J. FAC. SCI., SHINSHU UNIVERSITY Vol. 13, No. 2 Dec., 1978

Homotopy groups of Symmetric spaces Γ_n

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(Recieved December 25, 1978)

§ 1. Introduction

Γ_n denotes the symmetric space SO_{2n}/U_n. The homotopy groups $\pi_q(\Gamma_n)$ are called stable if $q<2n-1$. The stable homotopy groups of Γ_n have been determined by Bott [4], he showed that in the stable range, $\pi_q(\Gamma_n) \cong \pi_{q+1}(SO)$, i. e.,

$$\begin{aligned} \pi_q(\Gamma_n) &\cong Z && \text{for } q\equiv 2 \text{ mod. } 4, \\ \pi_q(\Gamma_n) &\cong Z_2 && \text{for } q\equiv 0, -1 \text{ mod. } 8, \\ \pi_q(\Gamma_n) &\cong 0 && \text{for all other values of } q. \end{aligned}$$

The first few unstable homotopy groups $\pi_{2n+r}(\Gamma_n)$ of the symmetric space Γ_n for $-1<r<1$ are calculated in [7] and [13].

In this paper, we continue the calculations of unstable homotopy groups of this space and determine further steps of unstable homotopy groups of Γ_n using of fibrations $\Gamma_{n+1} \longrightarrow S^{2n}$ with fiber Γ_n and of fibrations $SO_{2n} \longrightarrow \Gamma_n$ with fiber U_n. We rely heavily on Kervaire's calculations [9] and Matsunaga's calculations [11], [12].

The results are summarized in the following table :

Table of $\pi_{2n+r}(\Gamma_n)$ for $k>1$.

$r \backslash n$	$4k$	$4k+1$	$4k+2$	$4k+3$
-1	$Z+Z_2$	$Z_{(n-1)!}$	Z	$Z_{(n-1)!/2}$
0	Z_2+Z_2	0	Z_2	0
1	$Z_{n!}+Z_2$	Z	$Z_{n!}$ or $Z_{n!/2}+Z_2$	$Z+Z_2$
2	$G+Z_3$ ($k\equiv 1$ mod 3) G ($k\not\equiv 1$ mod 3)	$Z_{(24,n-1)/2}$	Z_6 ($k\equiv 0$ mod 3) Z_2 ($k\not\equiv 0$ mod 3)	$Z_{(24,n-1)}$
3	Z	$Z_{(n+1)!(24,n-1)/48}$	$Z+Z_2$ ($d=2$) Z ($d=1$)	$Z_{(n+1)!(24,n-1)/24}$
4	Z	Z_6 ($k\equiv 2$ mod 3) Z_2 ($k\not\equiv 2$ mod 3)	$Z_{(24,n)}$	$G+Z_3$ ($k\equiv 0$ mod 3) G ($k\not\equiv 0$ mod 3)
5			$Z_{(n+2)!(24,n)/24}$	

where $G\cong Z_4$ or Z_8.

(a, b) is the greatest common divisor of integers a and b. d $(=1$ or $2)$ is a integer such that $\pi_{8m+6}(SO_{8m+4}) \cong Z_4 + Z_{24d}$ (See [9]). In § 3, we shall discuss on the cohomology of Γ_n. The homotopy groups of Γ_n will be calculated in §§4–11. For lower values of n, we shall be calculated in § 12.

§ 2. Notions

The rotataion group SO_m is imbedded in SO_{m+r} as the upper left hand block and the unitary group U_n is imbedded in U_{n+r} as the upper left hand block. Let $i_m : SO_m \longrightarrow SO_{m+1}$ and $i'_n : U_n \longrightarrow U_{n+1}$ be inclusion maps and let $p_m : SO_{m+1} \longrightarrow SO_{m+1}/SO_m = S^m$ and $p'_n : U_{n+1} \longrightarrow U_{n+1}/U_n = S^{2n+1}$ be standard natural projections.

The unitary group U_n is imbedded in the rotation group SO_{2n} as the subset of matrices consisting of 2×2 blocks

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and let $j_n : U_n \longrightarrow SO_{2n}$ be the imbedding.

Let Γ_n be the homogeneous space SO_{2n}/U_n and $r_n : SO_{2n} \longrightarrow \Gamma_n$ be its quotient map. The natural map $SO_{2n+1}/U_n \longrightarrow SO_{2n+2}/U_{n+1} = \Gamma_{n+1}$ is bijection and the two manifolds have the same dimension. Thus we shall be used to identify these spaces and the fibration

$$SO_{2n}/U_n \longrightarrow SO_{2n+1}/U_n \longrightarrow S^{2n}$$

may be written as

$$\Gamma_n \xrightarrow{k_n} \Gamma_{n+1} \xrightarrow{q_n} S^{2n}. \tag{2.1}$$

We need the homotopy groups of SO_n ([9]) and U_n ([11]) in the subsequent calculations, so we give them in the following tables :

Table 1. The groups $\pi_{2n+i}(U_n)$

$i=$	0	1	2	3	4
n even	$Z_{n!}$	Z_2	$Z_{(n+1)!}+Z_2$	$Z_{(24,n)}$	$Z_{(n+2)!(24,n)/48}$
n odd	$Z_{n!}$	0	$Z_{(n+1)!/2}$	$Z_{(24,n+3)/2}$	$Z_{(n+2)!(24,n+3)/24}$

Table 2. The groups $\pi_{n+r}(SO_n)$ for $n>8$.

$r \backslash n$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$	$8s+7$
-1	$Z+Z_2$	Z_2+Z_2	$Z+Z_2$	Z_2	$Z+Z$	Z_2	Z	Z_2
0	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_4	Z	Z_2+Z_2	Z_2	Z_4	Z
1	$Z_2+Z_2+Z_2$	Z_8	Z	Z_2	Z_2+Z_2	Z_8	Z	Z_2+Z_2
2	$Z_{24}+Z_8$	$Z+Z_2$	Z_{12}	Z_2+Z_2	Z_4+Z_{24d}	$Z+Z_2$	$Z_{12}+Z_2$	Z_2+Z_2
3	$Z+Z_2$	0	Z_2	Z_{8d}	$Z+Z_2$	Z_2	Z_2	Z_8
4	0	Z_2	Z_{8d}	$Z+Z_2$	Z_2	Z_{12}	Z_8	$Z+Z_2$

In this table, d is anbiquously 1 or 2.

§ 3. Cohomology groups of $\Gamma_n=SO(n)/U(n)$

From [1, p. 203], Γ_n is trosion free ; additively, its integral cohomology groups are isomorphic to those of the following product of even dimensional spheres:

$$S^2\times S^4\times\cdots\cdots\times S^{2n-2}.$$

The cohomology ring $H^*(\Gamma_n; Z_2)$ has a simple system of generators, in the sense of Borel [1, p. 141], $\alpha_1,\cdots,\alpha_{n-1}$, with α_i of degree $2i$. In the fibre bundle $(B_{U(n)},\ p,\ B_{SO(2n)},\ \Gamma_n)$ with group $SO(2n)$, the generators $\alpha_1,\cdots,\alpha_{n-1}$ listed above are transgressive ; the transgression of the generator α_i is the Stifel-Whitney class W_{2i+1} modulo the ideal generated by $W_3,\cdots,W_{2i-1}$.

Since the transgression commutes with the Steenrod operation, using the formulas of W. T. Wu for the Steenrod operations of the Stiefel-Whitney classes, we have

$$Sq^2\alpha_i=\begin{cases}0 & \text{for } i \text{ even}\\ \alpha_{i+1} & \text{for } i \text{ odd}\end{cases} \tag{3.1}$$

modulo the ideal generated by $\alpha_1,\cdots,\alpha_i$ and

$$Sq^4\alpha_i=\begin{cases}\alpha_{i+2} & \text{for } i\equiv 2,\ 3 \bmod. 4\\ 0 & \text{for } i\equiv 0,\ 1 \bmod. 4\end{cases} \tag{3.2}$$

modulo the ideal generated by $\alpha_1,\cdots,\alpha_{i+1}$.

Consider the natural inclusion map $k_{n,t}:\Gamma_n\longrightarrow\Gamma_{n+t}$. By the standard construction, we may consider that $k_{n,t}$ is a fibre map. Let $\Gamma_{n,t}$ be its fibre.

By Bott [4], $k_{n,t}:\pi_i(\Gamma_n)\longrightarrow\pi_i(\Gamma_{n+t})$ is an isomorphism for $i<2n-1$ and an epimorphism for $i=2n-1$. Thus $\Gamma_{n,t}$ is $(2n-2)$-connected and $k^*_{n,t}: H^i(\Gamma_{n+t};Z_2)\longrightarrow H^i(\Gamma_n\ ;Z_2)$ is an isomorphism for $i<2n-1$ and a monomorphism for $i=2n-1$. But $H^{2n-1}(\Gamma_{n+t}\ ;Z_2)=H^{2n-1}(\Gamma_n\ ;Z_2)=0$. Thus $k^*_{n,t}: H^i(\Gamma_{n+t};Z_2)\longrightarrow H^i(\Gamma_n\ ;Z_2)$ is an isomorphism for $i \leqslant 2n-1$.

From the commutative diagram of fibre spaces

$$\begin{array}{ccccc} \Gamma_n & \longrightarrow & B_{U(n)} & \longrightarrow & B_{SO(2n)} \\ \downarrow k_{n,t} & & \downarrow & & \downarrow \\ \Gamma_{n+t} & \longrightarrow & B_{U(n+t)} & \longrightarrow & B_{SO(2n+2t)} \end{array}$$

we have

$$k^*_{n,t}(\alpha_i)=\alpha_i \qquad \text{for } i=1,\dots,n-1,$$
$$k^*_{n,t}(\alpha_{n+j})=0 \qquad \textit{for } j=0,\dots,t-1,$$

where $k^*_{n,t} : \mathrm{H}^*(\Gamma_{n+t} ; Z_2) \longrightarrow \mathrm{H}^*(\Gamma_n ; Z_2)$.

Let τ be the transgession in the spectral sequence of fibre space $k_{n,t} : \Gamma_n \longrightarrow \Gamma_{n+t}$. Then there exist elements $\beta_{n+j} \in \mathrm{H}^{2n+2j-1}(\Gamma_{n,t} ; Z_2)$ such that $\tau(\beta_{n+j})=\alpha_{n+j}$ for $j=0,\dots,t-1$.

From the dimensional argument,

$$\mathrm{H}^*(\Gamma_{n,t} ; Z_2) = \Lambda(\beta_n, \ \beta_{n+1},\dots,\beta_{n+t-1}) \tag{3.3}$$

as additive groups for degree $<4n-2$.

Since β_{n+i} is transgressive and the transgression commutes with the Steenrod operations, it follows from (3.1) and (3.2) that

$$Sq^2\beta_{n+i}=\begin{cases} 0 & \text{for } n+j \text{ even} \\ \beta_{n+j+1} & \text{for } n+j \text{ odd} \end{cases}$$

and

$$Sq^4\beta_{n+j}=\begin{cases} \beta_{n+j+2} & \text{for } n+j\equiv 2, \ 3 \text{ mod. } 4 \\ 0 & \text{for } n+j\equiv 0, \ 1 \text{ mod. } 4 \end{cases}$$

We denote by η_n and ν_n generators of $\pi_{2n+1}(S^n) \cong Z_2$ and 2-primary components Z_8 of $\pi_{2n+3}(S^n)$ respectively.

Consider a following cell complex $K_{n,3}$ for each case ;

(i) $n\equiv 0$ mod. 4,

$K_{n,3}=(S^{2n-1}\vee S^{2n+1})\cup_g e^{2n+3}$, $\qquad g=a\nu_{2n-1}\vee\eta_{2n+1} : S^{2n+2}\to S^{2n-1}\vee S^{2n+1}$

(ii) $n\equiv 1$ mod. 4,

$K_{n,3}=S^{2n-1}\cup_g C(S^{2n}\vee S^{2n+2})$, $\qquad g=\eta_{2n-1}\vee a\nu_{2n-1} : S^{2n}\vee S^{2n+2}\to S^{2n-1}$

(iii) $n\equiv 2$ mod. 4,

$K_{n,3}=(S^{2n-1}\vee S^{2n+1})\cup_g e^{2n+2}$, $\qquad g=\nu_{2n-1}\vee\eta_{2n+1} : S^{2n+2}\to S^{2n-1}\vee S^{2n+1}$

(iv) $n\equiv 3$ mod. 4,

$$K_{n,3}=S^{2n-1}\cup_g C(S^{2n}\vee S^{2n+2}), \qquad g=\eta_{2n-1}\vee\nu_{2n-1}: S^{2n}\vee S^{2n+2}\to S^{2n-1}$$

where $a\equiv 0$ mod. 2.

We denote by ${}^pG$ the direct sum of the free part and p-primary components of a group G. Then, from Lemma 2.5 of [8] and Proposition 8.1 of [16], we have the following Proposition.

Proposition 3.1 *The group ${}^2\pi_i(\Gamma_{n,3})$ is isomorphic to ${}^2\pi_i(K_{n,3})$ for $i<4n-2$ and $n>2$.*

Using the exact sequence of Lemma 3.1 of [8], we obtain

Proposition 3.2. *The group ${}^2\pi_{2n+i}(\Gamma_{n,3})$ is isomorphic to the corresponding groups*

i \ n	$4k$	$4k+1$	$4k+2$	$4k+3$
-1	Z	Z	Z	Z
0	Z_2	0	Z_2	0
1	$Z+Z_2$	Z	$Z+Z_2$	Z
2	Z_4 or Z_8		Z_2	0
3	Z		Z	$Z+Z_2$
4	Z_4	Z_8+Z_2	Z_4	Z_8+Z_2
5	Z_2		0	Z_2

§ 4. The groups $\pi_{2n}(\Gamma_{n-1})$ and $\pi_{2n}(\Gamma_{n-2})$ for $n\geq 8$.

For convenience, we will assume always that $n\equiv 0$ mod. 4 in §§4–11.

We consider the exact sequence

$$\pi_{2n}(SO_{2n-3})\xrightarrow{i_{2n-3}}\pi_{2n}(SO_{2n-2})\xrightarrow{p_{2n-2}}\pi_{2n}(S^{2n-3})\longrightarrow\pi_{2n-1}(SO_{2n-3})\longrightarrow\pi_{2n-1}(SO_{2n-2})$$

namely (table 1),

$$0\longrightarrow Z_2\xrightarrow{i_{2n-3}} Z_{12}+Z_2\xrightarrow{p_{2n-2}} Z_{24}\longrightarrow Z+Z_2\longrightarrow Z.$$

From the fact that image of $p_{2n-2}=2Z_{24}$, there exist generators x, y of $\pi_{2n}(SO_{2n-2})$ such that

$$(4.1)\qquad \begin{aligned} p_{2n-2}(x) &= 2\nu_{2n-3},\\ y &= i_{2n-3}(y') \end{aligned}$$

where x, y generate Z_{12}, Z_2 in $\pi_{2n}(SO_{2n-2}) \cong Z_{12}+Z_2$ resp., y' is a generator of $\pi_{2n}(SO_{2n-3}) \cong Z_2$ and ν_{2n-3} is a generator of $\pi_{2n}(S^{2n-3}) \cong Z_{24}$.

By [11], we have the exact sequence

$$0 \longrightarrow \pi_{2n}(U_{n-2}) \longrightarrow \pi_{2n}(U_{n-1}) \xrightarrow{p'_{n-1}} \pi_{2n}(S^{2n-3}) \longrightarrow \pi_{2n-1}(U_{n-2}) \longrightarrow 0$$

namely (see table 1),

$$0 \longrightarrow Z_{n!(24,n-2)/48} \longrightarrow Z_{n!/2} \xrightarrow{p'_{n-1}} Z_{24} \longrightarrow Z_{(24,n-2)} \longrightarrow 0.$$

Then we may shoose the generator θ of $\pi_{2n}(U_{n-1})$ such that

$$p'_{n-1}(\theta) = (24,\ n-2)\nu_{2n-3}. \tag{4.2}$$

Now consider the commutative diagram

$$\begin{tikzcd} \pi_{2n}(U_{n-1}) \arrow[rr, "j_{n-1}"] \arrow[dr, "p'_{n-1}"'] & & \pi_{2n}(SO_{2n-2}) \arrow[dl, "p_{2n-2}"] \\ & \pi_{2n}(S^{2n-3}) & \end{tikzcd}$$

Put $j_{n-1}(\theta)=ax+by$ $(0\leqslant a<12,\ 0\leqslant b\leqslant 1)$. From (4.1) and(4.2), we have $2a\equiv(24,n-2)$ mod. 24.

Consider the commutative diagram

$$\begin{tikzcd} \pi_{2n}(U_{n-1}) \arrow[r, "j_{n-1}"] \arrow[dr, "j'_{n-1}"'] & \pi_{2n}(SO_{2n-2}) \arrow[d, "i_{2n-2}"] & \\ & \pi_{2n}(SO_{2n-1}) \arrow[r, "r'_{n-1}"] & \pi_{2n}(SO_{2n-1}/U_{n-1}). \end{tikzcd}$$

Then, from [7], it follows that r'_{n-1} is an isomorphism onto. Thus j'_{n-1} is trivial. Since $i_{2n-2}(x)=0$ and $i_{2n-2}(y)\neq 0$ by [9], we obtain that

$$j_{n-1}(\theta)=((24,\ n-2)/2)x \tag{4.3}$$

where $(24,\ n-2)/2$ is an odd integer.

From the exact sequence

$$\pi_{2n}(U_{n-1}) \xrightarrow{j_{n-1}} \pi_{2n}(SO_{2n-2}) \longrightarrow \pi_{2n}(\Gamma'_{n-1}) \longrightarrow \pi_{2n-1}(U_{n-1})=0$$

and (4.3), we obtain that

$$\begin{aligned}\pi_{2n}(\Gamma'_{n-1}) &\cong \text{Cokernel of } j_{n-1}\\ &\cong Z_{(24,n-2)}.\end{aligned}$$

Consider the homotopy exact sequence associated to the fibration $q_{n-2}:\Gamma'_{n-1} \longrightarrow S^{2n-4}$;

$$\pi_{2n+1}(S^{2n-4}) \longrightarrow \pi_{2n}(\Gamma'_{n-2}) \longrightarrow \pi_{2n}(\Gamma'_{n-1}) \longrightarrow \pi_{2n}(S^{2n-4}).$$

Then, from $\pi_{2n+1}(S^{2n-4})=\pi_{2n}(S^{2n-4})=0$ for $n>8$, it follows that

$$\pi_{2n}(\Gamma'_{n-2}) \cong \pi_{2n}(\Gamma'_{n-1})$$

There fore we have

$$\begin{aligned}\pi_{2n}(\Gamma'_{n-2}) &\cong \pi_{2n}(\Gamma'_{n-1})\\ &\cong Z_{(24,n-2)}\end{aligned}$$

§ 5. The group $\pi_{2n+1}(\Gamma_{n-1})$ for $n\geq 8$.

Writting $\Gamma_n=SO_{2n-1}/U_{n-1}$, $\Gamma_{n-1}=SO_{2n-2}/U_{n-1}$, we have a commutative diagram

(5. 1)
$$\begin{tikzcd}
\pi_{2n+1}(U_{n-1}) \arrow[r] \arrow[d, equal] & \pi_{2n+1}(SO_{2n-2}) \arrow[r, "r_{n-1}"] \arrow[d, "i_{2n-2}"] & \pi_{2n+1}(\Gamma'_{n-1}) \arrow[r, "\Delta_{n-1}"] \arrow[d, "k_{n-1}"] & \pi_{2n}(U_{n-1}) \arrow[r, "j_{n-1}"] \arrow[d, equal] & \pi_{2n}(SO_{2n-2}) \arrow[d] \\
\pi_{2n+1}(U_{n-1}) \arrow[r] & \pi_{2n+1}(SO_{2n-1}) \arrow[r, "r'\pi"] & \pi_{2n+1}(\Gamma_n) \arrow[r, "\Delta'_n"] & \pi_{2n}(U_{n-1}) \arrow[r] & \pi_{2n}(SO_{2n-1}).
\end{tikzcd}$$

Since ${}^2\pi_{2n+1}(U_{n-1})=0$ and ${}^p\pi_{2n+2}(SO_{2n-2})={}^p\pi_{2n+1}(SO_{2n-1})=0$ for odd prime p, we have that r_{n-1} and r'_n are monomorphisms. Then, from (4. 3) and the exactness of the upper sequence of diagram (5. 1), it follows that there exists an element ξ of $\pi_{2n+1}(\Gamma_{n-1})$ such that

(5. 2)
$$\Delta_{n-1}(\xi)=(24/(24,\ n-2))\theta$$

where θ is a generator of $\pi_{2n}(U_{n-1})$.

On the other hand, by [7], we have the exact sequence

$$0 \longrightarrow \pi_{2n+1}(SO_{2n-1}) \longrightarrow \pi_{2n+1}(\Gamma'_n) \xrightarrow{\Delta'_n} \pi_{2n}(U_{n-1}) \longrightarrow 0,$$

namely

$$0 \longrightarrow Z_2+Z_2 \longrightarrow Z_2+Z_{n!} \xrightarrow{\Delta'_n} Z_{n!/2} \longrightarrow 0.$$

Then we may shoose generators λ, ζ of $\pi_{2n+1}(\Gamma'_n) \cong Z_2+Z_{n!}$ such that

(5. 3) $$\Delta'_n(\lambda)=0 \text{ and } \Delta'_n(\zeta)=\theta$$

where λ, ζ generate Z_2, $Z_{n!}$ in $\pi_{2n+1}(\Gamma'_n)$ respectively.

Now put $k_{n-1}(\xi)=a\lambda+b\zeta$, where $a=0$ or 1 and $0 \leqslant b < n!$. Then, from (5. 1) and (5. 3), we have

$$\Delta_{n-1}(\xi)=\Delta'_n k_{n-1}(\xi)=a\Delta'_n(\lambda)+b\Delta'_n(\zeta)=b\theta.$$

From (5. 2), we have

$$24/(24, n-2) \equiv b \text{ mod. } n!/2.$$

Thus we have

(5. 4) $$k_{n-1}(\xi)=a\lambda+(24/(24,\ n-2)+(n!/2)l)\zeta$$

for some integer l.

Since λ, ζ generate Z_2, $Z_{n!}$, we obtain that

$$2k_{n-1}(\xi)=(48/(24,\ n-2))\zeta \neq 0$$

for $n \geqslant 8$. Then we obtain that the order of ξ is $n!(24,\ n-2)/24$, since k_{n-1} is a monomorphism and ζ has order $n!$. On the other hand, from (4. 3) and the upper exact sequence of (5. 1), it follows that the group $\pi_{2n+1}(\Gamma'_{n-1})$ has order $n!(24, n-2)/24$. Therefore we have that

(5. 5) $$\pi_{2n+1}(\Gamma'_{n-1}) \cong Z_{n!(24,n-2)/24}.$$

§6. The groups $\pi_{2n+4}(\Gamma'_n)$ and $\pi_{2n+4}(\Gamma'_{n+1})$ for $n \geqslant 4$.

Consider the homotopy exact sequence of the fibration $SO_{2n}/U_n=\Gamma'_n$:

$$\pi_{2n+4}(SO_{2n}) \longrightarrow \pi_{2n+4}(\Gamma'_n) \longrightarrow \pi_{2n+3}(U_n) \xrightarrow{j_n} \pi_{2n+3}(SO_{2n}) \longrightarrow \pi_{2n+3}(\Gamma'_n)$$

namely

$$0 \longrightarrow \pi_{2n+4}(\Gamma'_n) \longrightarrow Z_{(24,n)} \longrightarrow Z+Z_2 \longrightarrow Z.$$

Then the above sequence shows that

(6. 1) $$\pi_{2n+4}(\Gamma_n) \cong \text{Kernel of } j_n \cong Z_{(24,n)/2}.$$

From the homotopy exact sequence of the fibration $q_n : \Gamma_{n+1} \longrightarrow S^{2n}$ and the fact that $\pi_{2n+4}(S^{2n}) = \pi_{2n+5}(S^{2n}) = 0$, it follows that

(6. 2) $$\pi_{2n+4}(\Gamma_n) \cong \pi_{2n+4}(\Gamma_{n+1}). \cong Z_{(24,n)/2}.$$

§ 7. The groups $\pi_{2n+2}(\Gamma_n)$ and $\pi_{2n+2}(\Gamma_{n-1})$ for $n \geqslant 8$.

Consider the homotopy exact sequence associated with the figration $q_{n-1} : \Gamma_n \longrightarrow S^{2n-2}$;

$$\pi_{2n+3}(S^{2n-2}) \longrightarrow \pi_{2n+2}(\Gamma_{n-1}) \longrightarrow \pi_{2n+2}(\Gamma_n) \longrightarrow \pi_{2n+2}(S^{2n-2}).$$

From the well known results that

(7. 1) $$\pi_{m+4}(S^m) = 0 \text{ for } m > 6 \text{ and } \pi_{m+5}(S^m) = 0 \text{ for } m > 7,$$

it follows that

$$\pi_{2n+2}(\Gamma_{n-1}) \cong \pi_{2n+2}(\Gamma_n)$$

for $n \geqslant 5$.

Consider the exact sequence of the fibration $SO_{2n-2}/U_{n-1} = \Gamma_{n-1}$;

(7. 2) $$\pi_{2n+2}(SO_{2n-2}) \longrightarrow \pi_{2n+2}(\Gamma_{n-1}) \longrightarrow \pi_{2n+1}(U_{n-1}) \longrightarrow \pi_{2n+1}(SO_{2n-2})$$

where $\pi_{2n+2}(SO_{2n-2}) \cong Z_8$ and $\pi_{2n+1}(SO_{2n-1}) \cong Z_2$.

Thus we have that

(7. 3) $$^{p}\pi_{2n+2}(\Gamma_{n-1}) \cong {}^{p}\pi_{2n+1}(U_{n-1}) \cong {}^{p}Z_{(24,n+2)/2}$$

for odd prime p.

Since $^{2}\pi_{2n+1}(U_{n-1}) = 0$, it follows from the exactness of (7. 2) that

(7. 4) $^{2}\pi_{2n+2}(\Gamma_{n-1})$ is a cyclic group of order at most 8.

Consider the homotopy exact sequence associated with the fibration $k_{n,3} : \Gamma_n \longrightarrow \Gamma_{n+3}$ with fibre $\Gamma_{n,3}$ in the sense of § 3;

$$\pi_{2n+3}(\Gamma_{n+3}) \longrightarrow \pi_{2n+2}(\Gamma_{n,3}) \longrightarrow \pi_{2n+2}(\Gamma_n) \longrightarrow \pi_{2n+2}(\Gamma_{n+3})$$

where $\pi_{2n+3}(\Gamma'_{n+3})=0$ and $\pi_{2n+2}(\Gamma'_{n+3}) \cong Z$.

From the exactness and (7. 4) we have

$$\pi_{2n+2}(\Gamma'_n) \cong \pi_{2n+2}(\Gamma'_{n,3}).$$

On the other hand, by Proposition 3. 1,

$$\begin{aligned} {}^2\pi_{2n+2}(\Gamma'_{n,3}) &\cong {}^2\pi_{2n+2}(K_{n,3}) \\ &\cong \begin{cases} Z_8 & \text{if } a=0 \text{ or } 4 \\ Z_4 & \text{if } a=2. \end{cases} \end{aligned}$$

Thus

$${}^2\pi_{2n+2}(\Gamma'_n) \cong Z_8 \text{ or } Z_4$$

Therefore we have

$$\begin{aligned} \pi_{2n+2}(\Gamma'_{n-1}) &\cong \pi_{2n+2}(\Gamma'_n) \\ &\cong \begin{cases} G+Z_3 & \text{for } n+2\equiv 0 \ (3) \\ G & \text{for } n+2\equiv 0 \ (3) \end{cases} \end{aligned}$$

where $G \cong Z_4$ or Z_8

§ 8. The group $\pi_{2n+5}(\Gamma'_{n+1})$ for $n \geqq 4$.

Consider the commutative diagram

(8. 1)
$$\begin{tikzcd} \pi_{2n+5}(\Gamma'_n) \arrow[r, "\Delta_n"] \arrow[d, "k_n"] & \pi_{2n+4}(U_n) \arrow[r] \arrow[d, "i'_n"] & \pi_{2n+4}(SO_{2n})=0 \arrow[d] \\ \pi_{2n+5}(\Gamma'_{n+1}) \arrow[r, "\Delta_{n+1}"] & \pi_{2n+4}(U_{n+1}) \arrow[r, "j_{n+1}"] & \pi_{2n+4}(SO_{2n+2}) \end{tikzcd}$$

with exact rows.

From [11], $\pi_{2n+4}(U_n) \cong Z_{(n+2)!(24,n)/48}$ and let δ_n be its generator. From the upper exact sequence of (8. 1), it follows that there exists an element γ of $\pi_{2n+5}(\Gamma'_n)$ such that $\Delta_n(\gamma)=\delta_n$. Since $i'_n : \pi_{2n+4}(U_n) \longrightarrow \pi_{2n+4}(U_{n+1})$ is a monomorphism, $i'_n\Delta_n(\gamma)$ has order $(n+2)!(24, n)/48$. From the commutativity of (8. 1), $\Delta_{n+1}k_n(\gamma)=i'_n\Delta_n(\gamma)$. Thus the element $k_n(\gamma)$ of $\pi_{2n+5}(\Gamma'_{n+1})$ is of order at least $(n+2)!(24, n)/48$.

Consider the homotopy exact sequence of the fibration $q_{n+1} : \Gamma_{n+2} \longrightarrow S^{2n+2}$;

$$\begin{aligned} 0=\pi_{2n+6}(S^{2n+2}) \longrightarrow \pi_{2n+5}(\Gamma_{n+1}) \longrightarrow \pi_{2n+5}(\Gamma_{n+2}) \longrightarrow \pi_{2n+5}(S^{2n+2}) \\ \longrightarrow \pi_{2n+4}(\Gamma_{n+1}) \xrightarrow{k_{n+1}} \pi_{2n+4}(\Gamma_{n+2}). \end{aligned}$$

Here, from [7], we have that $k_{n+1}: \pi_{2n+4}(\Gamma_{n+1}) \longrightarrow \pi_{2n+4}(\Gamma_{n+2})$ is trivial and the group $\pi_{2n+5}(\Gamma_{n+2})$ is of order $(n+2)!$. Hence, from (6.1) and (6.2), the above exact sequence shows that the group $\pi_{2n+5}(\Gamma_{n+1})$ is of order $(n+2)!(24, n)/48$.

Thus we obtain that the element $k_n(\gamma)$ must be of order $(n+2)!(24, n)/48$ and $\pi_{2n+5}(\Gamma_{n+1})$ is a cyclic group, i. e.,

(8.2) $$\pi_{2n+5}(\Gamma_{n+1}) \cong Z_{(n+2)!(24,n)/48}.$$

Now consider the exact sequence

$$\pi_{2n+5}(U_{n+1}) \xrightarrow{j_{n+1}} \pi_{2n+5}(SO_{2n+2}) \longrightarrow \pi_{2n+5}(\Gamma_{n+1}) \longrightarrow \pi_{2n+4}(U_{n+1})$$
$$\longrightarrow \pi_{2n+4}(SO_{2n+2}) \longrightarrow \pi_{2n+4}(\Gamma_{n+1}) \longrightarrow \pi_{2n+3}(U_{n+1})=0$$

of the fibration r_{n+1} ; $SO_{2n+2} \longrightarrow \Gamma_{n+1}$, where $\pi_{2n+4}(SO_{2n+2}) \cong Z_{12}$ and $\pi_{2n+4}(U_{n+1}) \cong Z_{(n+2)!/2}$. Thus, from (6.2) and (8.2), it follows that the homomorphism

(8.3) $$j_{n+1}: \pi_{2n+5}(U_{n+1}) \longrightarrow \pi_{2n+5}(SO_{2n+2})$$

is an epimorphism.

§ 9. The groups $\pi_{2n+6}(\Gamma_{n+1})$ and $\pi_{2n+6}(\Gamma_{n+2})$

From the homotopy exact sequence

$$\pi_{2n+7}(S^{2n+2}) \longrightarrow \pi_{2n+6}(\Gamma_{n+1}) \longrightarrow \pi_{2n+6}(\Gamma_{n+2}) \longrightarrow \pi_{2n+6}(S^{2n+2})$$

of the fibration $q_{n+1}: \Gamma_{n+2} \longrightarrow S^{2n+2}$ and (7.1), we obtain that

(9.1) $$\pi_{2n+6}(\Gamma_{n+1}) \cong \pi_{2n+6}(\Gamma_{n+2}).$$

Consider the exact sepuence

(9.2) $$\pi_{2n+6}(SO_{2n+2}) \longrightarrow \pi_{2n+6}(\Gamma_{n+1}) \longrightarrow \pi_{2n+5}(U_{n+1}) \xrightarrow{j_{n+1}} \pi_{2n+5}(SO_{2n+2})$$

of the fibration $r_{n+1}: SO_{2n+2} \longrightarrow \Gamma_{n+1}$ where $\pi_{2n+6}(SO_{2n+2}) \cong Z_{8d}$ ($d=1$ or 2) and $\pi_{2n+5}(SO_{2n+2}) \cong Z_2$. Thus we have

(9.3) $$\begin{aligned} {}^p\pi_{2n+6}(\Gamma_{n+1}) &\cong {}^p\pi_{2n+5}(U_{n+1}) \\ &\cong \begin{cases} 0 & n+1 \not\equiv 0 \mod 3 \\ Z_3 & n+1 \equiv 0 \mod 3. \end{cases} \end{aligned}$$

for odd prime p and

(9. 4) $\pi_{2n+6}(\Gamma'_{n+1})$ is finite.

Consider the exact sepuence

(9. 5) $$\pi_{2n+7}(\Gamma_{n+5}) \longrightarrow \pi_{2n+6}(\Gamma_{n+2,3}) \longrightarrow \pi_{2n+6}(\Gamma'_{n+2}) \longrightarrow \pi_{2n+6}(\Gamma'_{n+5})$$

of the fibration $k_{n+3,3} : \Gamma_{n+2} \longrightarrow \Gamma_{n+5}$ in the sense of § 3 where $\pi_{2n+7}(\Gamma_{n+5}) \cong Z_2$ and $\pi_{2n+6}(\Gamma_{n+5}) \cong Z$. From Proposition 3. 2,

$$\begin{aligned} {}^2\pi_{2n+6}(\Gamma_{n+2,3}) &\cong {}^2\pi_{2n+6}(K_{n+2,3}) \\ &\cong Z_2. \end{aligned}$$

Thus from (9. 4) and (9. 5),

(9. 6) ${}^2\pi_{2n+6}(\Gamma'_{n+2})$ is of order at most 2.

Let $n \not\equiv 0$ mod. 8. From (8. 3) and the exact sequence (9. 2), we obtain that the sequence

$$Z_{8d} \longrightarrow \pi_{2n+6}(\Gamma'_{n+1}) \longrightarrow Z_4 \xrightarrow{j_{n+1}} Z_2 \longrightarrow 0$$

is exact. Thus, from (9. 6), we have

(9. 7) ${}^2\pi_{2n+6}(\Gamma'_{n+1}) \cong Z_2$ for $n \not\equiv 0$ mod 8.

Let $n \equiv 0$ mod, 8. Writing $\Gamma'_{n+3} = SO_{2n+5}/U_{n+2}$. Then, from (8. 3) and (9. 2), we have a commutative diagram

(9. 8)
$$\begin{tikzcd}
\pi_{2n+6}(U_{n+1}) \arrow[r, "j_{n+1}"] \arrow[d, "i'_{n+1}"] & \pi_{2n+6}(SO_{2n+2}) \arrow[r] \arrow[d, "i''_{2n+2}"] & \pi_{2n+6}(\Gamma'_{n+1}) \arrow[r] & 0 \\
\pi_{2n+6}(U_{n+2}) \arrow[r, "j'_{n+2}"] & \pi_{2n+6}(SO_{2n+5}) \arrow[r] & \pi_{2n+6}(\Gamma'_{n+3})=0 & \\
& \pi_{2n+7}(S^{2n+5}) \arrow[ul, "\partial'"] \arrow[u, "\partial"'] & &
\end{tikzcd}$$

with rows exact.

From[9], we may shoose the generator w of $\pi_{2n+6}(SO_{2n+5})$ such that $\partial(\eta^2_{2n+5}) = 4w$, Since j'_{n+2} is an epimorphism and $\partial' : \pi_{2n+7}(S^{2n+5}) \longrightarrow \pi_{2n+6}(U_{n+2})$ is a split monomorphism, we can shoose generators β_{n+2}, β'_{n+2} of $\pi_{2n+6}(U_{n+2}) \cong Z_{(n+3)!} + Z_2$ such that

(9. 9) $j'_{n+2}(\beta_{n+2}) = w$ and $j'_{n+2}(\beta'_{n+2}) = 4w$.

From [11], there exists a generator δ_{n+1} of $\pi_{2n+6}(U_{n+1}) \cong Z_{(n+3)!/2}$ such that

$$i'_{n+1}(\delta_{n+1})=2\beta_{n+2}+\beta'_{n+2} \tag{9.10}$$

Thus, from the commutativity of (9. 8), (9. 9) and (9. 10),

$$\begin{aligned} i''_{2n+2}j_{n+1}(\delta_{n+1}) &= j'_{n+2}i'_{n+1}(\delta_{n+1}) \\ &= 6w. \end{aligned}$$

On the other hand, since $i''_{2n+2}: \pi_{2n+6}(SO_{2n+2}) \longrightarrow \pi_{2n+6}(SO_{2n+5})$ is an epimorphism and $\pi_{2n+6}(SO_{2n+2})$ is a cyclic group, $j_{n+1}:\pi_{2n+6}(U_{n+1}) \longrightarrow \pi_{2n+6}(SO_{2n+2})$ is not an epimorphism.

Thus, from (9. 5),

$$^2\pi_{2n+6}(\Gamma_{n+1}) \cong Z_2. \tag{9.11}$$

From (9. 1), (9. 3), (9. 7) and (9. 11), we have

$$\begin{aligned} \pi_{2n+6}(\Gamma_{n+1}) &\cong \pi_{2n+6}(\Gamma_{n+2}) \\ &\cong \begin{cases} Z_2 & \text{for } n+1 \not\equiv 0 \bmod 3 \\ Z_6 & \text{for } n+1 \equiv 0 \bmod 3. \end{cases} \end{aligned}$$

§ 10. The group $\pi_{2n+7}(\Gamma_{n+2})$ for $n \geqq 8$.

Writing $\Gamma_{n+3}=SO_{2n+6}/U_{n+3}$, $\Gamma_{n+4}=SO_{2n+7}/U_{n+3}$, we have a commutative diagram

(10. 1)
$$\begin{tikzcd} & \pi_{2n+7}(\Gamma_{n+3}) \arrow[r, "k_{n+3}"] & \pi_{2n+7}(\Gamma_{n+4}) \arrow[r] & \pi_{2n+7}(S^{2n+6}) \\ \pi_{2n+7}(SO_{2n+5}) \arrow[ur, "r'_{n+3}"] \arrow[r, "i_{2n+5}"] & \pi_{2n+7}(SO_{2n+6}) \arrow[u, "r_{n+3}"] \arrow[r] & \pi_{2n+7}(SO_{2n+7}) \arrow[u, "r_{n+4}"] & \end{tikzcd}$$

namely (table 1 and [7])

$$\begin{tikzcd} & Z+Z_2 \arrow[r] & Z+Z_2 \arrow[r] & Z_2 \\ Z+Z_2 \arrow[ur] \arrow[r] & Z \arrow[u] \arrow[r] & Z. \arrow[u] & \end{tikzcd}$$

From [7], the following sequence

$$0 \longrightarrow \pi_{2n+7}(\Gamma_{n+3}) \xrightarrow{k_{n+3}} \pi_{2n+7}(\Gamma_{n+4}) \longrightarrow \pi_{2n+7}(S^{2n+9}) \longrightarrow 0$$

is exact. Let ξ', ϕ' be generators of $\pi_{2n+7}(\Gamma_{n+4})$ which generate Z, Z_2 in $\pi_{2n+7}(\Gamma_{n+4})$

respectively. Then we can shoose generators ξ, ϕ of $\pi_{2n+7}(\Gamma'_{n+3}) \cong Z+Z_2$ such that

(10.2) $$k_{n+3}(\xi)=2\xi' \text{ and } k_{n+3}(\phi)=\phi'$$

where ξ, ϕ generate Z, Z_2 in $\pi_{2n+7}(\Gamma_{n+3})$ respectively.

Moreover, we can shoose a generator u_i of the free part of $\pi_{2n+7}(SO_{2n+i})$ ($i=5$, 6 and 7) such that

(10.3) $$i_{2n+i}(u_i)=u_{i+1} \qquad (i=5,\ 6)$$

by [9].

From [7], we have

(10.4) $$r_{n+4}(u_7)=((n+3)!/2)\xi'+\phi'.$$

From (10.2)–(10.4) and the commutativity of the diagram (10.1) we obtain that

(10.5) $$r'_{n+3}(u_5)=((n+3)!/4)\xi+\phi.$$

Consider the homotopy exact sequence associated to the fibration $p_{2n+4}: SO_{2n+5} \longrightarrow S^{2n+4}$;

$$0 \longrightarrow \pi_{2n+7}(SO_{2n+4}) \xrightarrow{i_{2n+4}} \pi_{2n+7}(SO_{2n+5}) \xrightarrow{p_{2n+4}} \pi_{2n+7}(S^{2n+4})$$

$$\longrightarrow \pi_{2n+6}(SO_{2n+4}) \longrightarrow \pi_{2n+6}(SO_{2n+5}) \longrightarrow 0$$

namely

$$0 \longrightarrow Z+Z_2 \longrightarrow Z+Z_2 \longrightarrow Z_{24} \longrightarrow Z_4+Z_{24d} \longrightarrow Z_8 \longrightarrow 0$$

where $d=1$ or 2. Then we can shoose generators u_4, u'_4 of $\pi_{2n+7}(SO_{2n+4}) \cong Z+Z_2$ such that generate Z, Z_2 respectively and

(10.6) $$\begin{cases} i_{2n+4}(u_4)=2u_5, & i_{2n+4}(u'_4)=u'_5 \quad \text{if } d=1 \\ i_{2n+4}(u_4)=u_5, & i_{2n+4}(u'_4)=u'_5 \quad \text{if } d=2. \end{cases}$$

From the commutative diagram

$$\begin{tikzcd} \pi_{2n+7}(\Gamma'_{n+3}) \arrow[rr, "q_{n+2}"] & & \pi_{2n+7}(S^{2n+4}) \\ & \pi_{2n+7}(SO_{2n+5}) \arrow[ul, "r'_{n+3}"] \arrow[ur, "p_{2n+4}"'] & \end{tikzcd}$$

and (10.6) we have

$$q_{n+2}r'_{n+3}(u_5)=p_{2n+4}(u_5)=\begin{cases} 0 & \text{if } d=2 \\ 12\nu_{2n+4} & \text{if } d=1. \end{cases}$$

From (10.5)

$$\begin{aligned} q_{n+2}r'_{n+3}(u_5) &= q_{n+2}((n+3)!/4)\xi+\phi) \\ &= ((n+3)!/4)q_{n+2}(\xi)+q_{n+2}(\phi) \\ &= q_{n+2}(\phi) \qquad \text{for } n\geqslant 8. \end{aligned}$$

(Since $(n+3)!/4\equiv 0 \bmod 24$ for $n\geqslant 8$).

Thus the element ϕ of order 2 is in the image of $k_{n+3}:\pi_{2n+7}(\Gamma'_{n+2})\longrightarrow \pi_{2n+7}(\Gamma_{n+3})$ if and only if $d=2$. Therefore, from the exact sepuence $0\longrightarrow \pi_{2n+7}(\Gamma'_{n+2})\longrightarrow \pi_{2n+7}(\Gamma_{n+3})\longrightarrow \pi_{2n+7}(S^{2n+4})$, we obtain that

$$\pi_{2n+7}(\Gamma'_{n+2})\cong\begin{cases} Z+Z_2 & \text{if } d=2 \\ Z & \text{if } d=1. \end{cases}$$

§ 11. The group π_{2n+1} (Γ_{n-2})

Consider the exact sequence

$$\pi_{2n+2}(S^{2n-4})\longrightarrow \pi_{2n+1}(\Gamma_{n-2})\longrightarrow \pi_{2n+1}(\Gamma_{n-1})\longrightarrow \pi_{2n+1}(S^{2n-4})=0$$

where $\pi_{2n+2}(S^{2n-4})\cong Z_2$. Thus we have

$$\begin{aligned} {}^p\pi_{2n+1}(\Gamma'_{n-1}) &\cong {}^p\pi_{2n+1}(\Gamma'_{n-1}) \\ &\cong {}^pZ_{n!(24,n-2)/24} \qquad \text{by (5.5)} \end{aligned}$$

for odd prime p.

Consider the homotopy exact sequence associated with the fibration $k_{n-2,3}:\Gamma_{n-2}\longrightarrow \Gamma_{n+1}$;

$$\begin{aligned} &{}^2\pi_{2n+1}(\Gamma_{n-2,3})\longrightarrow {}^2\pi_{2n+1}(\Gamma_{n-2})\longrightarrow {}^2\pi_{2n+1}(\Gamma_{n+1})\longrightarrow {}^2\pi_{2n}(\Gamma_{n-2,3}) \\ &\longrightarrow {}^2\pi_{2n}(\Gamma_{n-2})\longrightarrow {}^2\pi_{2n}(\Gamma_{n+1})\longrightarrow {}^2\pi_{2n-1}(\Gamma_{n-2,3}) \end{aligned}$$

where ${}^2\pi_{2n}(\Gamma_{n+1})\cong Z_2$ and ${}^2\pi_{2n+1}(\Gamma_{n+1})\cong {}^2Z_{n!}$.

From Proposition 3.2,

$$\begin{aligned} &{}^2\pi_{2n+1}(\Gamma_{n-2,3})\cong {}^2\pi_{2n+1}(K_{n-2,3})=0, \\ &{}^2\pi_{2n}(\Gamma_{n-2,3})\cong {}^2\pi_{2n}(K_{n-2,3})\cong Z_4, \end{aligned}$$

$$^2\pi_{2n-1}(\Gamma'_{n-2,3}) \cong {}^2\pi_{2n-1}(K_{n-2,3}) \cong Z$$

Thus from the above sequence we have

$$^2\pi_{2n+1}(\Gamma'_{n-2}) \cong {}^2Z_{n!(24,n-2)/24}.$$

Therefore we have

$$\pi_{2n+1}(\Gamma'_{n-2}) \cong Z_{n!(24,n-2)/24}.$$

§ 12. The homotopy group of Γ_n for lower values of n.

We compute the homotopy groups $\pi_{2n+i}(\Gamma_n)$ of Γ_n for lower values of n.

We can be identify the two manifolds SO_3/U_1 and SO_5/U_2. Thus the well known result $SO_2=U_1$ implies that

$$SO_4/U_2=S^2.$$

Therefor Γ_2 is a 2-sphere.

Consider the fibration $S^2=\Gamma_2 \longrightarrow \Gamma_3 \xrightarrow{p} S^4$ of (2.1). Then we have the following homotopy exact sequence associated with the fiber space $p: \Gamma_3 \longrightarrow S^4$;

$$\text{(12.1)} \qquad \longrightarrow \pi_i(\Gamma_3) \xrightarrow{p} \pi_i(S^4) \xrightarrow{\Delta} \pi_{i-1}(S^2) \longrightarrow \pi_{i-1}(\Gamma_3) \longrightarrow$$

where Δ is the boundary homomorphism. For the boundary homomorphism Δ, we have the formula

$$\text{(12.2)} \qquad \Delta(\alpha\circ E\beta)=(\Delta(\alpha))\circ\beta \qquad \text{for } \alpha\in\pi_{i+1}(S^4)$$

where $E:\pi_j(S^i) \longrightarrow \pi_{j+1}(S^{i+1})$ is the suspension homomorphism.

Consider the exact sequence

$$\pi_4(S^4) \longrightarrow \pi_3(S^2) \longrightarrow \pi_3(\Gamma_3).$$

Then $\pi_3(\Gamma_3)=0$ by [4] and $\pi_3(S^2)$ is an infinite cyclic group generated by Hopf map η_2. Thus we have

$$\text{(12.3)} \qquad \Delta(\iota_4)=\eta_2$$

where ι_4 is a generator of $\pi_4(S^4)\cong Z$.

Thus, from (12.2) and (12.3), we have

$$\text{(12.4)} \qquad \Delta(E\alpha)=\Delta(\iota_4)\alpha=\eta_{2*}(\alpha) \qquad \text{for } \alpha\in\pi_{i-1}(S^3).$$

The homomorphism

$$\eta_{2*} : \pi_i(S^3) \longrightarrow \pi_i(S^2)$$

induced by Hopf map η_2 is an isomorphism onto for $i \geqslant 3$.

Thus it follows from (12. 4) that the boundary homomorphism $\Delta : \pi_i(S^4) \longrightarrow \pi_{i-1}(S^2)$ is an split epimorphism for $i \geqslant 4$. Therefore we have an isomorphism

$$\pi_i(S^4) \cong \pi_i(\Gamma_3) \oplus \pi_{i-1}(S^2) \tag{12. 5}$$

for $i \geqslant 4$.

It is well known that there is map $h : S^7 \longrightarrow S^4$ whose Hopf invariant is ± 1 and that

$$E \oplus h_* : \pi_{i-1}(S^3) \oplus \pi_i(S^7) \longrightarrow \pi_i(S^4) \tag{12. 6}$$

an isomorphisms onto for all i.

Thus it follows from (12. 5) and (12. 6),

***Proposition* 12. 1.** *We have isomorphisms*

$$\pi_i(\Gamma_3) \cong \pi_i(S^7)$$

for $i \geqslant 4$.

Consider the fibration $p : E \longrightarrow S^m$ with fiber F. $\pi_m(S^m)$ is an infinite cyclic group generated by ι_m. Let $\alpha \in \pi_m(E)$. Then there exists an integer q such that $p_*(\alpha) = q\iota_m$ where $p_* : \pi_m(E) \longrightarrow \pi_m(S^m)$ is the homomorphism induced by the projection p. $\alpha \in \pi_m(E)$ is the homotopy class of a cross section of E if and only if $q=1$. The image of $p_* : \pi_m(E) \longrightarrow \pi_m(S^m)$ concides with the kernel of the boundary homomorphism $\Delta : \pi_m(S^m) \longrightarrow \pi_{m-1}(\mathrm{F})$.

Hence the fibration $p : E \longrightarrow S^m$ admits a cross section if and only if $\Delta(\iota_m) = 0$.

Now we consider the fibration $p : \Gamma_4 \longrightarrow S^6$ with a fiber Γ_3. From $\pi_5(\Gamma_3) = 0$ and the above remark, it follows that the fibration $p : \Gamma_4 \longrightarrow S^6$ have a cross section. Therefore we obtain the following proposition.

***Proposition* 12. 2.** *There exists the isomorphism onto*

$$\pi_i(\Gamma_4) \cong \pi_i(\Gamma_3) \oplus \pi_i(S^6)$$

for all i.

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