k, (w, \alpha)} f$ (or $\lim_{w \rightarrow 0} G^{p_k, a, (w, \alpha)} f$) exists, then

$$(28) \quad (\mathcal{A} + r^k D) \left(\lim_{w \rightarrow 0} G^{p_k, (w, \alpha)} f \right) = f, \quad (\mathcal{A} + r^k D) \left(\lim_{w \rightarrow 0} G^{p_k, a, (w, \alpha)} f \right) = f.$$

Corollary. Under the same assumptions as (iii), if $G^{p_k} f$ (or $G^{p_k, a} f$) exists, then as an L^2 -valued analytic function in w

$$(29) \quad \lim_{w \rightarrow 0} G^{p_k, (w, \alpha)} f = G^{p_k} f, \quad \lim_{w \rightarrow 0} G^{p_k, a, (w, \alpha)} f = G^{p_k, a} f.$$

Proof. By (28), to show (29), it is sufficient to show that $\lim_{w \rightarrow 0} G^{p_k, (w, \alpha)} f$ (or $\lim_{w \rightarrow 0} G^{p_k, a, (w, \alpha)} f$) satisfies the boundary condition. But this follows from the definition.

Definition. In $L^2(B_\alpha \times M, \pi^*(E))$ and in $L^2(\mathbb{R}^n \times M, \pi^*(E))$, we set

$$G_{k,\alpha,(w,\alpha)} = \sum_p G^p_{k,\alpha,(w,\alpha)}, \quad G_{k,(w,\alpha)} = \sum_p G^p_{k,(w,\alpha)}.$$

By note in n°6 and proposition 2', we have

Proposition 3. (i). If $\operatorname{Re} \alpha > 0$, $\operatorname{Re} w > n - 2$, $G_{k,(w,\alpha)}$ (resp. $G_{k,\alpha,(w,\alpha)}$) is defined on $\sum \mathcal{D}(G^p_{k,(w,\alpha)})$ (resp. $\sum \mathcal{D}(G^p_{k,\alpha,(w,\alpha)})$).

(ii). If $G_{k,(w,\alpha)} f$ (or $G_{k,\alpha,(w,\alpha)} f$) is defined on $\operatorname{Re} w > 0$ and as a holomorphic function of w , $\lim_{w \rightarrow 0} G_{k,(w,\alpha)} f$ (or $\lim_{w \rightarrow 0} G_{k,\alpha,(w,\alpha)} f$) exists, then

$$(28)' \quad (A+r^k D) \lim_{w \rightarrow 0} G_{k,(w,\alpha)} f = f, \quad (A+r^k D) \lim_{w \rightarrow 0} G_{k,\alpha,(w,\alpha)} f = f.$$

Definition. If $G_{k,(w,\alpha)} f$ (or $G_{k,\alpha,(w,\alpha)} f$) is continued analytically on the domain whose closure contains 0, as a function of w , and $\lim_{w \rightarrow 0} G_{k,(w,\alpha)} f$ (or $\lim_{w \rightarrow 0} G_{k,\alpha,(w,\alpha)} f$) exists, then we set

$$\lim_{w \rightarrow 0} G_{k,(w,\alpha)} f = G_k f, \quad \lim_{w \rightarrow 0} G_{k,\alpha,(w,\alpha)} f = G_{k,\alpha} f.$$

By definition and (29), if $\sum_p G^p k f$ (or $\sum_p G^p k, \alpha f$) exists, then

$$G_k f = \sum_p G^p k f, \quad G_{k,\alpha} f = \sum_p G^p k, \alpha f,$$

and we have

$$(29)' \quad (A+r^k D) G_k f = f, \quad (A+r^k D) G_{k,\alpha} f = f.$$

Note. We may define $G^p_{k,\alpha,(0),(w,\alpha)}$ and $G_{k,\alpha,(0),(w,\alpha)}$ similarly. If D has only finite number of positive proper values and $-\pi/2 < \arg p - w < \pi/2$, we get $\mathcal{D}(G^p_{k,\alpha,(0),(w,\alpha)}) = \mathcal{D}(G^p_{k,\alpha,(w,\alpha)})$ and $\mathcal{D}(G_{k,\alpha,(0),(w,\alpha)}) = \mathcal{D}(G_{k,\alpha,(w,\alpha)})$. We also note that, set $\mathcal{N}_{n,k;D} = \bigcup_{p \geq 0} \mathcal{N}_{n,p,k;D}$, $\mathcal{N}_{n,k;D}$ is an empty set if D has only finite number of positive proper values and does not depend on lower order selfadjoint perturbation of D .

§4. Trace of $G_{k,(w,\alpha)}$.

9. Definition. For D , we set

$$(30) \quad \zeta_{D,+}(s) = \sum_{\lambda \in \operatorname{Spec} D, \lambda > 0} \lambda^{-s}, \quad \zeta_{D,-}(s) = \sum_{\lambda \in \operatorname{Spec} D, \lambda < 0} (-\lambda)^{-s}.$$

$\zeta_{D,+}(s)$ and $\zeta_{D,-}(s)$ both exist if $\operatorname{Re} s$ is sufficiently large, and since

$$(31) \quad \eta(s) = \zeta_{D,+}(s) - \zeta_{D,-}(s), \quad \zeta_D(s) = \zeta_{D,+}(2s) + \zeta_{D,-}(2s),$$

they are both continued meromorphically on whole plane and at $s=0$, they have at

most poles of order 1 ([3], [10]).

Lemma 9. *If $\operatorname{Re} k > -2$, in the sense of analytic continuation, we have*

$$\begin{aligned}
 (32) \quad & \operatorname{tr} G^b_{k,a,(w,\alpha)} \\
 & = -\frac{1}{2} \frac{(n+p-3)!}{p!(n-2)!} (p+\alpha)^{-w} \dim \ker Da^2 + (p+\alpha)^{-\frac{k}{k+2}w} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \\
 & \quad \frac{(2p+n-2)(n+p-3)!}{p!} \left\{ C_{n,k,+}\zeta_{D,+}\left(\frac{2}{k+2}\right) + C_{n,k,-}\zeta_{D,-}\left(\frac{2}{k+2}\right) \right\} \\
 & \quad + o(1), \quad a \rightarrow \infty, \quad (n, p) \neq (2, 0), \\
 & \operatorname{tr} G^0_{k,a,(w,\alpha)} \\
 & = \frac{\alpha^{-w}}{2} \dim \ker Da^2 \log a + \alpha^{-\frac{k}{k+2}w} \frac{\Gamma\left(\frac{2}{k+2}\right)}{\Gamma\left(\frac{k}{k+2}\right)} \left\{ C_{2,k,+}\zeta_{D,+}\left(\frac{2}{k+2}\right) \right. \\
 & \quad \left. + C_{2,k,-}\zeta_{D,-}\left(\frac{2}{k+2}\right) \right\} + o(1), \quad a \rightarrow \infty, \quad n=2.
 \end{aligned}$$

Proof. By lemma 6, if $-\pi/2 < \arg p^{-w} < \pi/2$, we get if $\lambda \neq 0$

$$\operatorname{tr} G_{\lambda,k,(w,\alpha)} = (p+\alpha)^{-w} (n-2)! C_{n,k,\operatorname{sgn} \lambda} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n-2}{k+2}\right)} (p+\alpha)^{\frac{2w}{k+2}} |\lambda|^{-\frac{2}{k+2}}.$$

Hence we obtain (32) by (10), (20) and (30).

Corollary 1. *If $\zeta_{D^2}(0)$ exists, then*

$$\begin{aligned}
 (33) \quad & \lim_{n \rightarrow \infty} \operatorname{tr} G^b_{k,a,(w,\alpha)} \\
 & = -\frac{1}{2} \frac{(n+p-3)!}{p!(n-2)!} (p+\alpha)^{-w} \dim \ker Da^2 + \frac{2p+n-2}{2p+n} \zeta_{D^2}(0) + o(1), \quad (n, p) \neq (2, 0), \\
 & \lim_{k \rightarrow \infty} \operatorname{tr} G^0_{k,a,(w,\alpha)} = \frac{\alpha^{-w}}{2} \dim \ker Da^2 \log a - \frac{1}{2} \zeta_{D^2}(0) + o(1), \quad n=2.
 \end{aligned}$$

Proof. Since $C_{n,k,+} - C_{n,k,-} = O(k^{-2})$, $k \rightarrow \infty$ and $\lim_{k \rightarrow \infty} k^{-2} \zeta_{D,\pm}(2/(k+2)) = 0$, we get (33) by (32) and lemma 7.

Corollary 2. *If $\ker D = \{0\}$, then*

$$\begin{aligned}
 (32)' \quad & \operatorname{tr} G^b_k = \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \frac{(2p+n-2)(n+p-3)!}{p!} \\
 & \quad \left\{ C_{n,k,+}\zeta_{D,+}\left(\frac{2}{k+2}\right) + C_{n,k,-}\zeta_{D,-}\left(\frac{2}{k+2}\right) \right\}.
 \end{aligned}$$

$$(33)' \quad \lim_{k \rightarrow \infty} \operatorname{tr} G^{p_k} = -\frac{1}{2} \frac{(n+p-3)! (2p+n-2)}{p!(n-2)! (2p+n)} \zeta_D(0).$$

Note. If D has no positive proper values, then by (5), we get

$$\begin{aligned} \operatorname{tr} G^{p_k, a, (0), (w, \alpha)} &= -\frac{1}{2} \frac{(n+p-3)!}{p!(n-2)!} (p+\alpha)^{-w} \dim \ker Da^2 + (p+\alpha)^{-\frac{k}{k+2}-w} \\ &\quad \left\{ \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \frac{(2p+n-2)(n+p-3)!}{p!} C_{n, k, -} \zeta_{-D}\left(\frac{2}{k+2}\right) \right\} + o(1), \\ \operatorname{tr} G^{0_k, a, (0), (w, \alpha)} &= \frac{\alpha^{-w}}{2} \dim \ker Da^2 \log a \\ &\quad + \alpha^{-\frac{k}{k+2}-w} \frac{\Gamma\left(\frac{2}{k+2}\right)}{\Gamma\left(\frac{k}{k+2}\right)} C_{2, k, -} \zeta_{-D}\left(\frac{2}{k+2}\right) + o(1), \quad n=2. \end{aligned}$$

Here, $\zeta_{-D}(s)$ means the ζ -function of positive operator $-D$.

Definition. We denote $G^{p_k, (w, \alpha)}$, $G_{k, (w, \alpha)}$, etc., by $G^{p, +}_{k, (w, \alpha)}$, $G^{+}_{k, (w, \alpha)}$, etc., and the fundamental solutions of $\Delta_w a - r^k D = \Delta_w a + r^k (-D)$ constructed for the same boundary condition as (4)' by $G^{p, -}_{k, (w, \alpha)}$, $G^{-}_{k, (w, \alpha)}$, etc..

Lemma 10. If $\operatorname{Re} k > -2$, in the sense of analytic continuation, we have

$$\begin{aligned} (34) \quad & \operatorname{tr} G^{p, k, +}_{a, (w, \alpha)} - \operatorname{tr} G^{p, k, -}_{a, (w, \alpha)} \\ &= (p+\alpha)^{-\frac{k}{k+2}-w} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \frac{(2p+n-2)(n+p-3)!}{p!} \\ &\quad (C_{n, k, +} - C_{n, k, -}) \eta_D\left(\frac{2}{k+2}\right) + o(1). \end{aligned}$$

Proof. This follows from (32) and (31).

Corollary 1. If $\eta_D(0)$ exists, we have

$$\begin{aligned} (35) \quad & \lim_{k \rightarrow \infty} k^2 (\operatorname{tr} G^{p, k, +}_{a, (w, \alpha)} - \operatorname{tr} G^{p, k, -}_{a, (w, \alpha)}) \\ &= (p+\alpha)^{-w} \frac{\pi^2}{4} \frac{(2p+n-2)(n+p-3)!}{(n-2)! p! (2p+n)} \eta_D(0) + o(1), \end{aligned}$$

where $(2p+n-2)(n+p-3)!/p!$ means 1 if $(n, p) = (2, 0)$.

Corollary 2. We have

$$(34)' \quad \text{tr } G^{p,+}_{k,(w,\alpha)} - \text{tr } G^{p,-}_{k,(w,\alpha)}$$

$$= (p+\alpha)^{-\frac{k}{k+2}-w} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \frac{(2p+n-2)(n+p-3)!}{p!}$$

$$(C_{n,k,+} - C_{n,k,-}) \eta_D\left(\frac{2}{k+2}\right).$$

$$(35)' \quad \lim_{k \rightarrow \infty} k^2 (\text{tr } G^{p,+}_{k,(w,\alpha)} - \text{tr } G^{p,-}_{k,(w,\alpha)})$$

$$= (p+\alpha)^{-w} \frac{\pi^2}{4} \frac{(2p+n-2)(n+p-3)!}{(n-2)! p! (2p+n)} \eta_D(0).$$

$$(34)'' \quad \text{tr } G^{p,+}_k - \text{tr } G^{p,-}_k$$

$$= \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \frac{(2p+n-2)(n+p-3)!}{p!} (C_{n,k,+} - C_{n,k,-}) \eta_D\left(\frac{2}{k+2}\right).$$

$$(35)'' \quad \lim_{k \rightarrow \infty} k^2 (\text{tr } G^{p,+}_k - \text{tr } G^{p,-}_k) = \frac{\pi^2}{4} \frac{(2p+n-2)(n+p-3)!}{(n-2)! p! (2p+n)} \eta_D(0).$$

10. Definition. We set

$$(36) \quad \zeta_n(w, \alpha) = \sum_{p=1}^{\infty} \frac{(n+p-3)!}{(n-3)! p!} (p+\alpha)^{-w}, \quad n \geq 4,$$

$$\zeta_3(w, \alpha) = \zeta(w, \alpha) = \sum_{p=1}^{\infty} (p+\alpha)^{-w}, \quad \zeta_2(w, \alpha) = \sum_{p=1}^{\infty} \frac{1}{p} (p+\alpha)^{-w}.$$

$$(36)_k \quad \zeta_{n,k}(w, \alpha) = \sum_{p=1}^{\infty} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k-2}{k+2}\right)} \frac{(2p+n-2)(n+p-3)!}{p!} (p+\alpha)^{-w}.$$

$$(36)_{\infty} \quad \zeta_{\infty,n}(w, \alpha) = \sum_{p=1}^{\infty} \frac{(2p+n-2)(n+p-3)!}{(2p+n)p!} (p+\alpha)^{-w}.$$

By definition, for $\operatorname{Re} \alpha > 0$, $\zeta_n(w, \alpha)$ and $\zeta_{\infty,n}(w, \alpha)$ converges if $\operatorname{Re} w > n-2$ and holomorphic in w and α . On the other hand, Since we get at $p \rightarrow \infty$, $(2p+n-2)(n+p-3)!/p! = O(p^{n-2})$, $\Gamma((2p+n)/(k+2))/\Gamma((2p+n+k-2)/(k+2)) = ((2p+n)/(k+2))^{-(k-2)/(k+2)} [1 + 2(k-2)/(2p+n)(k+2) + O(((k+2)/(2p+n))^2)]$, $\zeta_{n,k}(w, \alpha)$ converges if $\operatorname{Re} w > n-2-\operatorname{Re}((k-2)/(k+2))$.

Using these functions, we obtain by lemma 9 and lemma 10

Proposition 4. If $\operatorname{Re} k > -2$, $\operatorname{Re} w > n - 2 - \operatorname{Re}((k-2)/(k+2))$, we have at $a \rightarrow \infty$,

$$(37) \quad \begin{aligned} \operatorname{tr} G_{k,a,(w,\alpha)} &= -\frac{a^2}{2(n-2)} \{ \zeta_n(w, \alpha) + \alpha^{-w} \} \dim \ker D + \left\{ \zeta_{n,k} \left(\frac{k}{k+2} w, \alpha \right) \right. \\ &\quad \left. + \alpha^{-w} (n-2)! \frac{\Gamma \left(\frac{n}{k+2} \right)}{\Gamma \left(\frac{n+k-2}{k+2} \right)} \right\} \left\{ C_{n,k,+} \zeta_{D,+} \left(\frac{2}{k+2} \right) \right. \\ &\quad \left. + C_{n,k,-} \zeta_{D,-} \left(\frac{2}{k+2} \right) \right\} + o(1), \quad n \geq 3, \\ \operatorname{tr} G_{k,a,(w,\alpha)} &= \alpha^{-w} \frac{a^2 \log a}{2} \dim \ker D - \frac{a^2}{2} \zeta_2(w, \alpha) \dim \ker D \\ &\quad + \left\{ \zeta_{2,k} \left(\frac{k}{k+2} w, \alpha \right) + \alpha^{-w} \frac{\Gamma \left(\frac{2}{k+2} \right)}{\Gamma \left(\frac{k}{k+2} \right)} \right\} \\ &\quad \left\{ C_{2,k,+} \zeta_{D,+} \left(\frac{2}{k+2} \right) + C_{2,k,-} \zeta_{D,-} \left(\frac{2}{k+2} \right) \right\} + o(1), \quad n = 2. \end{aligned}$$

Corollary. Under the same assumptions about k and α as above, if $\zeta_{D^2}(0)$ exists, we have at $a \rightarrow \infty$,

$$(38) \quad \begin{aligned} \lim_{k \rightarrow \infty} \operatorname{tr} G_{k,a,(w,\alpha)} &= -\frac{a^2}{2(n-2)} \{ \zeta_n(w, \alpha) + \alpha^{-w} \} \dim \ker D \\ &\quad - \frac{1}{2} \left(\frac{1}{(n-2)!} \zeta_{\infty,n}(w, \alpha) + \frac{1}{n} \alpha^{-w} \right) \zeta_{D^2}(0) + o(1), \quad n \geq 3, \\ \lim_{k \rightarrow \infty} \operatorname{tr} G_{k,a,(w,\alpha)} &= \alpha^{-w} \frac{a^2 \log a}{2} \dim \ker D - \frac{a^2}{2} \zeta_2(w, \alpha) \dim \ker D \\ &\quad - \frac{1}{2} \left\{ \zeta_{\infty,2}(w, \alpha) + \frac{1}{2} \alpha^{-w} \right\} \zeta_{D^2}(0) + o(1), \quad n = 2. \end{aligned}$$

Proposition 4'. Under the same assumptions about k and α as in proposition 4, we have at $a \rightarrow \infty$,

$$(39) \quad \begin{aligned} \operatorname{tr} G_{k,a,(w,\alpha)}^+ - \operatorname{tr} G_{k,a,(w,\alpha)}^- &= \left\{ \zeta_{n,k} \left(\frac{k}{k+2} w, \alpha \right) + \alpha^{-w} (n-2)! \frac{\Gamma \left(\frac{n}{k+2} \right)}{\Gamma \left(\frac{n+k-2}{k+2} \right)} \right\} \\ &\quad \{ C_{n,k,+} - C_{n,k,-} \} \eta_D \left(\frac{2}{k+2} \right) + o(1), \end{aligned}$$

Corollary. *Under the same assumptions about k and α , if $\eta_D(0)$ exists, we have at $a \rightarrow \infty$,*

$$(40) \quad \lim_{k \rightarrow \infty} k^2 (\operatorname{tr} G^+_{k, a, (w, \alpha)} - \operatorname{tr} G^-_{k, a, (w, \alpha)}) \\ = \frac{\pi^2}{4} \left\{ \frac{1}{(n-2)!} \zeta_{\infty, n}(w, \alpha) + \frac{1}{n} \alpha^{-w} \right\} \eta_D(0) + o(1).$$

Note. By (39) and (40), we may set

$$(39)' \quad \operatorname{tr} G^+_{k, (w, \alpha)} - \operatorname{tr} G^-_{k, (w, \alpha)} \\ = \left\{ \zeta_{n, k} \left(\frac{k}{k+2} w, \alpha \right) + \alpha^{-w} (n-2)! \frac{\Gamma \left(\frac{n}{k+2} \right)}{\Gamma \left(\frac{n+k-2}{k+2} \right)} \right\} \{C_{n, k, +} - C_{n, k, -}\} \eta_D \left(\frac{2}{k+2} \right).$$

$$(40)' \quad \lim_{k \rightarrow \infty} k^2 (\operatorname{tr} G^+_{k, (w, \alpha)} - \operatorname{tr} G^-_{k, (w, \alpha)}) = \frac{\pi^2}{4} \left\{ \frac{1}{(n-2)!} \zeta_{\infty, n}(w, \alpha) + \frac{1}{n} \alpha^{-w} \right\} \eta_D(0).$$

If D has no positive proper values, we have under the same assumptions about k and α as above

$$\operatorname{tr} G_{k, a, (0), (w, \alpha)} = -\frac{a^2}{2(n-2)} \zeta_n(w, \alpha) + \alpha^{-w} \dim \ker D + \left\{ \zeta_{n, k} \left(\frac{k}{k+2} w, \alpha \right) \right. \\ \left. + \alpha^{-w} (n-2)! \frac{\Gamma \left(\frac{n}{k+2} \right)}{\Gamma \left(\frac{n+k-2}{k+2} \right)} \right\} C_{n, k, -} \zeta_{-D} \left(\frac{2}{k+2} \right) + o(1), \quad n \geq 3, \\ \operatorname{tr} G_{k, a, (0), (w, \alpha)} = \alpha^{-w} \frac{a^2 \log a}{2} \dim \ker D - \frac{a^2}{2} \zeta_2(w, \alpha) \dim \ker D \\ + \left\{ \zeta_{2, k} \left(\frac{k}{k+2} w, \alpha \right) + \alpha^{-w} \frac{\Gamma \left(\frac{2}{k+2} \right)}{\Gamma \left(\frac{k}{k+2} \right)} \right\} C_{2, k, -} \zeta_{-D} \left(\frac{2}{k+2} \right) + o(1), \quad n=2.$$

§5. Analytic continuations of $\zeta_n(w, \alpha)$, $\zeta_{n, k}(w, \alpha)$ and $\zeta_{\infty, n}(w, \alpha)$.

II. If $\operatorname{Re} w > n-2$, $\operatorname{Re} \alpha > 0$, we have

$$\zeta_2(w, \alpha) \Gamma(w) = - \int_0^\infty e^{-t} \log(1-e^{-t}) t^{w-1} dt, \\ \zeta_n(w, \alpha) \Gamma(w) = \int_0^\infty e^{-t} \frac{1}{(1-e^{-t})^{n-2}-1} t^{w-1} dt, \quad n \geq 3.$$

Hence, to denote $\int_\infty^{(0, +)} f(t) dt$ the integral of f along the path starting from ∞ ,

rounds 0 in the negative direction and ends at ∞ , we get

$$(41) \quad \begin{aligned} \zeta_2(w, \alpha) &= \frac{1}{(1-e^{2\pi\sqrt{-1}w}) \Gamma(w)} \left[\int_{-\infty}^{(0,+)} \{ -e^{-\alpha t} \log(1-e^{-t}) \} t^{w-1} dt - 2\pi\sqrt{-1} \alpha^{-w} \Gamma(w) \right], \\ \zeta_n(w, \alpha) &= \frac{1}{(1-e^{2\pi\sqrt{-1}w}) \Gamma(w)} \int_{-\infty}^{(0,+)} e^{-\alpha t} \frac{1}{(1-e^{-t})^{n-2}-1} t^{w-1} dt, \quad n \geq 3. \end{aligned}$$

By (41), we obtain

Lemma 11. $\zeta_n(w, \alpha)$ allows analytic continuation on whole w -plane and 1-valued if $n \geq 3$.

Since we know

$$(42) \quad \lim_{w \rightarrow 0} (1-e^{2\pi\sqrt{-1}w}) \Gamma(w) = -2\pi\sqrt{-1},$$

to determin the branch of α^{-w} to satisfy $\lim_{w \rightarrow 0} \alpha^{-w} = 1$, we get by (41)

$$(43)_2 \quad \lim_{w \rightarrow 0} w \zeta_2(w, \alpha) = 1.$$

In general, denote $B^{(n-2)}_m(-1, \dots, -1)$ the Bernoulli number of order m , we know ([6], I)

$$(1-e^{-t})^{2-n} = \sum_{m=0}^{\infty} \frac{B^{(n-2)}(-1, \dots, -1)}{m!} t^{m+2-n}, \quad n \geq 3.$$

This shows $\zeta_n(w, \alpha)$ has poles at $w=1, \dots, n-2$ if $n \geq 3$ and

$$(43) \quad \lim_{w \rightarrow n-2} (w+2-n) \zeta_n(w, \alpha) = \frac{1}{(n-3)!}, \quad n \geq 3.$$

On the other hand, by (42), we get

$$\begin{aligned} \zeta_n(0, \alpha) &= \frac{\sqrt{-1}}{2\pi} \int_{-\infty}^{(0,+)} e^{-\alpha t} \frac{1}{(1-e^{-t})^{n-2}-1} \frac{dt}{t} \\ &= \text{Res}_{t=0} \frac{1}{t} e^{-\alpha t} \frac{1}{(1-e^{-t})^{n-2}-1}, \quad n \geq 3. \end{aligned}$$

Hence we obtain

$$(44) \quad \begin{aligned} \zeta_n(0, \alpha) &= \sum_{m=0}^{n-2} (-1)^m \frac{B^{(n-2)}_{n-2-m}(-1, \dots, -1)}{m!(n-2-m)!} \alpha^m - 1, \\ \zeta_n(0, 0) &= \frac{B^{(n-2)}_{n-2}(-1, \dots, -1)}{(n-2)!} - 1, \\ \frac{d^{n-2}}{d\alpha^{n-2}} \zeta_n(0, \alpha) &= (-1)^n, \quad n \geq 3. \end{aligned}$$

Summarizing these, we obtain

Lemma 12. (i). $\zeta_n(w, \alpha)$ has a pole of order 1 at $w=n-2$ and the residue at $w=n-2$ does not depend on α .

(ii). If $n \geq 3$, $\zeta_n(w, \alpha)$ is holomorphic at $w=0$ and $\zeta_n(0, \alpha)$ is a polynomial of degree $n-2$ in α .

12. We set $\varphi^{p_0, n, k}(x) = \sum_{p=p_0}^{\infty} \frac{\Gamma((2p+n)/(k+2))/\Gamma(2p+n+k-2)/(k+2))}{x^{(2p+n+k-2)/(k+2)-1}} \varphi^{p_0, n, k}(x)$ converges for $|x| < 1$, and since $\int_0^x (x-s)^{\mu} s^{\nu} ds = B(\mu, \nu)$ $x^{\mu+\nu-1}$, we have

$$(45) \quad \varphi^{p_0, n, k}(x) = -\frac{1}{\Gamma\left(\frac{k-2}{k+2}\right)} \int_0^x (x-s)^{\frac{k-2}{k+2}} s^{\frac{n+2p_0}{k+2}} \frac{ds}{1-s^{\frac{2}{k+2}}}, \quad \text{Re } k > 0, \quad k \neq 2, \quad |x| < 1.$$

The right hand side of (45) has meaning unless $x^{2/(k+2)} = 1$ if k is not an irrational real number. In the rest, we assume k is not an irrational real number and $\text{Re } k > 0$, $k \neq 2$. $\varphi^{p_0, n, k}(x)$ may be many valued with the branching point at 0. But if k is not a real number, we consider $\varphi^{p_0, n, k}(x)$ only on $\mathbb{C}^* = \mathbb{C} - \{0\}$.

By (45), $\varphi^{p_0, n, k}(x)$ has branching points only on $\{x | x^{2/(k+2)} = 1\}$ except $x=0$. Then, since we get for an integer m

$$\text{Res}_{s=e^{mk\pi\sqrt{-1}}} \varphi^{p_0, n, k}(x) = -\frac{k+2}{2} (x - e^{mk\pi\sqrt{-1}})^{\frac{k-2}{k+2}} e^{(n+k)m\pi\sqrt{-1}},$$

denote the generator of $\pi_1(\mathbb{C} - \{e^{mk\pi\sqrt{-1}}\})$ by σ_m , we get

$$(\varphi^{p_0, n, k}(x))^{\sigma_m} = \varphi^{p_0, n, k}(x) - \frac{k+2}{\Gamma\left(\frac{k-2}{k+2}\right)} \pi\sqrt{-1} e^{(n+k)m\pi\sqrt{-1}} (x - e^{mk\pi\sqrt{-1}})^{\frac{k-2}{k+2}}.$$

Then, since $(\hat{\sigma}_m; n, k(x))^{\sigma_m} = e^{((k-2)/(k+2))2\pi\sqrt{-1}} \hat{\sigma}_m; n, k(x)$, where $\hat{\sigma}_m; n, k(x) = -((k+2)/\Gamma((k-2)/(k+2))) \pi\sqrt{-1} e^{(n+k)m\pi\sqrt{-1}} (x - e^{mk\pi\sqrt{-1}})^{(k-2)/(k+2)}$, we obtain

$$(46) \quad (\varphi^{p_0, n, k}(x))^{\sigma_m} u = \varphi^{p_0, n, k}(x) + \frac{1 - e^{\left(\frac{k-2}{k+2}\right)2\mu\pi\sqrt{-1}}}{1 - e^{\left(\frac{k-2}{k+2}\right)2\pi\sqrt{-1}}} \hat{\sigma}_m; n, k(x).$$

Here, μ is a positive integer and x is fixed to be $0 \leq \arg(x - e^{mk\pi\sqrt{-1}}) < 2\pi$.

Next, we set $\xi = x^{2/(k+2)}$, where the branch of $x^{2/(k+2)}$ is chosen to be $1^{2/(k+2)} = 1$. Then we have

$$(47) \quad \xi^{2-\frac{n}{2}} \varphi^{p_0, n, k}(\xi) = \sum_{p=p_0}^{\infty} \frac{\Gamma\left(\frac{2p+n}{k+2}\right)}{\Gamma\left(\frac{2p+n+k+2}{k+2}\right)} \xi^p, \quad |\xi| < 1.$$

Since the right hand side of (47) is holomorphic at $\xi=0$, $\varphi^{p_0, k}(\xi)$ is 1-valued near the origin if n is even and 2-valued if n is odd. Therefore $\varphi^{p_0, k}(\xi)$ branches only at $\xi=1$ if n is even and branches at $\xi=0$ and $\xi=\pm 1$ if n is odd. The branching point of $\varphi^{p_0, k}(\xi)$ at $\xi=0$ is cancelled by multiplying $\xi^{2-n/2}$. Summarizing these, we have

Lemma 13. $\xi^{2-n/2}\varphi^{p_0, k}(\xi)$ is continued analytically on whole complex plane with the branching point at $\xi=1$ if n is even and branching points at $\xi=\pm 1$ if n is odd. To set

$$\sigma_{0; n, k, 0}(\xi) = -\frac{k+2}{\Gamma\left(\frac{k-2}{k+2}\right)} \pi \sqrt{-1} \xi^{2-\frac{n}{2}} (\xi^{\frac{k+2}{2}} - 1)^{\frac{k-2}{k+2}}, \quad n \text{ is an integer},$$

$$\sigma_{1; n, k, 0}(\xi) = -\frac{k+2}{\Gamma\left(\frac{k-2}{k+2}\right)} \pi \sqrt{-1} e^{(k+1)\pi\sqrt{-1}} \xi^{2-\frac{n}{2}} (\xi^{\frac{k+2}{2}} + 1)^{\frac{k-2}{k+2}},$$

n is an odd integer,

and denote the generators of $\pi_1(C - \{1\})$ and $\pi_1(C - \{-1\})$ by σ_0 and σ_1 , we get

$$(46)' \quad (\xi^{2-\frac{n}{2}} \varphi^{p_0, k}(\xi))^{\sigma_i \mu} = \xi^{2-\frac{n}{2}} \varphi^{p_0, k}(\xi) + \frac{1 - e^{\left(\frac{k-2}{k+2}\right) 2\mu\pi\sqrt{-1}}}{1 - e^{\left(\frac{k-2}{k+2}\right) 2\pi\sqrt{-1}}} \sigma_{i; n, k, 0}(\xi), \quad i=0, 1.$$

Note. Denote the generator of $\pi_1(C - \{0\})$ by τ , for an odd integer n , we get $(\sigma_{0; n, k, 0}(\xi))^\tau = \sigma_{1; n, k, 0}(\xi)$.

We set $F_{n, k}(\xi) = \sum_{p=1}^{\infty} \{ \Gamma((2p+n)/(k+2))/\Gamma((2p+n+k-2)/(k+2)) \} (2p+n-3)(p+n-3)!/p! \cdot \xi^p$. Then we have by (47),

$$F_{2, k}(\xi) = 2 \int_0^{\xi} \varphi_{0, k}^0(\xi) d\xi,$$

$$F_{n, k}(\xi) = 2\xi \frac{d^{n-2}}{d\xi^{n-2}} \left\{ \xi^{2-\frac{n}{2}} \varphi_{n-1, k}^0(\xi) \right\} + (n-2) \frac{d^{n-3}}{d\xi^{n-3}} \left\{ \xi^{2-\frac{n}{2}} \varphi_{n-2, k}^0(\xi) \right\}, \quad n \geq 3.$$

Hence we obtain by lemma 13

Lemma 13'. $F_{n, k}(\xi)$ is continued analytically on whole complex plane with the branching points at $\xi=1$ if n is even and at $\xi=\pm 1$ if n is odd. Under the action of $\sigma = \sigma_0$, the generator of $\pi_1(C - \{1\})$, we have

$$(48) \quad (F_{n, k}(\xi))^\sigma = F_{n, k}(\xi)^\sigma = F_{n, k}(\xi) + \sigma_{n, k}(\xi),$$

where $\sigma_{2, k}(\xi) = 2 \int_0^{\xi} \eta \sigma_{0; 2, k, 0}(\eta) d\eta$ and $\sigma_{n, k}(\xi) = 2\xi d^{n-2} \sigma_{0; n, k, 0}(\xi)/d\xi^{n-2} + (n-2) d^{n-3} \sigma_{0; n, k, 0}(\xi)/d\xi^{n-3}$, $n \geq 3$.

Lemma 14. *If $\operatorname{Re} \alpha > 3n/2 - 5$, we have*

$$(49) \quad \xi_{n,k}(w, \alpha) = \frac{1}{(1-e^{2\pi\sqrt{-1}w})I(w)} \int_{-\infty}^{(0, +)} e^{-\alpha t} [F_{n,k}(e^{-t}) - \frac{\sigma_{n,k}(e^{-t})}{1-e^{(w+\frac{k-2}{k+2})2\pi\sqrt{-1}}}] t^{w-1} dt.$$

Proof. By the definition of $F_{n,k}(\xi)$, we get

$$\zeta_{n,k}(w, \alpha) = \frac{1}{I(w)} \int_0^\infty e^{-\alpha t} F_{n,k}(e^{-t}) t^{w-1} dt, \quad \operatorname{Re} w > n-2 - \operatorname{Re} \frac{k-2}{k+2}, \quad \operatorname{Re} \alpha > 0.$$

On the other hand, since $(\sigma_{n,k}(\xi))^\alpha = e^{((k-2)/(k+2))2\pi\sqrt{-1}} \sigma_{n,k}(\xi)$, we get for $\operatorname{Re} \alpha > 3n/2 - 5$

$$\int_0^\infty e^{-\alpha t} \sigma_{n,k}(e^{-t}) t^{w-1} dt = \frac{1}{1-e^{(w+\frac{k-2}{k+2})2\pi\sqrt{-1}}} \int_{-\infty}^{(0, +)} e^{-\alpha t} \sigma_{n,k}(e^{-t}) t^{w-1} dt,$$

if $\operatorname{Re} w > 3n/2 - 4$. But, since this right hand side has meanings if $w+(k-2)/(k+2)$ is not an integer, $\int_0^\infty e^{-\alpha t} \sigma_{n,k}(e^{-t}) t^{w-1} dt$ is continued analytically on whole w -plane. Therefore we obtain (49) by (48).

Corollary. $\zeta_{n,k}(w, \alpha)$ can be continued analytically on whole w -plane if $\operatorname{Re} \alpha > 3n/2 - 5$ and it is holomorphic at $w=0$ if $(k-2)/(k+2)$ is not an integer. If $(k-2)/(k+2)$ is not an integer, $\zeta_{n,k}(0, \alpha)$ is given by

$$\zeta_{n,k}(0, \alpha) = \frac{\sqrt{-1}}{2\pi} \int_{-\infty}^{(0, +)} e^{-\alpha t} \left[F_{n,k}(e^{-t}) - \frac{\sigma_{n,k}(e^{-t})}{1-e^{(\frac{k-2}{k+2})2\pi\sqrt{-1}}} \right] \frac{dt}{t}.$$

13. By the definitions of $\zeta_{\infty,n}(w, \alpha)$ and $\zeta_n(w, \alpha)$, we have $\zeta_{\infty,n}(w, \alpha) = \zeta_n(w, \alpha-1)$ and

$$\zeta_{\infty,n}(w, \alpha) = (n-3)! \zeta_n(w, \alpha) - 2 \sum_{p=1}^{\infty} \frac{(n+p-3)!}{(2p+n)p!} (p+\alpha)^{-w}, \quad n \geq 3.$$

Hence, for $n \geq 3$, to study $\zeta_{\infty,n}(w, \alpha)$, it is sufficient to study

$\sum_{p=1}^{\infty} \frac{(n+p-3)!}{(2p+n)p!} (p+\alpha)^{-w}$. For this purpose, we use

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{2p+n} x^{2p+n} &= -\frac{1}{2} \log(1-x^2) - \left(\frac{x^2}{2} + \dots + \frac{x^{2k}}{2k} + \dots + \frac{x^n}{n} \right), \quad n \text{ is even}, \\ &= \frac{1}{2} \log \frac{1+x}{1-x} - \left((x + \dots + \frac{x^{2k+1}}{2k+1} + \dots + \frac{x^n}{n}) \right), \quad n \text{ is odd}. \end{aligned}$$

By this formula, to set $f_{\infty,n}(y) = 2 \sum_{p=1}^{\infty} ((p+n-3)!/(2p+n)p!) y^p$, $n \geq 3$, we get

$$(50) \quad f_{\infty, n}(y) = -\frac{d^{n-3}}{dy^{n-3}} \left[y^{\frac{n}{2}-3} \left\{ \log(1-y) + (y + \dots + \frac{y^k}{k} + \dots + \frac{2}{n} y^{\frac{n}{2}}) \right\} \right], \quad n \text{ is even},$$

$$= \frac{d^{n-3}}{dy^{n-3}} \left[y^{\frac{n}{2}-3} \left\{ \log \frac{1+\sqrt{y}}{1-\sqrt{y}} - \left(2\sqrt{y} + \dots + \frac{2\sqrt{y}}{2k+1} y^k + \dots + \frac{2}{n} y^{\frac{n}{2}} \right) \right\} \right],$$

n is odd.

Because we have $(d^{n-3}/dy^{n-3})(\sum_{p=1}^{\infty} 1/(2p+n)y^{p+n-3}) = \sum_{p=1}^{\infty} (p+n-3)!/(2p+n)p!y^p$, $n \geq 3$ and $\sum_{p=1}^{\infty} 1/(2p+n)y^{p+n-3} = y^{n/2-3} \sum_{p=1}^{\infty} 1/(2p+n)y^{p+n/2} = y^{n/2-3} \sum_{p=1}^{\infty} 1/(2p+n)x^{2p+n}$, $y=x^2$.

Lemma 15. *For $n \geq 3$, we have*

$$(51) \quad \sum_{p=1}^{\infty} \frac{2(n+p-3)!}{(2p+n)p!} (p+\alpha)^{-w} = \frac{1}{(1-e^{2\pi\sqrt{-1}w})\Gamma(w)} \int_{\infty}^{(0, +)} e^{-\alpha t} f_{\infty, n}(e^{-t}) t^{w-1} dt,$$

n is even, $n \geq 6$,

$$= \frac{1}{(1-e^{2\pi\sqrt{-1}w})\Gamma(w)} \left\{ \int_{\infty}^{(0, +)} e^{-\alpha t} f_{\infty, 4}(e^{-t}) t^{w-1} dt \right.$$

$$\left. - 2\pi\sqrt{-1}(\alpha+2)^{-w} \Gamma(w) \right\}, \quad n=4,$$

$$= \frac{1}{(1-e^{2\pi\sqrt{-1}w})\Gamma(w)} \left[\int_{\infty}^{(0, +)} e^{-\alpha t} f_{\infty, n}(e^{-t}) t^{w-1} dt \right.$$

$$\left. + \frac{\sin \frac{n\pi}{2}}{2} (4-n)(2-n)(\Gamma \frac{n}{2} - 2)^2 \sqrt{-1} (\alpha + \frac{n}{2})^{-w} \Gamma(w) \right],$$

n is odd.

Proof. By (50), $f_{\infty, n}(y)$ is continued analytically on whole plane and $n/2-3$ is an integer if n is even and $n-3 \geq n/2-3 \geq 0$ if $n \geq 6$. Hence $f_{\infty, n}(y)$ is 1-valued if n is even and $n \geq 6$. This shows the first equality. Since $f_{\infty, 4}(y)-1/y^2 \log(1-y)$ is 1-valued, we get the second equality.

If n is odd, $y^{\pm 1/2} dk (y^{n/2-3})/dy^k$, $k \geq 3$ is 1-valued and since $d \log((1+\sqrt{y})/(1-\sqrt{y}))/dy = 1/\{\sqrt{y}(1-y)\}$, except the term $(d^{n-3} y^{n/2-3}/dy^{n-3}) \cdot \log((1+\sqrt{y})/(1-\sqrt{y}))$, the terms in $f_{\infty, n}(y)$ is 1-valued if n is odd. Therefore $f_{\infty, n}(y) - \sin(n\pi/2)/4\pi(4-n)(2-n)(\Gamma(n/2-2))^2 y^{-n/2} \log((1+\sqrt{y})/(1-\sqrt{y}))$ is 1-valued because $d^{n-3} y^{n/2-3}/dy^{n-3} = \Gamma(n/2-2)/\Gamma(1-n/2)y^{-n/2} = \Gamma(n/2-2)/\{\Gamma(3-n/2)/(2-n/2)(1-n/2)\} y^{-n/2}$. This shows the last equality.

Corollary. (i). $\zeta_{\infty, n}(w, \alpha)$ is continued analytically on whole w -plane.

(ii). $\zeta_{\infty, n}(w, \alpha)$ is holomorphic at $w=0$ if n is even and $n \geq 6$. In this case, we have

$$(44)' \quad \zeta_{\infty, n}(0, \alpha) = (n-3)! \left\{ \zeta_n(0, \alpha) - \frac{2}{n} \right\} - \sum_{k=0}^{\frac{n}{2}-3} \frac{(n-k-2)(n-3)! \left(\frac{n}{2}\right) - 3!}{k! \left(\frac{n}{2}-k-3\right)!} \\ \left\{ \sum_{l=0}^{n-k-3} (-1)^l \cdot \left(\alpha + \frac{n}{2} - k - 3 \right) \frac{ {}^l B^{(n-k-3)}_{n-l-k-3}(-1, \dots, -1)}{l!(n-l-k-3)!} \right\}, \\ \frac{d^{n-2}}{d\alpha^{n-2}} \zeta_{\infty, n}(0, \alpha) = (-1)^n (n-3)!.$$

(iii). $\zeta_{\infty, n}(w, \alpha)$ has a pole of order 1 at $w=0$ if $n=2, 4$ or n is odd. Their residues do not depend on α and given by

$$(43)' \quad \lim_{w \rightarrow 0} w \zeta_{\infty, n}(w, \alpha) = 1, \quad n=2, 4,$$

$$\lim_{w \rightarrow 0} w \zeta_{\infty, n}(w, \alpha) = -\frac{\sin \frac{n\pi}{2}}{4\pi} (4-n)(2-n) \left\{ \Gamma \left(\frac{n}{2} - 2 \right) \right\}^2, \quad n \text{ is odd.}$$

Proof. (i) follows from (50) and (51).

If n is even and $n \geq 6$, we get $\zeta_{\infty, n}(0, \alpha) = (n-3)! \zeta_n(0, \alpha) + \text{rest}_{t=0} e^{-\alpha t} f_{\infty, n}(e^{-t})/t$ and since

$$f_{\infty, n}(y) = - \sum_{k=0}^{\frac{n}{2}-3} \frac{(n-3)!(n-k-2)}{k!} \left(\frac{n}{2} - 3 \right) \cdots \left(\frac{n}{2} - k - 2 \right) \frac{y^{\frac{n}{2}-k-3}}{(1-y)^{n-k-3}} - \frac{2}{n} (n-3)!,$$

in this case, we have the first equality of (44)', because $(1-e^{-t})^{3+k-n} = \sum_m B_m (n-k-3) \cdots (-1, \dots, -1)/m! t^{m+3+k-n}$. Moreover, since the degree of $\text{rest}_{t=0} e^{-\alpha t} f_{\infty, n}(e^{-t})/t$ in α is $n-3$, we obtain $d^{n-2} \zeta_{\infty, n}(0, \alpha)/d\alpha^{n-2} = (n-3)! d^{n-2} \zeta_n(0, \alpha)/d\alpha^{n-2}$. This shows the second equality of (44)'.

If $n=2$, $\zeta_{\infty, 2}(w, \alpha) = \zeta_2(w, \alpha+1)$ and we obtain (43)' for $n=2$ by (43)₂.

By (51) and lemma 12, we have

$$\lim_{w \rightarrow 0} w \zeta_{\infty, n}(w, \alpha) = \lim_{w \rightarrow 0} w \frac{1}{(1-e^{2\pi\sqrt{-1}w})\Gamma(w)} 2\pi\sqrt{-1} (\alpha+2)^{-w} \Gamma(w), \quad n=4,$$

$$\lim_{w \rightarrow 0} w \zeta_{\infty, n}(w, \alpha) = \lim_{w \rightarrow 0} -w \frac{1}{(1-e^{2\pi\sqrt{-1}w})\Gamma(w)} \frac{\sin \frac{n\pi}{2}}{2} (4-n)(2-n) \left\{ \Gamma \left(\frac{n}{2} - 2 \right) \right\}^2, \\ \sqrt{-1} (\alpha + \frac{n}{2})^{-w} \Gamma(w), \quad n \text{ is odd},$$

we obtain (43)' for other n .

§6. The spectre difference.

14. Theorem 1. (i). Let $n \geq 3$, if $\operatorname{Re} k > 0$ and not an irrational real number, $(k-2)/(k+2)$ is not an integer, $2/(k+2)$ is not a half of integer, then we have at $a \rightarrow \infty$

$$(52) \quad \begin{aligned} \operatorname{tr} G_{k,a}(0, \alpha) = & -\frac{a^2}{2(n-2)!} \{\zeta_n(0, \alpha) + 1\} \dim \ker D + \{\zeta_{n,k}(0, \alpha) \\ & + (n-2)! \frac{\Gamma\left(\frac{n}{k+2}\right)}{\Gamma\left(\frac{n+k-2}{k+2}\right)}\} \left\{ C_{n,k,+} \zeta_{D,+} \left(\frac{2}{k+2} \right) \right. \\ & \left. + C_{n,k,-} \zeta_{D,-} \left(\frac{2}{k+2} \right) \right\} + o(1). \end{aligned}$$

(ii). Under the same assumptions about k , we have for $n \geq 2$

$$(53) \quad \lim_{w \rightarrow n-2} (w+2-n) \operatorname{tr} G_{k,a}(w, \alpha) = -\frac{a^2}{2(n-2)!} \dim \ker D.$$

(iii). If $\zeta_{D^2}(0)$ exists and n is even, $n \geq 6$, we have at $a \rightarrow \infty$

$$(54) \quad \begin{aligned} \lim_{k \rightarrow \infty} \operatorname{tr} G_{k,a}(0, \alpha) = & -\frac{a^2}{2(n-2)!} \{\zeta_n(0, \alpha) + 1\} \dim \ker D \\ & - \frac{1}{2} \left\{ \frac{1}{(n-2)!} \zeta_{\infty,n}(0, \alpha) + \frac{1}{n} \right\} \zeta_{D^2}(0) + o(1), \\ \lim_{k \rightarrow \infty} \frac{\partial^{n-2}}{\partial \alpha^{n-2}} \operatorname{tr} G_{k,a}(0, \alpha) = & -\frac{a^2}{2(n-2)!} \dim \ker D - \frac{1}{2(n-2)} \zeta_{D^2}(0) + o(1). \end{aligned}$$

(iv). If $\zeta_{D^2}(0)$ exists and $n=2, 4$ or odd, we have

$$(55) \quad \begin{aligned} \lim_{a \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w \operatorname{tr} G_{k,a}(w, \alpha) = & -\frac{1}{n} \zeta_{D^2}(0), \quad n=2, 4, \\ \lim_{a \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w \operatorname{tr} G_{k,a}(w, \alpha) = & -\frac{\sin \frac{n\pi}{2}}{8\pi} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2}-2\right) \right\}^2 \zeta_{D^2}(0), \\ n \text{ is odd.} \end{aligned}$$

Proof. These follows from proposition 4, its corollary and the results of §5.

Note 1. (52) and (53) hold although $2/(k+2)$ is a half of an integer if D has no negative proper values.

Note 2. If D has no positive proper values, (52) and (53) also hold for $G_{k,a}(0)$, (w, α) .

Similarly by proposition 4', we obtain

Theorem 1'. (i). Under the same assumptions about k as in theorem 1, (i), we have at $a \rightarrow \infty$

$$(56) \quad \begin{aligned} & \text{tr} G^+_{k, a, (0, \alpha)} - \text{tr} G^-_{k, a, (0, \alpha)} \\ &= \left\{ \zeta_{n, k}(0, \alpha) + (n-2)! \frac{\Gamma\left(\frac{n}{k+2}\right)}{\Gamma\left(\frac{n+k-2}{k+2}\right)} \right\} (C_{n, k, +} - C_{n, k, -}) \eta_D\left(\frac{2}{k+2}\right) + o(1). \end{aligned}$$

(ii). If $\eta_D(0)$ exists and n is even, $n \geq 6$, we have

$$(57) \quad \begin{aligned} \lim_{k \rightarrow \infty} k^2 \{ \text{tr} G^+_{k, (0, \alpha)} - \text{tr} G^-_{k, (0, \alpha)} \} &= -\frac{\pi^2}{2} \left\{ \zeta_{\infty, n}(0, \alpha) + \frac{1}{n} \right\} \eta_D(0), \\ \lim_{k \rightarrow \infty} k^2 \frac{\partial^{n-2}}{\partial \alpha^{n-2}} \{ \text{tr} G^+_{k, (0, \alpha)} - \text{tr} G^-_{k, (0, \alpha)} \} &= -\frac{\pi^2}{2(n-2)} \eta_D(0). \end{aligned}$$

(iii). If $\eta_D(0)$ exists and $n = 2, 4$ or odd, we have

$$(58) \quad \begin{aligned} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 \{ \text{tr} G^+_{k, (w, \alpha)} - \text{tr} G^-_{k, (w, \alpha)} \} &= -\frac{\pi^2}{2n} \eta_D(0), \quad n = 2, 4, \\ \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 \{ \text{tr} G^+_{k, (w, \alpha)} - \text{tr} G^-_{k, (w, \alpha)} \} \\ &= -\frac{\sin \frac{n\pi}{2}}{16} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2}-2\right) \right\}^2 \pi \eta_D(0), \quad n \text{ is odd}. \end{aligned}$$

Note. In the integral form, (57) and (58) are written as

$$(57)' \quad \begin{aligned} \frac{\sqrt{-1}}{2\pi} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} k \{ \text{tr} G^+_{k, (0, \alpha)} - \text{tr} G^-_{k, (0, \alpha)} \} dk &= -\frac{\pi^2}{2} \left\{ \zeta_{\infty, n}(0, \alpha) + \frac{1}{n} \right\} \eta_D(0), \\ \frac{\sqrt{-1}}{2\pi} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} k \frac{\partial^{n-2}}{\partial \alpha^{n-2}} \{ \text{tr} G^+_{k, (0, \alpha)} - \text{tr} G^-_{k, (0, \alpha)} \} dk &= -\frac{\pi^2}{2n} \eta_D(0), \\ n \text{ is even}, n \geq 6. \end{aligned}$$

$$(58)' \quad \begin{aligned} \frac{1}{4\pi^2} \int_{|w|=r} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} k \{ \text{tr} G^+_{k, (w, \alpha)} - \text{tr} G^-_{k, (w, \alpha)} \} dk dw \\ &= -\frac{\pi^2}{2n} \eta_D(0), \quad n = 2, 4, \\ \frac{1}{4\pi^2} \int_{|w|=r} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} k \{ \text{tr} G^+_{k, (w, \alpha)} - \text{tr} G^-_{k, (w, \alpha)} \} dk dw \end{aligned}$$

$$= -\frac{\sin \frac{n\pi}{2}}{16} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2}-2\right) \right\}^2 \pi \eta_D(0), \quad n \text{ is odd.}$$

15. Let $\theta: L^2(M, E) \rightarrow L^2(M, E)$ be a selfadjoint operator such that each proper function ϕ_λ of D belongs in $\mathcal{D}(\theta)$ and for some positive constant δ ,

$$(59) \quad ||\theta\phi_\lambda|| = O(|\lambda|^{1-\delta}).$$

Then by a theorem of Weyl ([18], [5]), $D+\theta$ has only point spectre and the number of different proper values of D and $D+\theta$ are finite. We define the numbers $d_+ = d_{+,1}(D, \theta)$ and $d_- = d_{-,1}(D, \theta)$ by

$$d_+ = d_{+,1} - d_{+,2}, \quad d_- = d_{-,1} - d_{-,2}.$$

Here, $d_{+,1} = d_{+,1}(D, \theta)$ is the number of positive proper values of $D+\theta$ which are not the proper value of D , $d_{+,2} = d_{+,2}(D, \theta)$ is the number of positive proper values of D which are not the proper value of $D+\theta$, $d_{-,1} = d_{-,1}(D, \theta)$ is the number of negative proper values of $D+\theta$ which are not the proper values of D , and $d_{-,2} = d_{-,2}(D, \theta)$ is the number of negative proper values of D which are not the proper values of $D+\theta$. By definitions, we have, although $\zeta_{D^2}(0)$ or $\eta_D(0)$ does not exist,

$$(60) \quad \lim_{s \rightarrow 0} \zeta_{(D+\theta)^2}(s) - \zeta_{D^2}(s) = d_+ + d_-,$$

$$\lim_{s \rightarrow 0} \eta_{D+\theta}(s) - \eta_D(s) = d_+ - d_-.$$

By (60), (55) and (58), denoting $G_{k,a,(w,\alpha);D}$, etc., the fundamental solutions of $A_{w,\alpha} + r^k D$, etc., in $L^2(B_a \times M, \pi^*(E))$, etc., with the boundary condition (4)', we obtain

$$(61)_+ \quad \lim_{a \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w \{ \operatorname{tr} G_{k,a,(w,\alpha);D+\theta} - \operatorname{tr} G_{k,a,(w,\alpha);D} \}$$

$$= -\frac{1}{n} (d_+ + d_-), \quad n=2, 4,$$

$$\lim_{a \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w \{ \operatorname{tr} G_{k,a,(w,\alpha);D+\theta} - \operatorname{tr} G_{k,a,(w,\alpha);D} \}$$

$$= \frac{\sin \frac{n\pi}{2}}{8\pi} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2}-2\right) \right\}^2 (d_+ + d_-), \quad n \text{ is odd.}$$

$$(61)_- \quad \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 [\{ \operatorname{tr} G_{k,(w,\alpha);D+\theta}^+ - \operatorname{tr} G_{k,(w,\alpha);D+\theta}^- \}$$

$$- \{ \operatorname{tr} G_{k,(w,\alpha);D}^+ - \operatorname{tr} G_{k,(w,\alpha);D}^- \}] = \frac{\pi^2}{2n} (d_+ - d_-), \quad n=2, 4,$$

$$\begin{aligned}
& \lim_{w \leftarrow 0} \lim_{k \rightarrow \infty} w k^2 [\{\operatorname{tr} G^+_{k, (w, \alpha); D+\theta} - \operatorname{tr} G^-_{k, (w, \alpha); D+\theta}\} \\
& - \{\operatorname{tr} G^+_{k, (w, \alpha); D} - \operatorname{tr} G^-_{k, (w, \alpha); D}\}] \\
& = -\frac{\sin \frac{n\pi}{2}}{16} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma \left(\frac{n}{2} - 2 \right) \right\}^2 \pi (d_+ - d_-), \quad n \text{ is odd.}
\end{aligned}$$

We note that, since (60) holds although $\zeta_D(0)$ or $\eta_D(0)$ does not exist, (61) holds although $\zeta_{D^2}(0)$ or $\eta_D(0)$ does not exist.

On the other hand, since θ is selfadjoint, by (59), we have

$$(62) \quad \| \theta G^p_{\lambda, k, a, (w, \alpha)} \| = O(|\lambda|^{1-\delta-\frac{2}{k+2}}), \quad \lambda \rightarrow \infty,$$

if k , w and α are real numbers. Hence, denote $L^2(M, E)_D$ the completion of $(\ker D)^\perp \cap C_\infty(M, E)$ by the norm $\|f\|_D^2 = (Df, Df)$, as an operator on $L^2([0, a])$, $r^{n-1}dr \otimes \mathcal{H}^{p, n} \otimes L^2(M, E)_D$, $\theta G^p_{k, a, (w, \alpha)}$ is bounded and $(\theta G^p_{k, a, (w, \alpha)})^m$ is compact if m is sufficiently large. Therefore, if $a < 1$ and k is sufficiently large, in the space $L^2([0, a], r^{n-1}dr) \otimes \mathcal{H}^{p, n} \otimes L^2(M, E)_D$, $(I + r^k \theta G^p_{k, a, (w, \alpha)})^{-1}$ exists uniquely, and given by the Neumann series

$$(I + r^k \theta G^p_{k, a, (w, \alpha)})^{-1} = I - r^k \theta G^p_{k, a, (w, \alpha)} + \cdots + (-1)^m (r^k \theta G^p_{k, a, (w, \alpha)})^m + \cdots,$$

if k , w and α are real numbers. Moreover, by the definition of $G^p_{k, a, (w, \alpha)}$, this Neumann series also converges on $L^2([0, a], r^{n-1}dr) \otimes \mathcal{H}^{p, n} \otimes \ker D$ if k is sufficiently large. Then, since

$$(\mathcal{A}_{w, \alpha} + r^k(D + \theta))G_{k, a, (w, \alpha); D} f = f + r^k \theta G_{k, a, (w, \alpha); D} f,$$

we get

$$(63)' \quad G_{k, a, (w, \alpha); D+\theta} = G_{k, a, (w, \alpha); D} (I + r^k \theta G_{k, a, (w, \alpha); D})^{-1},$$

because, since the estimate (62) does not depend on p , $(I + r^k \theta G_{k, a, (w, \alpha); D})^{-1}$ exists if $a < 1$ and k is sufficiently large, and under the boundary condition (4)', $\ker(\mathcal{A}_{w, \alpha} + r^k(D + \theta))\{0\}$. By (63)', we obtain

$$(64) \quad (\mathcal{A}_{w, \alpha} + r^k D)G_{k, a, (w, \alpha); D+\theta} = (I + r^k \theta G_{k, a, (w, \alpha); D})^{-1}.$$

But since the left hand side of (64) is defined and analytic for complex w , α , $\operatorname{Re} \alpha > 0$, and $a > 0$, and the limit at $a \rightarrow \infty$ exists, we obtain

Lemma 16. $(I + r^k \theta G_{k, a, (w, \alpha); D})^{-1}$ and $(I + r^k \theta G_{k, (w, \alpha); D})^{-1}$ are densely defined in $L^2(B_a \times M, \pi^*(E))$ and in $L^2(\mathbb{R}^n \times M, \pi^*(E))$, and we have

$$(63) \quad G_{k, (w, \alpha); D+\theta} = G_{k, (w, \alpha); D} (I + r^k \theta G_{k, (w, \alpha); D})^{-1},$$

$$G_{k, a, (w, \alpha); D+\theta} = G_{k, a, (w, \alpha); D} (I + r^k \theta G_{k, a, (w, \alpha); D})^{-1}.$$

16. Theorem 2. Under the above assumptions about θ , denoting $G_{k,(w,\alpha);D}$, etc., by $G_{k,(w,\alpha)}$, etc., we obtain

$$(65)_+ \quad \lim_{\alpha \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w [\operatorname{tr}\{G_{k,a,(w,\alpha)} r^k \theta G_{k,a,(w,\alpha)} (I + r^k \theta G_{k,a,(w,\alpha)})^{-1}\}]$$

$$= \frac{1}{n} (d_+ + d_-), \quad n = 2, 4,$$

$$\lim_{\alpha \rightarrow \infty} \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} [w \operatorname{tr}\{G_{k,a,(w,\alpha)} r^k \theta G_{k,a,(w,\alpha)} (I + r^k \theta G_{k,a,(w,\alpha)})^{-1}\}]$$

$$= -\frac{\sin \frac{n\pi}{2}}{8\pi} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2} - 2\right) \right\}^2 (d_+ + d_-), \quad n \text{ is odd.}$$

$$(65)_- \quad \lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 [\operatorname{tr}\{G^{+k,(w,\alpha)} r^k \theta G^{+k,(w,\alpha)} (I + r^k \theta G^{+k,(w,\alpha)})^{-1}\}]$$

$$+ \operatorname{tr}\{G^{-k,(w,\alpha)} r^k \theta G^{-k,(w,\alpha)} (I - r^k \theta G^{-k,(w,\alpha)})^{-1}\}]$$

$$= -\frac{\pi^2}{2n} (d_+ - d_-), \quad n = 2, 4,$$

$$\lim_{w \rightarrow 0} \lim_{k \rightarrow \infty} w k^2 [\operatorname{tr}\{G^{+k,(w,\alpha)} r^k \theta G^{+k,(w,\alpha)} (I + r^k \theta G^{+k,(w,\alpha)})^{-1}\}]$$

$$+ \operatorname{tr}\{G^{-k,(w,\alpha)} r^k \theta G^{-k,(w,\alpha)} (I - r^k \theta G^{-k,(w,\alpha)})^{-1}\}]$$

$$= -\frac{\sin \frac{n\pi}{2}}{16} \frac{(4-n)(2-n)}{(n-2)!} \left\{ \Gamma\left(\frac{n}{2} - 2\right) \right\}^2 \pi (d_+ - d_-), \quad n \text{ is odd.}$$

Proof. Since we know

$$(I + r^k \theta G_{k,(w,\alpha)})^{-1} - I = -r^k \theta G_{k,(w,\alpha)} (I + r^k \theta G_{k,(w,\alpha)})^{-1},$$

we get (65)₊ by lemma 16 and (61).

Since $(\Delta_{w,\alpha} - r^k(D + \theta))G^{-k,(w,\alpha)}f = f - r^k \theta G^{-k,(w,\alpha)}f$, we get

$$G^{-k,(w,\alpha);D+\theta} = G^{-k,(w,\alpha);D} (I - r^k \theta G^{-k,(w,\alpha);D})^{-1},$$

$$G^{-k,a,(w,\alpha);D+\theta} = G^{-k,a,(w,\alpha);D} (I - r^k \theta G^{-k,a,(w,\alpha);D})^{-1},$$

where $(I - r^k \theta G^{-k,(w,\alpha);D})^{-1}$ and $(I - r^k \theta G^{-k,a,(w,\alpha);D})^{-1}$ are densely defined in $L^2(\mathbb{R}^n \times M, \pi^*(E))$ and in $L^2(B_a \times M, \pi^*(E))$ by the same reason as lemma 16. Then, since

$$(I - r^k \theta G_{k,(w,\alpha)})^{-1} - I = r^k \theta G_{k,(w,\alpha)} (I - r^k \theta G_{k,(w,\alpha)})^{-1},$$

we get (65)₋.

Corollary. The set of selfadjoint operators of $L^2(M, E)$ which are defined on the set of proper functions of D and satisfy (59) and the condition $\ker (\Delta_{w,\alpha} + r^k D) \cap \mathcal{R}(G_{k,a,(w,\alpha);D+\theta}) = \{0\}$, has at least $\mathbb{Z} \oplus \mathbb{Z}$ connected components, with the strong topology.

Proof. By a theorem of Weyl ([18], 5), the sets of integers which can be expressed as d_+ and d_- are both equal to \mathbb{Z} . On the other hand, by (65), $d_+ + d_-$ and $d_+ - d_-$ are both expressed by moving θ and the value of the left hand sides of (65) depend continuously on. Hence we have the corollary.

Note. Similarly, for even $n, n \geq 6$, we obtain

$$(65)_+': \lim_{\alpha \rightarrow \infty} \lim_{k \rightarrow \infty} \left[\frac{\partial^{n-2}}{\partial \alpha^{n-2}} \operatorname{tr} G_{k, a, (0, \alpha)} r^k \theta G_{k, a, (0, \alpha)} (I + r^k \theta G_{k, a, (0, \alpha)})^{-1} \right. \\ \left. + \frac{\alpha^2}{2(n-2)} (\dim \ker(D + \theta) - \dim \ker D) \right] = -\frac{1}{2(n-2)} (d_+ + d_-), \\ \lim_{k \rightarrow \infty} k^2 \frac{\partial^{n-2}}{\partial \alpha^{n-2}} [\operatorname{tr} G_{k, (0, \alpha)}^+ r^k \theta G_{k, (0, \alpha)}^+ (I + r^k \theta G_{k, (0, \alpha)}^+)^{-1} \\ + \operatorname{tr} G_{k, (0, \alpha)}^- r^k \theta G_{k, (0, \alpha)}^- (I - r^k \theta G_{k, (0, \alpha)}^-)^{-1}] = -\frac{1}{2(n-2)} (d_+ - d_-).$$

On the other hand, if we use $G_{k, a}^p$, etc., instead of $G_{k, a, (w, \alpha)}$, etc., we get

$$(66)_+ \quad \lim_{\alpha \rightarrow \infty} \lim_{k \rightarrow \infty} [\operatorname{tr} G_{k, a}^p r^k \theta G_{k, a}^p (I + r^k \theta G_{k, a}^p)^{-1} + \frac{(n+p-3)! \alpha^2}{2(p!) (n-2)!} (\dim \ker(D + \theta) \\ - \dim \ker D)] = -\frac{(2p+n-2)(n+p-3)!}{2(2p+n)(p!)(n-2)!} (d_+ + d_-), \quad n \geq 3,$$

$$\lim_{\alpha \rightarrow \infty} \lim_{k \rightarrow \infty} [\operatorname{tr} G_{k, a}^p r^k \theta G_{k, a}^p (I + r^k \theta G_{k, a}^p)^{-1} \\ + \frac{\alpha^2 \log \alpha}{2} (\dim \ker(D + \theta) - \dim \ker D)] = -\frac{1}{4} (d_+ + d_-), \quad n = 2. \\ (66)_- \quad \lim_{k \rightarrow \infty} k^2 [\operatorname{tr} G_{k, (0, \alpha)}^+ r^k \theta G_{k, (0, \alpha)}^+ (I + r^k \theta G_{k, (0, \alpha)}^+)^{-1} + \operatorname{tr} G_{k, (0, \alpha)}^- r^k \theta G_{k, (0, \alpha)}^- (I - r^k \theta G_{k, (0, \alpha)}^-)^{-1}] \\ = -\frac{\pi^2}{4} \frac{(2p+n-2)(n+p-3)!}{(2p+n)(p!)(n-2)!} (d_+ - d_-).$$

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