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Subgroups of type A_2-series in the exceptional Lie groups G_2, F_4 and E_6

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It is known that the simply connected compact exceptional Lie groups G_2, F_4 and E_6 have subgroups of type A_2, $A_2 \oplus A_2$ and $A_2 \oplus A_2 \oplus A_2$ respectively as one of their subgroups of maximal rank [2], and these groups must be exactly isomorphic to the groups

$$SU(3) \subset G_2,$$

$$(SU(3) \times SU(3))/\boldsymbol{Z}_3 \subset F_4,$$

$$(SU(3) \times SU(3) \times SU(3))/\boldsymbol{Z}_3 \subset E_6$$

respectively [1]. In this paper, we give these embeddings explicitly.

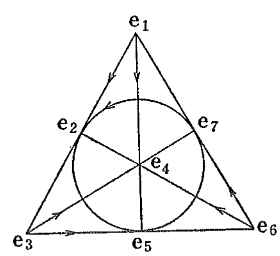
1. Preliminaries.

Let $\mathfrak{C}$ denote the Cayley division algebra over the field of real numbers $\boldsymbol{R}$. This algebra $\mathfrak{C}$ has a basis $\{e_0, e_1, e_2, \cdots, e_7\}$ with the following multiplication :

$e_0 = 1$, $e_k{}^2 = -1$, $k = 1, 2, \cdots, 7$,

$e_k e_l = -e_l e_k$, $k \neq l$, $k, l = 1, 2, \cdots, 7$,

$e_1 e_2 = e_3$, $e_4 e_2 = e_6$, $e_2 e_5 = e_7$, $\cdots$.



Let $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$ denote the exceptional Jordan algebra of all 3×3 Hermitian matrices X with entries in $\mathfrak{C}$

$$X = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i \in \boldsymbol{R}, \ x_i \in \mathfrak{C}$$

with respect to the multiplication $X \circ Y = \frac{1}{2}(XY + YX)$. In $\mathfrak{J}$, the crossed product $X \times Y$, the inner product (X, Y) and the determinant $\det X$ are defined respectively by

$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(X \circ Y))E),$$

$$(X,\ Y)=\mathrm{tr}(X\circ Y),\quad \det X=\frac{1}{3}(X\times X,\ X)$$

where E is the unit matrix.

In $\mathfrak{J}$, we adopt the following notations.

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$F_1(x)=\begin{pmatrix}0&0&0\\0&0&x\\0&\bar{x}&0\end{pmatrix},\quad F_2(x)=\begin{pmatrix}0&0&\bar{x}\\0&0&0\\x&0&0\end{pmatrix},\quad F_3(x)=\begin{pmatrix}0&x&0\\\bar{x}&0&0\\0&0&0\end{pmatrix}.$$

Then these elements generate $\mathfrak{J}$ additively and the table of the crossed product among them are given as follows :

$$E_i\times E_i=0,\qquad E_i\times E_{i+1}=\frac{1}{2}E_{i+2},$$

$$E_i\times F_i(x)=-\frac{1}{2}F_i(x),\qquad E_i\times F_j(x)=0,\quad i\neq j,$$

$$F_i(x)\times F_i(y)=-(x,\ y)E_i,\qquad F_i(x)\times F_{i+1}(y)=\frac{1}{2}F_{i+2}(\overline{xy})$$

where the indexes are considered as mod 3.

Let $\mathfrak{J}^C=\mathfrak{J}(3,\ \mathfrak{C}^C)$ be the split exceptional Jordan algebra over the field of complex numbers $\boldsymbol{C}$. This algebra $\mathfrak{J}^C$ may be considered as the complexification of the Jordan algebra $\mathfrak{J}$:

$$\mathfrak{J}^C=\{\ X_1+iX_2\ |\ X_1,\ X_2\in\mathfrak{J},\ i^2=-1\ \}.$$

In $\mathfrak{J}^C$ also, the crossed product $X\times Y$, the inner product $(X,\ Y)$ and the determinant $\det X$ are naturally defined. The complex conjugate in $\mathfrak{J}^C$ is denoted by τ :

$$\tau(X_1+iX_2)=X_1-iX_2,\qquad X_1,\ X_2\in\mathfrak{J}.$$

Finally, we define the positive definite Hermitian inner product $\langle X,\ Y\rangle$ in $\mathfrak{J}^C$ by

$$\langle X,\ Y\rangle=(\tau X,\ Y).$$

2. Subgroup of type A_2 in the group G_2.

The group G_2 is defined as the autmorphism group of $\mathfrak{C}$:

$$G_2=\mathrm{Aut}(\mathfrak{C})=\{\ \alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C},\ \mathfrak{C})\ |\ \alpha(xy)=\alpha(x)\alpha(y)\ \}.$$

Then it is a simply connected compact Lie group of type G_2.

The field of complex numbers $\boldsymbol{C}$ is contained in $\mathfrak{C}$ as $\{\ a+be_1\ |\ a,\ b\in\boldsymbol{R}\ \}$ and any Cayley number $x\in\mathfrak{C}$ can be represented by

$$x = a_0 + a_1e_2 + a_2e_4 + a_3e_6, \quad a_i \in \boldsymbol{C},$$

hence $\mathfrak{C}$ is a 4-dimensional left vector space over $\boldsymbol{C}$. Furthermore we use the following identification :

$$x = a_0 + a_1e_2 + a_2e_4 + a_3e_6 \quad \longleftrightarrow \quad x = a_0 + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = x_C + \boldsymbol{x}_F \in \boldsymbol{C} \oplus \boldsymbol{C}^3.$$

Theorem 1. *The group G_2 contains a subgroup which is isomorphic to the special unitary group $SU(3) = \{A \in M(3, \boldsymbol{C}) \mid AA^* = E, \det A = 1\}$.*

Proof. We difine a homomorphism $\phi : SU(3) \longrightarrow G_2$ by

$$\begin{aligned} \phi(A)(x) &= \phi(A)(x_C + \boldsymbol{x}_F) \qquad x \in \mathfrak{C} \\ &= x_C + A\boldsymbol{x}_F. \end{aligned}$$

First of all, we have to show that $\alpha = \phi(A) \in G_2$. In order to prove this, it is sufficient to show

$$\alpha(e_k)\alpha(e_l) = \alpha(e_ke_l), \qquad k, l = 1, 2, \cdots, 7.$$

For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in SU(3)$, we have

$$\begin{aligned} \alpha(e_2)\alpha(e_3) &= (a_{11}e_2 + a_{21}e_4 + a_{31}e_6)(a_{11}e_3 + a_{21}e_5 + a_{31}e_7) \\ &= |a_{11}|^2e_1 - \overline{a_{21}a_{11}}e_7 + \overline{a_{31}a_{11}}e_5 + \overline{a_{11}a_{21}}e_7 + |a_{21}|^2e_1 - \overline{a_{31}a_{21}}e_3 - \overline{a_{11}a_{31}}e_5 + \overline{a_{21}a_{31}}e_3 + |a_{31}|^2e_1 \\ &= (|a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2)e_1 \\ &= e_1 = \alpha(e_1), \\ \alpha(e_2)\alpha(e_4) &= (a_{11}e_2 + a_{21}e_4 + a_{31}e_6)(a_{12}e_2 + a_{22}e_4 + a_{32}e_6) \\ &= -(a_{11}\overline{a}_{12} + a_{21}\overline{a}_{22} + a_{31}\overline{a}_{32}) + (\overline{a_{31}a_{22} - a_{21}a_{32}})e_2 + (\overline{a_{11}a_{32} - a_{31}a_{12}})e_4 + (\overline{a_{21}a_{12} - a_{11}a_{22}})e_6 \\ &= 0 - a_{13}e_2 - a_{23}e_4 - a_{33}e_6 \\ &= -\alpha(e_6) = \alpha(e_2e_4). \end{aligned}$$

The others are also proved by calculations similar to the above. Finally, $\ker\phi = \{E\}$ is easily obtained. Thus the proof of Theorem 1 is completed.

3. Subgroup of type $A_2 \oplus A_2$ in the group F_4.

The group F_4 is defined as the automorphism group of $\mathfrak{J}$:

$$F_4=\mathrm{Aut}(\mathfrak{J})=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})\mid \alpha(X\circ X)=\alpha X\circ\alpha Y\ \}$$
$$=\{\ \alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})\mid \alpha(X\times Y)=\alpha X\times\alpha Y\ \}.$$

Then it is a simply connected compact simple Lie group of type F_4. The group G_2 is a subgroup of F_4 by the correspondence $\alpha\in G_2\longrightarrow\alpha'\in F_4$,

$$\alpha'\begin{pmatrix}\xi_1 & x_3 & \overline{x_2}\\ \overline{x_3} & \xi_2 & x_1\\ x_2 & \overline{x_1} & \xi_3\end{pmatrix}=\begin{pmatrix}\xi_1 & \alpha(x_3) & \overline{\alpha(x_2)}\\ \overline{\alpha(x_3)} & \xi_2 & \alpha(x_1)\\ \alpha(x_2) & \overline{\alpha(x_1)} & \xi_3\end{pmatrix}.$$

From now on, we use the following identification :

$$X=\begin{pmatrix}\xi_1 & x_3 & \overline{x_2}\\ \overline{x_3} & \xi_2 & x_1\\ x_2 & \overline{x_1} & \xi_3\end{pmatrix}\in\mathfrak{J}\quad\longleftrightarrow\quad X=\begin{pmatrix}\xi_1 & x_{3C} & \overline{x_{2C}}\\ \overline{x_{3C}} & \xi_2 & x_{1C}\\ x_{2C} & \overline{x_{1C}} & \xi_3\end{pmatrix}+\begin{pmatrix}\boldsymbol{x}_{1F} & \boldsymbol{x}_{2F} & \boldsymbol{x}_{3F}\end{pmatrix}$$
$$=X_C+X_F\in\mathfrak{J}(3,\ \boldsymbol{C})\oplus M(3,\ \boldsymbol{C}).$$

Theorem 2. *The group F_4 contains a subgroup which is isomorphic to the group $(SU(3)\times SU(3))/\boldsymbol{Z}_3$, where $\boldsymbol{Z}_3=\{\ (E,\ E),\ (\omega E,\ \omega E),\ (\omega^2E,\ \omega^2E)\ \},\omega\in\boldsymbol{C},\ \omega^3=1,\ \omega\neq 1.$*

Proof. we define a homomorphism $\phi: SU(3)\times SU(3)\longrightarrow F_4$ by

$$\phi(A,\ B)(X)=\phi(A,\ B)(X_C+X_F)\qquad X\in\mathfrak{J}$$
$$=BX_CB^*+AX_FB^*.$$

First of all, we have to prove that $\phi(A,\ B)\in F_4$. Since $\alpha=\phi(A,\ E)\in G_2\subset F_4$, we show only $\beta=\phi(E,\ B)\in F_4$, *i.e.*,

$$\beta(X\times Y)=\beta X\times\beta Y,\qquad X,\ Y\in\mathfrak{J}.$$

In order to prove this, it is sufficient to show

$$\beta E_i\times\beta E_i=0,\qquad\qquad \beta E_i\times\beta E_{i+1}=\frac{1}{2}\beta E_{i+2},$$

$$\beta E_i\times\beta F_i(e_k)=-\frac{1}{2}\beta F_i(e_k),\qquad\qquad \beta E_i\times\beta F_j(e_k)=0,\qquad i\neq j,$$

$$\beta F_i(e_k)\times\beta F_i(e_l)=-(e_k,\ e_l)\beta E_i,\qquad\qquad \beta F_i(e_k)\times\beta F_{i+1}(e_l)=\frac{1}{2}\beta F_{i+2}(\overline{e_ke_l}).$$

For $B=\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}\in SU(3)$, we have

$$2\beta E_1\times\beta F_1(e_2)$$

$$=2(|b_{11}|^2E_1+|b_{21}|^2E_2+|b_{31}|^2E_3+F_1(b_{21}\overline{b_{31}})+F_2(b_{31}\overline{b_{11}})+F_3(b_{11}\overline{b_{21}}))$$

$$\times(F_1(\overline{b_{11}}e_2)+F_2(\overline{b_{21}}e_2)+F_3(\overline{b_{31}}e_2))$$
$$=F_1(-|b_{11}|^2(\overline{b_{11}}e_2)+\overline{(\overline{b_{21}}e_2)(b_{11}\overline{b_{21}})}+\overline{(b_{31}\overline{b_{11}})(\overline{b_{31}}e_2)})+F_2(*)+F_3(*)$$
$$=F_1(-|b_{11}|^2\overline{b_{11}}e_2-\overline{b_{21}}\overline{b_{11}}b_{21}e_2-b_{31}\overline{b_{11}}\overline{b_{31}}e_2)+F_2(*)+F_3(*)$$
$$=F_1(-(|b_{11}|^2+|b_{21}|^2+|b_{31}|^2)\overline{b_{11}}e_2)+F_2(*)+F_3(*)$$
$$=F_1(-\overline{b_{11}}e_2)+F_2(-\overline{b_{21}}e_2)+F_3(-\overline{b_{31}}e_2)$$
$$=-\beta F_1(e_2),$$

$$2\beta F_1(e_3)\times\beta F_2(e_4)$$
$$=2(F_1(\overline{b_{11}}e_3)+F_2(\overline{b_{21}}e_3)+F_3(\overline{b_{31}}e_3))\times(F_1(\overline{b_{12}}e_4)+F_2(\overline{b_{22}}e_4)+F_3(\overline{b_{32}}e_4))$$
$$=F_1(\overline{(\overline{b_{21}}e_3)(\overline{b_{32}}e_4)}+\overline{(\overline{b_{22}}e_4)(\overline{b_{31}}e_3)})+F_2(*)+F_3(*)$$
$$=F_1((-b_{21}b_{32}+b_{22}b_{31})e_7)+F_2(*)+F_3(*)$$
$$=F_1(-\overline{b_{13}}e_7)+F_2(-\overline{b_{23}}e_7)+F_3(-\overline{b_{33}}e_7)$$
$$=\beta F_3(-e_7)=\beta F_3(\overline{e_3e_4}).$$

The others are also proved by calculations similar to the above. Finally, $\ker\phi=\boldsymbol{Z}_3=\{(E, E), (\omega E, \omega E), (\omega^2E, \omega^2E)\}$, $\omega\in\boldsymbol{C}$, $\omega^3=1$, $\omega\neq1$, is easily obtained. Thus the proof of Theorem 2 is completed.

4. Subgroup of type $A_2\oplus A_2\oplus A_2$ in the group E_6.

The group E_6 is defined by

$$E_6=\{\alpha\in\mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C)\,|\,\det\alpha X=\det X,\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}$$
$$=\{\alpha\in\mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C)\,|\,\alpha X\times\alpha Y=\tau\alpha\tau(X\times Y),\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}.$$

Then it is a simply connected compact simple Lie group of type E_6 [3]. The group F_4 is a subgroup of E_6 by the correspondence $\alpha\in F_4\longrightarrow\alpha^C\in E_6$,

$$\alpha^C(X_1+iX_2)=\alpha X_1+i\alpha X_2,\qquad X_1,\ X_2\in\mathfrak{J}.$$

Let $\boldsymbol{C}^C=\{p+iq\mid p, q\in\boldsymbol{C}, i^2=-1\}$ be the complexification algebra of the complex numbers $\boldsymbol{C}\subset\mathfrak{C}$.

Lemma 3. *The algebra $\boldsymbol{C}^C$ is isomorphic to the direct product of two algebras $\boldsymbol{C}$:*

$$\boldsymbol{C}^C=\boldsymbol{C}\times\boldsymbol{C}.$$

Furthermore this isomorphism is naturally extended to the following isomorphism as algebra

$$M(n,\ C^C)=M(n,\ C)\times M(n,\ C).$$

If $D=(B,\ C)$ *under the isomorphism, then we have*

$$\widetilde{D}=(B^*,\ C^*)$$

where $\widetilde{D}=\tau D^*={}^t(\tau\overline{D})$ *and*

$$\det D=(\det B,\ \det C).$$

Proof. It is easy to verify that mappings $f: C\times C\longrightarrow C^C$ and $f: M(n,\ C)\times M(n,\ C)\longrightarrow M(n,\ C^C)$,

$$f(b,\ c)=\frac{b+c}{2}+i\frac{(b-c)e_1}{2},$$

$$f(B,\ C)=\frac{B+C}{2}+i\frac{(B-C)e_1}{2}$$

give the isomorphisms as algebra between them and satisfy the required results.

we define a group $SU^C(n)$ by

$$SU^C(n)=\{\ D\in M(n,\ C^C)\mid D\widetilde{D}=E,\ \det D=1\ \}.$$

Proposition 4. *The group* $SU^C(n)$ *is isomorphic to the direct product of two special unitary groups* $SU(n)=\{\ A\in M(n,\ C)\mid AA^*=E,\ \det A=1\ \}$:

$$SU^C(n)=SU(n)\times SU(n).$$

Proof. The restriction of the isomorphism f of Lemma 3 to $SU(n)\times SU(n)$ obviously gives the required isomorphism.

Analogously in the case F_4, any element $X\in\mathfrak{J}^C$ can be represented by

$$X=X_C+X_F\in\mathfrak{J}(3,\ C^C)\oplus M(3,\ C^C).$$

Theorem 5. *The group* E_6 *contains a subgroup which is isomorphic to the group* $(SU(3)\times SU(3)\times SU(3))/Z_3$, *where* $Z_3=\{\ (E,\ E,\ E),\ (\omega E,\ \omega E,\ \omega E),\ (\omega^2E,\ \omega^2E,\ \omega^2E)\ \}$, $\omega\in C,\ \omega^3=1,\ \omega\neq 1$.

Proof. we define a homomorphism $\phi: SU(3)\times SU^C(3)\longrightarrow E_6$ *by*

$$\phi(A,\ D)(X)=\phi(A,\ D)(X_C+X_F)\qquad X\in\mathfrak{J}^C$$
$$=DX_CD^*+AX_F\widetilde{D}.$$

First of all, we have to prove that $\phi(A,\ D)\in E_6$. Since $\phi(A,\ E)\in G_2\subset F_4\subset E_6$, we show only $\delta=\phi(E,\ D)\in E_6$, *i.e.*,

$$\delta X\times\delta Y=\tau\delta\tau(X\times Y),\qquad \langle\delta X,\ \delta Y\rangle=\langle X,\ Y\rangle.$$

As for the first formula, the proof is quite similar to the case F_4. In fact, for $D=\begin{pmatrix} d_{11} & d_{13} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \in SU^C(3)$, we have, for example,

$2\delta E_1 \times \delta F_1(e_2)$ (Denote the complex conjugate in $\boldsymbol{C}^C$ also by τ and $\tau\overline{d}$ by $\widetilde{d}$)

$$=2(d_{11}\overline{d_{11}}E_1+d_{21}\overline{d_{21}}E_2+d_{31}\overline{d_{31}}E_3+F_1(d_{21}\overline{d_{31}})+F_2(d_{31}\overline{d_{11}})+F_3(d_{11}\overline{d_{21}}))$$

$$\times(F_1(\widetilde{d}_{11}e_2)+F_2(\widetilde{d}_{21}e_2)+F_3(\widetilde{d}_{31}e_2))$$

$$=F_1(-(d_{11}\overline{d_{11}})(\widetilde{d}_{11}e_2)+\overline{(\widetilde{d}_{21}e_2)(d_{11}\overline{d_{21}})}+\overline{(d_{31}\overline{d_{11}})(\widetilde{d}_{31}e_2)})+F_2(*)+F_3(*)$$

$$=F_1(-d_{11}\overline{d}_{11}\widetilde{d}_{11}e_2-\widetilde{d}_{21}\overline{d}_{11}d_{21}e_2-d_{31}\overline{d}_{11}\widetilde{d}_{31}e_2)+F_2(*)+F_3(*)$$

$$=F_1(-(d_{11}\widetilde{d_{11}}+d_{21}\widetilde{d_{21}}+d_{31}\widetilde{d_{31}})\overline{d_{11}}e_2)+F_2(*)+F_3(*)$$

$$=F_1(-\overline{d_{11}}e_2)+F_2(-\overline{d_{21}}e_2)+F_3(-\overline{d_{31}}e_2)$$

$$=\tau\delta F_1(-e_2)=2\tau\delta(E_1\times F_1(e_2))=2\tau\delta\tau(E_1\times F_1(e_2)),$$

$2\delta F_1(ie_3)\times\delta F_2(e_4)$

$$=2i(F_1(\widetilde{d}_{11}e_3)+F_2(\widetilde{d}_{21}e_3)+F_3(\widetilde{d}_{31}e_3))\times(F_1(\widetilde{d}_{12}e_4)+F_2(\widetilde{d}_{22}e_4)+F_3(\widetilde{d}_{32}e_4))$$

$$=i(F_1(\overline{(\widetilde{d}_{21}e_3)(\widetilde{d}_{32}e_4)}+\overline{(\widetilde{d}_{22}e_4)(\widetilde{d}_{31}e_3)})+F_2(*)+F_3(*))$$

$$=i(F_1((\tau(-d_{21}d_{32}+d_{22}d_{31}))e_7)+F_2(*)+F_3(*))$$

$$=i(F_1(-\overline{d_{13}}e_7)+F_2(-\overline{d_{23}}e_7)+F_3(-\overline{d_{33}}e_7))$$

$$=i\tau\delta F_3(-e_7)=2i\tau\delta(F_1(e_3)\times F_2(e_4))=2\tau\delta\tau(F_1(ie_3)\times F_2(e_4)).$$

Next, we shall prove $\langle\delta X, \delta Y\rangle=\langle X, Y\rangle$. Obviously, for $D\in SU^C(3)$, X, $Y\in\mathfrak{J}^C$,

$$\langle DX_CD^*, DY_CD^*\rangle=\langle X_C, Y_C\rangle, \qquad \langle X_C, Y_F\rangle=0$$

are satisfied. Moreover we have

$$\langle X_F\widetilde{D}, Y_F\widetilde{D}\rangle=\langle X_F, Y_F\rangle.$$

In fact, for example,

$$\langle F_1(e_2)\widetilde{D}, F_1(e_2)\widetilde{D}\rangle=\langle F_1(\widetilde{d}_{11}e_2)+F_2(\widetilde{d}_{21}e_2)+F_3(\widetilde{d}_{31}e_2), F_1(\widetilde{d}_{11}e_2)+F_2(\widetilde{d}_{21}e_2)+F_3(\widetilde{d}_{31}e_2)\rangle$$

$$=(\overline{d_{11}}, \widetilde{d_{11}})+(\overline{d_{21}}, \widetilde{d_{21}})+(\overline{d_{31}}, \widetilde{d_{31}})$$

$$=1=\langle F_1(e_2), F_1(e_2)\rangle.$$

$$\langle F_1(e_k)\widetilde{D}, F_2(e_l)\widetilde{D}\rangle=\ldots=0=\langle F_1(e_k), F_2(e_l)\rangle.$$

Using these formulae, we have $\langle\delta X, \delta Y\rangle=\langle X, Y\rangle$. Thus we see that $\delta=\phi(E, D)\in E_6$, so that $\phi(A, D)\in E_6$. Finally, $\ker\phi=\boldsymbol{Z}_3=\{(E, E), (\omega E, \omega E), (\omega^2 E, \omega^2 E)\}$

$\omega \in C$, $\omega^3=1$, $\omega \neq 1$, is easily obtained. Together with Proposition 4, we have proved Theorem 5.

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