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Non-compact Simple Lie Groups $E_{6(-14)}$ and $E_{6(2)}$ of Type E_6

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It is known that there exist five simple Lie groups of type E_6 up to local isomorphism, one of them is compact and the others are non-compact. The compact simple Lie group is given by

$$E_6=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)\,|\,\det\alpha X=\det X,\ \langle\alpha X,\ \alpha Y\rangle=\langle X,\ Y\rangle\}$$

where $\mathfrak{J}^C$ is the split exceptional Jordan algebra over the complex numbers $\boldsymbol{C}$ and $\langle X,\ Y\rangle$ the positive definite Hermitian inner product in $\mathfrak{J}^C$, and it is simply connected and its center is $\boldsymbol{Z}_3$ [8]. Two of the non-compact simple Lie groups are given respectively by

$$E_{6(-26)}=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})\,|\,\det\alpha X=\det X\},$$

$$E_{6(6)}=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}_2,\ \mathfrak{J}_2)\,|\,\det\alpha X=\det X\}$$

where $\mathfrak{J}$ (resp. $\mathfrak{J}_2$) is the exceptional (resp. split exceptional) Jordan algebra over the real numbers $\boldsymbol{R}$, and their polar decompositions are given respectively by

$$E_{6(-26)}\simeq F_4\times \boldsymbol{R}^{26}, \qquad E_{6(6)}\simeq Sp(4)/\boldsymbol{Z}_2\times \boldsymbol{R}^{42},$$

and both centers are trivial [1], [3], [5].

In this paper, we find out explicitly the two other non-compact simple Lie groups. The results are as follows. These groups are given respectively by

$$E_{6,\sigma}=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)\,|\,\det\alpha X=\det X,\ \langle\alpha X,\ \alpha Y\rangle_\sigma=\langle X,\ Y\rangle_\sigma\},$$

$$E_{6,\gamma}=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)\,|\,\det\alpha X=\det X,\ \langle\alpha X,\ \alpha Y\rangle_\gamma=\langle X,\ Y\rangle_\gamma\}$$

where $\langle X,\ Y\rangle_\sigma$ and $\langle X,\ Y\rangle_\gamma$ are the Hermitian inner products in $\mathfrak{J}^C$. Their polar decompositions are given respectively by

$$E_{6,\sigma}\simeq(U(1)\times Spin(10))/\boldsymbol{Z}_4\times \boldsymbol{R}^{32},$$

$$E_{6,\gamma}\simeq(Sp(1)\times SU(6))/\boldsymbol{Z}_2\times \boldsymbol{R}^{40}$$

and both centers are given by the cyclic group $\boldsymbol{Z}_3=\{1, \omega 1, \omega^2 1\}$, $\omega\in\boldsymbol{C}$, $\omega^3=1$, $\omega\neq 1$, of order 3:

$$z(E_{6,\sigma})=\boldsymbol{Z}_3, \qquad z(E_{6,\tau})=\boldsymbol{Z}_3.$$

I. Non-compact simple Lie group $E_{6,\sigma}$ of type E_6

1. Jordan algebras $\mathfrak{J}$ and $\mathfrak{J}_1$.

Let $\mathfrak{C}$ be the Cayley algebra over the real numbers $\boldsymbol{R}$. In this algebra $\mathfrak{C}=\boldsymbol{H}\oplus\boldsymbol{H}e$ (where $\boldsymbol{H}$ is the quaternion field over $\boldsymbol{R}$), the multiplication xy, the conjugate $\bar{x}$, the scalar part $\mathrm{t}(x)$, the inner product (x, y) and the norm $|x|$ are defined respectively by

$$(a+bc)(c+de)=(ac-\bar{d}b)+(b\bar{c}+da)e,$$

$$\overline{a+be}=\bar{a}-be, \qquad \mathrm{t}(x)=x+\bar{x},$$

$$(a+be,\ c+de)=(a,\ c)+(b,\ d), \qquad |x|=\sqrt{(x,\ x)}.$$

Let $\mathfrak{C}^C=\{x_1+ix_2\,|\,x_1, x_2\in\mathfrak{C}\}$ be the complexification algebra of $\mathfrak{C}$. In $\mathfrak{C}^C$, the conjugate $\bar{x}$, the scalar part $\mathrm{t}(x)$ and the inner product (x, y) are also defined naturally.

Let $\mathfrak{J}=\mathfrak{J}(3, \mathfrak{C})$ be the Jordan algebra consisting of all 3×3 Hermitian matrices with entries in $\mathfrak{C}$

$$X=X(\xi, x)=\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i\in\boldsymbol{R}, x_i\in\mathfrak{C}$$

with respect to the multiplication

$$X\circ Y=\frac{1}{2}(XY+YX).$$

In $\mathfrak{J}$, the inner product (X, Y), the crossed product $X\times Y$, the cubic form (X, Y, Z) and the determinant $\det X$ are defined respectively by

$$(X,\ Y)=\mathrm{tr}(X\circ Y),$$

$$X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X,\ Y))E),$$

$$(X,\ Y,\ Z)=(X\times Y,\ Z)=(X,\ Y\times Z),$$

$$\det X=\frac{1}{3}(X,\ X,\ X)=\xi_1\xi_2\xi_3+\mathrm{t}(x_1x_2x_3)-\xi_1x_1\bar{x}_1-\xi_2x_2\bar{x}_2-\xi_3x_3\bar{x}_3,$$

where $X=X(\xi,\ x)$ and E is the 3×3 unit matrix.

Let $\mathfrak{J}^C=\mathfrak{J}(3, \mathfrak{C}^C)$ be the split exceptional Jordan algebra over the complex numbers $\boldsymbol{C}$. This Jordan algebra $\mathfrak{J}^C$ may be considered as the complexification of the Jordan algebra $\mathfrak{J}$. Especially any element X of $\mathfrak{J}^C$ can be uniquely represented by the form

$$X = X_1 + iX_2, \qquad X_1,\ X_2 \in \mathfrak{J},\ i^2 = -1.$$

In $\mathfrak{J}^C$, the inner product (X, Y), the crossed product $X \times Y$, the cubic form (X, Y, Z) and the determinant $\det X$ are also defined naturally. Moreover we define a mapping, called the complex conjugation, $\tau : \mathfrak{J}^C \to \mathfrak{J}^C$ by

$$\tau(X_1 + iX_2) = X_1 - iX_2, \qquad X_1,\ X_2 \in \mathfrak{J}$$

and the positive definite Hermitian inner product $\langle X, Y \rangle$ in $\mathfrak{J}^C$ by

$$\langle X, Y \rangle = (\tau X, Y).$$

Next, let $\mathfrak{J}_1$ be the Jordan algebra consisting of all 3×3 Γ-Hermitian matrices, i. e. $\Gamma X^* \Gamma = X$, where $\Gamma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with entries in $\mathfrak{C}$

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i \in \boldsymbol{R},\ x_i \in \mathfrak{C}$$

with respect to the multiplication $X \circ Y = \frac{1}{2}(XY + YX)$. In $\mathfrak{J}_1$ also, the inner product (X, Y), the crossed product $X \times Y$, the cubic form (X, Y, Z) and the determinant $\det X$ are defined by the quite analogous formulae as in $\mathfrak{J}$ (e. g. $\det X = \frac{1}{3}(X, X, X) = \xi_1\xi_2\xi_3 + \mathrm{t}(x_1x_2x_3) - \xi_1 x_1\overline{x}_1 + \xi_2 x_2\overline{x}_2 + \xi_3 x_3\overline{x}_3$).

Furthermore let $\mathfrak{J}_1^C$ be the complexification of the Jordan algebra $\mathfrak{J}_1$ and also in $\mathfrak{J}_1^C$ the inner product (X, Y), the crossed product $X \times Y$, the cubic form (X, Y, Z) and the determinant $\det X$ are naturally defined. Finally we define the Hermitian inner product $\langle X, Y \rangle$ in $\mathfrak{J}_1^C$ by

$$\langle X, Y \rangle = (\tau X, Y)$$

where $\tau(X_1 + iX_2) = X_1 - iX_2$ for $X_1,\ X_2 \in \mathfrak{J}_1$.

From now on, we will use the same notations for the same operations in $\mathfrak{J}$ and $\mathfrak{J}_1$, but as occasion demands the notations in $\mathfrak{J}_1$ will be indexed by the figure 1.

Proposition 1. *$\mathfrak{J}_1^C$ is isomorphic to $\mathfrak{J}^C$ as Jordan algebra over $\boldsymbol{C}$ by an isomorphism $f : \mathfrak{J}_1^C \to \mathfrak{J}^C$ defined as follows*:

$$fX = \Gamma_1 X \Gamma_1^*, \qquad \Gamma_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And f satisfies the following properties.

(i) $(X, Y)_1=(fX, fY)$,

(ii) $\det X=\det fX$,

(iii) $\langle X, Y\rangle_1=\langle fX, fY\rangle_\sigma$

where $\sigma:\mathfrak{J}^C\to\mathfrak{J}^C$ *is the linear involution defined by*

$$\sigma\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}=\begin{pmatrix}\xi_1 & -x_3 & -\bar{x}_2\\ -\bar{x}_3 & \xi_2 & x_1\\ -x_2 & \bar{x}_1 & \xi_3\end{pmatrix}$$

and the inner product $\langle X, Y\rangle_\sigma$ *in* $\mathfrak{J}^C$ *is defined by*

$$\langle X, Y\rangle_\sigma=\langle\sigma X, Y\rangle.$$

Proof. It is easy to see that f is a linear isomorphism over C and satisfies $f(X\circ Y)=fX\circ fY$. And

(i) $(X, Y)_1=\mathrm{tr}(X\circ Y)=\mathrm{tr}(f(X\circ Y))=\mathrm{tr}(fX\circ fY)=(fX, fY)$.

(ii) We have immediately $\det X=\det fX$.

(iii) Since we have $f\tau X=\tau\sigma fX$, we have

$$\langle X, Y\rangle_1=(\tau X, Y)_1=(f\tau X, fY)=(\tau\sigma fX, fY)=\langle\sigma fX, fY\rangle=\langle fX, fY\rangle_\sigma.$$

2. Groups of type E_6 and F_4.

The group $E_{6,\sigma}$ is defined to be the group of linear isomorphisms of $\mathfrak{J}^C$ leaving the determinant $\det X$ and the Hermitian inner product $\langle X, Y\rangle_\sigma$ invariant:

$$E_{6,\sigma}=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)\mid\det\alpha X=\det X,\ \langle\alpha X, \alpha Y\rangle_\sigma=\langle X, Y\rangle_\sigma\}$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)\mid(\alpha X, \alpha Y, \alpha Z)=(X, Y, Z),\ \langle\alpha X, \alpha Y\rangle_\sigma=\langle X, Y\rangle_\sigma\}$$

and $F_{4,\sigma}$ the subgroup of $E_{6,\sigma}$ preserving the inner product (X, Y):

$$F_{4,\sigma}=\{\alpha\in E_{6,\sigma}\mid(\alpha X, \alpha Y)=(X, Y)\}$$
$$=\{\alpha\in E_{6,\sigma}\mid\alpha E=E\}.$$

Next, to consider the group $E_{6,\sigma}$ we need to define the group $E_{6,1}$ and the subgroup $F_{4,1}$ of $E_{6,1}$:

$$E_{6,1}=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}_1{}^C,\mathfrak{J}_1{}^C)\mid\det\alpha X=\det X,\ \langle\alpha X,\alpha Y\rangle=\langle X, Y\rangle\}$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}_1{}^C,\mathfrak{J}_1{}^C)\mid(\alpha X, \alpha Y, \alpha Z)=(X, Y, Z),\langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\},$$

$$F_{4,1}=\{\alpha\in E_{6,1}\mid(\alpha X, \alpha Y)=(X, Y)\}$$
$$=\{\alpha\in E_{6,1}\mid\alpha E=E\}.$$

Finally we shall recall the compact group E_6 and the compact subgroup F_4 of E_6:

$$E_6=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)\mid\det\alpha X=\det X,\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}$$
$$=\{\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)\mid(\alpha X, \alpha Y, \alpha Z)=(X, Y, Z),\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\},$$

$$F_4=\{\alpha\in E_6\,|\,(\alpha X,\ \alpha Y)=(X,\ Y)\}$$
$$=\{\alpha\in E_6\,|\,\alpha E=E\}.$$

Lemma 2. *The group $F_{4,1}$ is homeomorphic to $Spin(9)\times\boldsymbol{R}^{16}$ and a simple (in the sense of the center $z(F_{4,1})=1$) Lie group of type F_4.*

Proof. We define the group $F_{4(-20)}$ by

$$F_{4(-20)}=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}_1,\ \mathfrak{J}_1)\,|\,\alpha(X\circ Y)=\alpha X\circ\alpha Y\}$$
$$=\{\alpha\in E^1{}_{6(-26)}\,|\,(\alpha X,\ \alpha Y)=(X,\ Y)\}$$
$$=\{\alpha\in E^1{}_{6(-26)}\,|\,\alpha E=E\}$$

where $E^1{}_{6(-26)}=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}_1,\ \mathfrak{J}_1)\,|\det\alpha X=\det X\}$. Then the argument used in the proof of Proposition 1 of [8] shows that $F_{4(-20)}$ is isomorphic to $F_{4,1}$ by the complexification $\alpha\rightarrow\alpha^C$ (which means $\alpha^C(X_1+iX_2)=\alpha X_1+i\alpha X_2,\ X_1, X_2\in\mathfrak{J}_1$). Recall now that $F_{4(-20)}$ is homeomorphic to $\mathrm{Spin}(9)\times\boldsymbol{R}^{16}$ and a simple (in the sense of the center $z(F_{4(-20)})=1$) Lie group of type F_4 (Theorem 8 and 11 [6]), then results follow.

Proposition 3. *The group $E_{6,\sigma}$ is isomorphic to the group $E_{6,1}$ and also $F_{4,\sigma}$ to $F_{4,1}$. In particular, $F_{4,\sigma}$ is homeomorphic to $Spin(9)\times\boldsymbol{R}^{16}$ and a simple (in the sense of the center $z(F_{4,\sigma})=1$) Lie group of type F_4.*

Proof. By using the isomorphism $f:\mathfrak{J}_1{}^C\rightarrow\mathfrak{J}^C$ in Proposition 1, we define a mapping $\phi: E_{6,\sigma}\rightarrow E_{6,1}$ by

$$\phi(\alpha)X=f^{-1}\alpha fX,\qquad X\in\mathfrak{J}_1{}^C.$$

Then from Proposition 1 it is easily obtained that ϕ gives an isomorphism between $E_{6,\sigma}$ and $E_{6,1}$. Furthermore we can readily show that the restriction $\phi|F_{4,\sigma}$ gives an isomorphism between $F_{4,\sigma}$ and $F_{4,1}$.

Remark. Let the group $E_{6(-26)}$ and its subgroup $F'_{4(-20)}$ be defined respectively by

$$E_{6(-26)}=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\mathfrak{J})\,|\det\alpha X=\det X\},$$
$$F'_{4(-20)}=\{\alpha\in E_{6(-26)}\,|\,(\alpha X,\ \alpha Y)_\sigma=(X,\ Y)_\sigma\}$$
$$=\{\alpha\in E_{6(-26)}\,|\,\alpha\Gamma=\Gamma\}$$

where $(X,\ Y)_\sigma=(\sigma X,\ Y)$. Then we have already known that

$$E_{6(-26)}\simeq F_4\times\boldsymbol{R}^{26},\qquad z(E_{6(-26)})=1\qquad([1],\ [3]),$$
$$F'_{4(-20)}\simeq Spin(9)\times\boldsymbol{R}^{16},\qquad z(F'_{4(-20)})=1\qquad([6]).$$

Now, define a mapping $g:\mathfrak{J}_1\rightarrow\mathfrak{J}$ by

$$g\begin{pmatrix} \xi_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_1 & -x_3 & \bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

then g is a linear isomorphism over $\boldsymbol{R}$ and satisfies the properties $\det X = -\det gX$ and $(X, Y)_1 = (gX, gY)_\sigma$. We see therefore that the mapping $\psi' : E_{6(-26)} \to E^1{}_{6(-26)}$ defined by

$$\psi'(\alpha)X = g^{-1}\alpha gX, \qquad X \in \mathfrak{J}_1$$

gives an isomorphism between $E_{6(-26)}$ and $E^1{}_{6(-26)}$ and that the restriction $\psi' | F'_{4(-20)}$ gives one between $F'_{4(-20)}$ and $F_{4(-20)}$.

3. Lie algebra $\mathfrak{e}_{6,\sigma}$ of $E_{6,\sigma}$.

We consider the Lie algebra $\mathfrak{e}_{6,\sigma}$ of $E_{6,\sigma}$:

$$\mathfrak{e}_{6,\sigma} = \{\zeta \in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\zeta X, X, X) = 0,\ \langle \zeta X, Y\rangle_\sigma = -\langle X, \zeta Y\rangle_\sigma\}.$$

Theorem 4. *Any element ζ of the Lie algebra $\mathfrak{e}_{6,\sigma}$ of the group $E_{6,\sigma}$ is uniquely represented by the form*

$$\zeta = \delta + \widetilde{S}, \qquad \delta \in \mathfrak{f}_{4,\sigma},\ S = \begin{pmatrix} 0 & s_3 & \bar{s}_2 \\ \bar{s}_3 & 0 & 0 \\ s_2 & 0 & 0 \end{pmatrix} + i\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & s_1 \\ 0 & \bar{s}_1 & \sigma_3 \end{pmatrix},$$

where $\sum \sigma_i = 0$, $\sigma_i \in \boldsymbol{R}$, $s_i \in \mathfrak{C}$ and $\mathfrak{f}_{4,\sigma} = \{\delta \in \mathfrak{e}_{6,\sigma} \mid (\delta X, Y) = -(X, \delta Y)\} = \{\delta \in \mathfrak{e}_{6,\sigma} \mid \delta E = 0\}$ *is the Lie algebra of the group $F_{4,\sigma}$ and, for S, $\widetilde{S} \in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C)$ is defined by $\widetilde{S}X = S \circ X$. In particular, the type of the Lie group $E_{6,\sigma}$ is E_6.*

Proof. It is easily seen by the analogous argument as in the proof of Theorem 2 of [8].

4. Compact subgroup $(E_{6,\sigma})_K$ of $E_{6,\sigma}$.

We shall consider the following subgroup $(E_{6,\sigma})_K$ of $E_{6,\sigma}$:

$$\begin{aligned}(E_{6,\sigma})_K &= \{\alpha \in E_{6,\sigma} \mid \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\} \\ &= \{\alpha \in E_6 \mid \langle \alpha X, \alpha Y\rangle_\sigma = \langle X, Y\rangle_\sigma\}.\end{aligned}$$

To do this, we need some preparations. Following [8], we first define the subgroups E_σ of E_6 and $E_{\sigma,1}$ of E_σ by

$$E_\sigma = \{\alpha \in E_6 \mid \sigma\alpha\sigma = \alpha\},$$

$$E_{\sigma,1} = \{\alpha \in E_\sigma \mid \alpha E_1 = E_1\}$$

where $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we have already known the following

Lemma 5. (Proposition 11 [8]). *The group $E_{\sigma,1}$ is isomorphic to the spinor group $Spin(10)$.*

From now on, we identify the group $E_{\sigma,1}$ with the group $Spin(10)$.

We next define the subgroup $U(1)$ of $E_{\sigma,1}$ by

$$U(1)=\{\phi(\theta)\,|\,\phi(\theta)X(\xi,\ x)=\begin{pmatrix} \theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2 \\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1 \\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3 \end{pmatrix},\ \theta\in\mathrm{C},\ |\theta|=1\}.$$

It is obvious that the group $U(1)$ is isomorphic to the usual unitary group $U(1)=\{\theta\in C\,|\,|\theta|=1\}$. Furthermore we have known that the subgroups $U(1)$ and $Spin(10)$ of E_6 commute elementwisely (Lemma 12 [8]).

Finally we denote by α^* and $\hat{\alpha}$ the transpose of $\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)$ relative to $\langle X,\ Y\rangle$ and $\langle X,\ Y\rangle_\sigma$ respectively:

$$\langle \alpha X,\ Y\rangle=\langle X,\ \alpha^* Y\rangle, \qquad \langle \alpha X,\ Y\rangle_\sigma=\langle X,\ \hat{\alpha}Y\rangle_\sigma.$$

Then it holds generally

$$\hat{\alpha}=\sigma\alpha^*\sigma, \qquad \alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C),$$

since we have $\langle X,\ \hat{\alpha}Y\rangle=\langle\sigma X,\ \hat{\alpha}Y\rangle_\sigma=\langle\alpha\sigma X,\ Y\rangle_\sigma=\langle\sigma\alpha\sigma X,\ Y\rangle=\langle X,\ \sigma\alpha^*\sigma Y\rangle$, noting that $\sigma=\sigma^*=\hat{\alpha}$.

Proposition 6. *The group $(E_{6,\sigma})_K$ is isomorphic to the group $(U(1)\times Spin(10))/\boldsymbol{Z}_4$ where $\boldsymbol{Z}_4=\{(1,\phi(1)),\ (-1,\ \phi(-1)),\ (\mathrm{i},\ \phi(-i)),\ (-\mathrm{i},\ \phi(i))\}$.*

Proof. First we shall show that $(E_{6,\sigma})_K=E_\sigma$. Let α be an element of $(E_{6,\sigma})_K$, that is, $\alpha\alpha^*=\alpha\hat{\alpha}=1$, then from $\hat{\alpha}=\sigma\alpha^*\sigma$ we have $\sigma\alpha\sigma=\alpha$, that is, $\alpha\in E_\sigma$. Conversely, let α be an element of E_σ, then we have $\alpha\hat{\alpha}=\alpha\sigma\alpha^*\sigma=\sigma\alpha\alpha^*\sigma=\sigma\sigma=1$, that is, $\alpha\in(E_{6,\sigma})_K$. Now, we have already known that a homomorphism $\varphi: U(1)\times Spin(10)\to E_\sigma=(E_{6,\sigma})_K$ defined by $\varphi(\theta,\beta)=\phi(\theta)\beta$ induces an isomorphism $(E_{6,\sigma})_K\cong(U(1)\times Spin(10))/\boldsymbol{Z}_4$ (Theorem 13 [8]). Thus Proposition 6 is proved.

5. Polar decomposition of $E_{6,\sigma}$.

To give a polar decomposition of $E_{6,\sigma}$, we use the following

Lemma 7 ([2] pp. 345). *Let G be a pseudoalgebraic subgroup of the general linear group $GL(n,\ \boldsymbol{C})$ such that the condition $A\in G$ implies $A^*\in G$. Then G is homeomorphic to the topological product of $G\cap U(n)$ (which is a maximal compact subgroup of G) and a Euclidean space $\boldsymbol{R}^d$:*

$$G\simeq(G\cap U(n))\times\boldsymbol{R}^d, \qquad d=\dim G-\dim(G\cap U(n))$$

where $U(n)$ is the unitary subgroup of $GL(n,\ \boldsymbol{C})$.

To use the above Lemma, first of all we show the following

Lemma 8. *$E_{6,\sigma}$ is a pseudoalgebraic subgroup of the general linear group $GL(27,\ \boldsymbol{C})=\mathrm{Iso}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)$ and satisfies the condition $\alpha\in E_{6,\sigma}$ implies $\alpha^*\in E_{6,\sigma}$.*

Proof. Since $\hat{\alpha}=\sigma\alpha^*\sigma, \alpha\hat{\alpha}=1$ for $\alpha\in E_{6,\sigma}$, we have $\alpha^*=\sigma\alpha^{-1}\sigma\in E_{6,\sigma}$. It is obvious that $E_{6,\sigma}$ is pseudoalgebraic, because $E_{6,\sigma}$ is defined by the pseudoalgebraic relations $\det \alpha X = \det X$ and $\langle \alpha X, \alpha Y\rangle_\sigma = \langle X, Y\rangle_\sigma$.

Let $U(\mathfrak{J}^C)$ be the unitary subgroup of $\mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C)$:

$$U(27) = U(\mathfrak{J}^C) = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\}.$$

Then we have

$$E_{6,\sigma} \cap U(\mathfrak{J}^C) = (E_{6,\sigma})_K \cong (U(1) \times Spin(10))/\boldsymbol{Z}_4$$

by Proposition 6. Finally we shall determine the dimension of the Euclidean part of $E_{6,\sigma}$. Since $E_{6,\sigma}$ is a simple Lie group of type E_6 by Theorem 4, the dimension d is obtained by

$$d = \dim E_{6,\sigma} - \dim(U(1) \times Spin(10)) = 78 - 46 = 32.$$

Thus we get the following

Theorem 9. *The group $E_{6,\sigma}$ is homeomorphic to the topological product of the group $(U(1) \times Spin(10))/\boldsymbol{Z}_4$ and a 32-dim. Eudlidean space $\boldsymbol{R}^{32}$:*

$$E_{6,\sigma} \simeq (U(1) \times Spin(10))/\boldsymbol{Z}_4 \times \boldsymbol{R}^{32}.$$

In particular, $E_{6,\sigma}$ is a connected (but not simply connected) Lie group.

6. Center $z(E_{6,\sigma})$ of $E_{6,\sigma}$.

Lemma 10. *For $a \in \mathfrak{C}$, $a \neq 0$, the mapping $\alpha(a) : \mathfrak{J}^C \to \mathfrak{J}^C$ defined by $\alpha(a)X(\xi, x) = Y(\eta, y)$ belongs to $E_{6,\sigma}$, where*

$$\begin{cases} \eta_1 = \dfrac{\xi_1 - \xi_3}{2} + \dfrac{\xi_1 + \xi_3}{2}\cosh|a| + \dfrac{(a, x_2)}{|a|}\sinh|a|, \\ \eta_2 = \xi_2, \\ \eta_3 = -\dfrac{\xi_1 - \xi_3}{2} + \dfrac{\xi_1 + \xi_3}{2}\cosh|a| + \dfrac{(a, x_2)}{|a|}\sinh|a|, \end{cases}$$

$$\begin{cases} y_1 = x_1 \cosh\dfrac{|a|}{2} + \dfrac{\overline{a x_3}}{|a|}\sinh\dfrac{|a|}{2}, \\ y_2 = x_2 + \dfrac{2(a, x_2)a}{|a|^2}\sinh^2\dfrac{|a|}{2} + \dfrac{(\xi_1 + \xi_2)a}{2|a|}\sinh|a|, \\ y_3 = x_3 \cosh\dfrac{|a|}{2} + \dfrac{\overline{x_1 a}}{|a|}\sinh\dfrac{|a|}{2}. \end{cases}$$

Proof. Since, for $F_2(a) = \begin{pmatrix} 0 & 0 & \overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}$, $\widetilde{F}_2(a)$ is an element of $\mathfrak{e}_{6,\sigma}$ by Theorem 4, it follows $\alpha(a) = \exp \widetilde{F}_2(a) \in E_{6,\sigma}$.

Theorem 11. *The center $z(E_{6,\sigma})$ of the group $E_{6,\sigma}$ is isomorphic to the cyclic group $\boldsymbol{Z}_3$ of order* 3:

$$z(E_{6,\sigma})=\boldsymbol{Z}_3=\{1,\ \omega 1,\ \omega^2 1\},\qquad \omega\in\boldsymbol{C},\ \omega^3=1,\ \omega\neq 1.$$

proof. Let $\alpha\in z(E_{6,\sigma})$. From the commutativity with $\sigma\in E_{6,\sigma}$, we have $\sigma\alpha=\alpha\sigma$, that is, $\alpha\in(E_{6,\sigma})_K$. Hence there exists an element $(\theta,\ \beta)\in U(1)\times Spin(10)$ such that $\alpha=\varphi(\theta,\beta)=\phi(\theta)\beta$ by Proposition 6. Moreover we see that β is an element of the center $z(Spin(10))$, noting that the groups $U(1)$ and $Spin(10)$ commute elementwisely. In fact, it holds $\phi(\theta)\beta\beta'=\beta'\phi(\theta)\beta=\phi(\theta)\beta'\beta$, hence $\beta\beta'=\beta'\beta$ for all $\beta'\in Spin(10)$. Now, as is well known, the order of $z(Spin(10))$ is 4 and obviously $\phi(\varepsilon)\in z(Spin(10))$ for $\varepsilon=\pm 1,\ \pm i$, therefore we have

$$z(Spin(10))=\{\phi(1),\ \phi(-1),\ \phi(\mathrm{i}),\ \phi(-\mathrm{i})\}\subset U(1).$$

Hence $\alpha=\phi(\theta')\in U(1)$ for some $\theta'\in\boldsymbol{C},\ |\theta'|=1$. Next, from the commutativity with $\alpha(a)\in E_{6,\sigma}$ as in Lemma 10, we have $\alpha\alpha(a)E=\alpha(a)\alpha E$, that is,

$$\begin{pmatrix}\lambda\cosh|a| & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\cosh|a|\end{pmatrix}$$

$$=\begin{pmatrix}\dfrac{\lambda-\mu}{2}+\dfrac{\lambda+\mu}{2}\cosh|a| & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & -\dfrac{\lambda-\mu}{2}+\dfrac{\lambda+\mu}{2}\cosh|a|\end{pmatrix}$$

where we denote θ' by λ and θ'^{-2} by μ. Hence we have $\lambda=\mu(=\omega)$, that is,

$$\alpha E=\omega E$$

where $\omega\in\boldsymbol{C}$ and $\omega^3=\det\alpha E=\det E=1$. Since $\omega 1\in z(E_{6,\sigma})$, we have $\omega^{-1}\alpha\in z(E_{6,\sigma})$ and $\omega^{-1}\alpha E=E$, hence $\omega^{-1}\alpha\in z(F_{4,\sigma})$. Therefore it follows that $\omega^{-1}\alpha=1$, that is, $\alpha=\omega 1$, since $z(F_{4,\sigma})=1$ by proposition 3. Thus the proof of Theorem 11 is completed.

II. Non-compact simple Lie group $E_{6,\tau}$ of type E_6

7. Split Jordan algebra $\mathfrak{J}_2$.

Let $\mathfrak{C}'$ be the split Cayley algebra over $\boldsymbol{R}$. In $\mathfrak{C}'=\boldsymbol{H}\oplus\boldsymbol{H}e'$, the multiplication xy, the conjugate $\bar{x}$, the scalar part $\mathrm{t}(x)$ and the inner product $(x,\ y)'$ are defined respectively by

$$(a+be')(c+de')=(ac+\bar{d}b)+(b\bar{c}+da)e',$$

$$\overline{a+be'}=\bar{a}-be',\qquad \mathrm{t}(x)=x+\bar{x},$$

$$(a+be',\ c+de')'=(a,\ c)-(b,\ d).$$

Let $\mathfrak{C}'^C$ be the complexification algebra of $\mathfrak{C}'$. In $\mathfrak{C}'^C$, the conjugate $\overline{x}$, the scalar part $\mathrm{t}(x)$ and the inner product $(x, y)'$ are also defined naturally. The mapping $k : \mathfrak{C}'^C \to \mathfrak{C}'^C$ defined by

$$k((a+be')+i(c+de'))=(a+de)+i(c-be)$$

gives an isomorphism as algebra over C and satisfies

$$k(\overline{x})=\overline{k(x)}, \qquad (x, y)'=(k(x), k(y)).$$

Let $\mathfrak{J}_2=\mathfrak{J}(3,\mathfrak{C}')$ be the Jordan algebra consisting of all 3×3 Hermitian matrices with entries in $\mathfrak{C}'$

$$X=X(\xi, x)=\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i\in \boldsymbol{R}, x_i\in\mathfrak{C}'$$

with respect to the multiplication $X\circ Y=\dfrac{1}{2}(XY+YX)$. In $\mathfrak{J}_2$ also, the inner product (X, Y), the crossed product $X\times Y$, the cubic form (X, Y, Z) and the determinant $\det X$ are defined by the quite same formulae in $\mathfrak{J}$.

Furthermore the complexification $\mathfrak{J}_2{}^C$ of $\mathfrak{J}_2$ and the several operations in $\mathfrak{J}_2{}^C$ are also similar to the definitions in the section 1.

From now on, we will use the same notations for the same operations in $\mathfrak{J}$ and $\mathfrak{J}_2$, but as occasion demands the notations in $\mathfrak{J}_2$ will be indexed by the figure 2.

Proposition 12. *$\mathfrak{J}_2{}^C$ is isomorphic to $\mathfrak{J}^C$ as Jordan algebra over C by an isomorphism $h : \mathfrak{J}_2{}^C\to\mathfrak{J}^C$ defined as follows:*

$$hX(\xi, x)=X(\xi, k(x)).$$

And h satisfies the following properties.

(i) $(X, Y)_2=(hX, hY)$,

(ii) $\det X=\det hX$,

(iii) $\langle X, Y\rangle_2=\langle hX, hY\rangle_\gamma$

where $\gamma : \mathfrak{J}^C\to\mathfrak{J}^C$ is the linear involution defined by

$$\gamma X(\xi, a+be)=X(\xi, a-be)$$

where $\xi\in C$, a, $b\in \boldsymbol{H}^C$ and the inner product $\langle X, Y\rangle_\gamma$ in $\mathfrak{J}^C$ is defined by

$$\langle X, Y\rangle_\gamma=\langle\gamma X, Y\rangle.$$

Proof. It is easy to see that h is a linear isomorphism over C and satisfies $h(X\circ Y)=hX\circ hY$. The properties (i), (ii) and (iii) are shown similarly in the proof of Proposition 1.

8. Groups of type E_6 and F_4.

The group $E_{6,\gamma}$ is defined to be the group of linear isomorphisms of $\mathfrak{J}^C$ leaving the determinant $\det X$ and the Hermitian inner product $\langle X, Y\rangle_\gamma$ invariant:

$$E_{6,\gamma}=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)\,|\,\det\alpha X=\det X,\ \langle\alpha X, \alpha Y\rangle_\gamma=\langle X, Y\rangle_\gamma\}$$
$$=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)\,|\,(\alpha X, \alpha Y, \alpha Z)=(X, Y, Z),\ \langle\alpha X, \alpha Y\rangle_\gamma=\langle X, Y\rangle_\gamma\}$$

and $F_{4,\gamma}$ the subgroup of $E_{6,\gamma}$ preserving the inner product (X, Y):

$$F_{4,\gamma}=\{\alpha\in E_{6,\gamma}\,|\,(\alpha X, \alpha Y)=(X, Y)\}$$
$$=\{\alpha\in E_{6,\gamma}\,|\,\alpha E=E\}.$$

Next, to consider the group $E_{6,\gamma}$ we need to define the group $E_{6,2}$ and the subgroup $F_{4,2}$ of $E_{6,2}$:

$$E_{6,2}=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}_2{}^C,\mathfrak{J}_2{}^C)\,|\,\det\alpha X=\det X,\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}$$
$$=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}_2{}^C,\mathfrak{J}_2{}^C)\,|\,(\alpha X, \alpha Y, \alpha Z)=(X, Y, Z),\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\},$$
$$F_{4,2}=\{\alpha\in E_{6,2}\,|\,(\alpha X, \alpha Y)=(X, Y)\}$$
$$=\{\alpha\in E_{6,2}\,|\,\alpha E=E\}.$$

Lemma 13. *The group $F_{4,2}$ is homeomorphic to $(Sp(1)\times Sp(3))/Z_2\times R^{28}$ and a simple (in the sense of the center $z(F_{4,2})=1$) Lie group of type F_4.*

Proof. We define the group $F'_{4(4)}$ by

$$F'_{4(4)}=\{\alpha\in \mathrm{Iso}_R(\mathfrak{J}_2,\ \mathfrak{J}_2)\,|\,\alpha(X\circ Y)=\alpha X\circ\alpha Y\}$$
$$=\{\alpha\in E'_{6(6)}\,|\,(\alpha X, \alpha Y)=(X, Y)\}$$
$$=\{\alpha\in E'_{6(6)}\,|\,\alpha E=E\}$$

where $E'_{6(6)}=\{\alpha\in \mathrm{Iso}_R(\mathfrak{J}_2,\mathfrak{J}_2)\,|\,\det\alpha X=\det X\}$. Then the argument used in the proof of Proposition 1 of [8] shows that $F'_{4(4)}$ is isomorphic to $F_{4,2}$ by the complexification $\alpha\to\alpha^C$. Recall now that $F'_{4(4)}$ is homeomorphic to $(Sp(1)\times Sp(3))/Z_2\times R^{28}$ and a simple (in the sense of the center $z(F'_{4(4)})=1$) Lie group of type F_4 [7], then the results follow.

Proposition 14. *The group $E_{6,\gamma}$ is isomorphic to the group $E_{6,2}$ and also $F_{4,\gamma}$ to $F_{4,2}$. In particular, $F_{4,\gamma}$ is homeomorphic to $(Sp(1)\times Sp(3))/Z_2\times R^{28}$ and a simple (in the sense of the center $z(F_{4,\gamma})=1$) Lie group of type F_4.*

Proof. By using the isomorphism $h:\mathfrak{J}_2{}^C\to\mathfrak{J}^C$ in Proposition 12, we define a mapping $\phi: E_{6,\gamma}\to E_{6,2}$ by

$$\phi(\alpha)X=h^{-1}\alpha hX, \qquad X\in\mathfrak{J}_2{}^C.$$

Then from proposition 12 it is easily obtained that ϕ gives an isomorphim between

$E_{6,\gamma}$ and $E_{6,2}$. Furthermore we can readily show that the restriction $\psi|F_{4,\gamma}$ gives an isomorphism between $F_{4,\gamma}$ and $F_{4,2}$.

9. Lie algebra $\mathfrak{e}_{6,\gamma}$ of $E_{6,\gamma}$.

We consider the Lie algebra $\mathfrak{e}_{6,\gamma}$ of $E_{6,\gamma}$:

$$\mathfrak{e}_{6,\gamma}=\{\zeta\in \mathrm{Hom}_C(\mathfrak{J}^C,\mathfrak{J}^C)\,|\,(\zeta X,\ X,\ X)=0,\ \langle\zeta X,\ Y\rangle_\gamma=-\langle X,\ \zeta Y\rangle_\gamma\}.$$

Theorem 15. *Any element ζ of the Lie algebra $\mathfrak{e}_{6,\gamma}$ of theg roup $E_{6,\gamma}$ is uniquely represented by the form*

$$\zeta=\delta+\tilde{S},\qquad \delta\in\mathfrak{f}_{4,\gamma},\ S=\begin{pmatrix} 0 & s_3e & -s_2e \\ -s_3e & 0 & s_1e \\ s_2e & -s_1e & 0 \end{pmatrix}+i\begin{pmatrix} \tau_1 & t_3 & \overline{t}_2 \\ \overline{t}_3 & \tau_2 & t_1 \\ t_2 & \overline{t}_1 & \tau_3 \end{pmatrix},$$

where $\sum\tau_i=0$, $\tau_i\in \boldsymbol{R}$, s_i, $t_i\in \boldsymbol{H}$ and $\mathfrak{f}_{4,\gamma}=\{\delta\in\mathfrak{e}_{6,\gamma}\,|\,(\delta X,\ Y)=-(X,\ \delta Y)\}=\{\delta\in\mathfrak{e}_{6,\gamma}\,|\,\delta E=0\}$ *is the Lie algebra of the group $F_{4,\gamma}$ and, for $S,\tilde{S}\in\mathrm{Hom}_C(\mathfrak{J}^C,\ \mathfrak{J}^C)$ is defined by $\tilde{S}X=S\circ X$. In particular, the type of the Lie group $E_{6,\gamma}$* is E_6.

Proof. It is easily seen by the analogous argument as in the proof of Theorem 2 of [8].

10. Compact subgroup $(E_{6,\gamma})_K$ of $E_{6,\gamma}$.

We shall consider the following subgroup $(E_{6,\gamma})_K$ of $E_{6,\gamma}$:

$$\begin{aligned}(E_{6,\gamma})_K&=\{\alpha\in E_{6,\gamma}\,|\,\langle\alpha X,\ \alpha Y\rangle=\langle X,\ Y\rangle\}\\ &=\{\alpha\in E_6\,|\,\langle\alpha X,\ \alpha Y\rangle_\gamma=\langle X,\ Y\rangle_\gamma\}.\end{aligned}$$

To do this, we need some preparations. Following [8], we first define the subgroup E_γ of E_6 by

$$E_\gamma=\{\alpha\in E_6\,|\,\gamma\alpha\gamma=\alpha\}.$$

Next we denote by ${}'\alpha$ the transpose of $\alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C)$ relative to $\langle X,\ Y\rangle_\gamma$: $\langle\alpha X,\ Y\rangle_\gamma=\langle X,\ {}'\alpha Y\rangle_\gamma$. Then it holds similarly in the section 4,

$${}'\alpha=\gamma\alpha^*\gamma,\qquad \alpha\in\mathrm{Iso}_C(\mathfrak{J}^C,\mathfrak{J}^C),$$

noting that $\gamma=\gamma^*={}'\gamma$.

Proposition 16. *The group $(E_{6,\gamma})_K$ is isomorphic to the group* $(Sp(1)\times SU(6))/\boldsymbol{Z}_2$ *where* $\boldsymbol{Z}_2=\{(1,\ E),\ (-1,\ -E)\}$.

Proof. By the proof similar to that of Proposition 6, it follows that $(E_{6,\gamma})_K=E_\gamma$. On the other hand, we have already known that E_γ is isomorphic to the group $(Sp(1)\times SU(6))/\boldsymbol{Z}_2$ (Theorem 16 [8]). Thus Proposition 16 is proved.

11. Polar decomposition of $E_{6,\gamma}$.

To use Lemma 7, first of all we show the following

Lemma 17. *$E_{6,\gamma}$ is a pseudoalgebraic subgroup of the general linear group $GL(n, C)=\mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C)$ and satisfies the condition $\alpha\in E_{6,\gamma}$ implies $\alpha^*\in E_{6,\gamma}$.*

Proof. Since $'\alpha=\gamma\alpha^*\gamma, \alpha'\alpha=1$ for $\alpha\in E_{6,\gamma}$. we have $\alpha^*=\gamma\alpha^{-1}\gamma\in E_{6,\gamma}$. It is obvious that $E_{6,\gamma}$ is pseudoalgebraic, because $E_{6,\gamma}$ is defined by the pseudoalgebraic relations $\det\alpha X=\det X$ and $\langle\alpha X, \alpha Y\rangle_\gamma=\langle X, Y\rangle_\gamma$.

Next, let $U(\mathfrak{J}^C)$ be the unitary subgroup of $\mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C)$ as in the section 5, then we have

$$E_{6,\gamma}\cap U(\mathfrak{J}^C)=(E_{6,\gamma})_K\cong(Sp(1)\times SU(6))/\boldsymbol{Z}_2$$

by Proposition 16. Finally we shall determine the dimension of the Euclidean part of $E_{6,\gamma}$. Since $E_{6,\gamma}$ is a simple Lie group of type E_6 by Theorem 15, the dimension d is obtained by

$$d=\dim E_{6,\gamma}-\dim(Sp(1)\times SU(6))=78-38=40.$$

Thus we get the following

Theorem 18. *The group $E_{6,\gamma}$ is homeomorphic to the topological product of the group $(Sp(1)\times SU(6))/\boldsymbol{Z}_2$ and a 40-dim. Euclidean space $\boldsymbol{R}^{40}$:*

$$E_{6,\gamma}\simeq(Sp(1)\times SU(6))/\boldsymbol{Z}_2\times\boldsymbol{R}^{40}.$$

In particular, $E_{6,\gamma}$ is a connected (but not simply connected) Lie group.

12. Center $z(E_{6,\gamma})$ of $E_{6,\gamma}$.

Theorem 19. *The center $z(E_{6,\gamma})$ of the group $E_{6,\gamma}$ is isomorphic to the cyclic group $\boldsymbol{Z}_3$ of order 3:*

$$z(E_{6,\gamma})=\boldsymbol{Z}_3=\{1, \omega 1, \omega^2 1\}, \qquad \omega\in\boldsymbol{C}, \ \omega^3=1, \ \omega\neq 1.$$

Proof. We define the linear transformations $\beta_i, i=1,2,3$ of $\mathfrak{J}^C$ by

$$\beta_1 X=\begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \quad \beta_2 X=\begin{pmatrix} \xi_1 & -x_3 & \overline{x}_2 \\ -\overline{x}_3 & \xi_2 & -x_1 \\ x_2 & -\overline{x}_1 & \xi_3 \end{pmatrix}, \quad \beta_3 X=\begin{pmatrix} \xi_2 & x_1 & \overline{x}_3 \\ \overline{x}_1 & \xi_3 & x_2 \\ x_3 & \overline{x}_2 & \xi_1 \end{pmatrix}$$

for $X=X(\xi, x)\in\mathfrak{J}^C$. Then as readily seen they are elements of $E_{6,\gamma}$. Now, let $\alpha\in z(E_{6,\gamma})$. From the commutativity with the above $\beta_i, i=1, 2, 3$, that is, $\beta_i\alpha E=\alpha\beta_i E=\alpha E$, we have

$$\alpha E=\omega E, \qquad \omega\in\boldsymbol{C}, \omega^3=1.$$

Thus, since $z(F_{4,\gamma})=1$ by Proposition 14, the result follows similarly in the proof of Theorem 11.

Since the fundamental group of $E_{6,\gamma}$ is $\boldsymbol{Z}_2$ from Theorem 18 and the center $z(E_{6,\gamma})$ of $E_{6,\gamma}$ is $\boldsymbol{Z}_3$, we have the following

Theorem 20. *The center* $z(\widetilde{E}_{6,\gamma})$ *of the simply connected non-compact Lie group* $\widetilde{E}_{6,\gamma}=E_{6(2)}$ *is isomorphic to the cyclic group* $\boldsymbol{Z}_6$ *of order* 6.

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