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Homotopy groups of homogeneous space $SU(n)/SO(n)$

Dedicated to Professor A. Komatu on his 70th birthday

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§ 1. Introduction

Let $SU(n)$, $SO(n)$ denote the special unitary, special orthgonal group respectively and Y_n the homogeneous space $SU(n)/SO(n)$. In [2], B. Harris showed that the homotopy exact sequence of the fibrations

$$SO(2n+1) \longrightarrow SU(2n+1) \longrightarrow Y_{2n+1}$$

reduces to the following direct sum decompsitions modulo 2-primary components ; i. e., there exist the following $\mathscr{C}$-isomorphisms

$$\pi_i(SU(2n+1)) \cong \pi_i(SO(2n+1)) \oplus \pi_i(Y_{2n+1})$$

for all i, where $\mathscr{C}$ denotes the class of 2-primary abelian groups.

If $r<0$, then the homotopy group $\pi_{n+r}(Y_n)$ is called stable and has been determined by Bott [1] ; he showed that in the stable range,

$$\pi_q(Y_n) \cong Z \qquad \text{for } q \equiv 1,\ 5 \bmod 8,$$

$$\pi_q(Y_n) \cong Z_2 \qquad \text{for } q \equiv 2,\ 3 \bmod 8,$$

$$\pi_q(Y_n) = 0 \qquad \text{for } q \equiv 0,\ 4,\ 6,\ 7 \bmod 8.$$

In this paper we calculate the first few unstable homotopy groups of the homogeneous spaces Y_n. The homotopy groups $\pi_{n+r}(Y_n)$ are given in the following table valid for $s \geqslant 1$:

Table of $\pi_{n+r}(Y_n)$

$r \backslash n$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$	$8s+7$
0	Z	$Z+Z_2$	$Z+Z_2$	Z_2+Z_2	Z	$Z+Z_2$	Z	Z_2
1	$Z+Z_2+Z_2$	Z_2+Z_2	Z_2+Z_4	0	$Z+Z_2+Z_2$	Z_2	Z_4	0
2	$Z_2+Z_2+Z_2$	Z_2+Z_8	0	$Z+Z_2$	Z_2+Z_2	Z_8	0	$Z+Z_2$
3	$Z_2+Z_{24}+Z_8$	Z_2	$Z+Z_2$	Z_2+Z_2	Z_4+Z_{24d}	Z_2	$Z+Z_{12}$	Z_2+Z_2
4	Z_2	Z	Z_2	Z_{8d}	Z_2	Z	Z_2	Z_2+Z_8
5	Z	Z_2	Z_{8d}	Z_2	Z	Z_2	Z_2+Z_8	Z_2

where $d=1$ or 2 (if $s=1$, then $d=1$).

§ 2. Preliminaries

Let $k_n : SO(n) \longrightarrow SU(n)$ be the inclusion map and $p_n : SU(n) \longrightarrow Y_n = SU(n)/SO(n)$ the projection. Inparticular we put $SU(\infty) = SU$ and $SO(\infty) = SO$. Let $j_n : SO(n) \longrightarrow SO$ and $r_n : SU(n) \longrightarrow SU$ be the natural inclusion maps. Then we have the commutative diagram

(2. 1)
$$\begin{array}{ccc} SO(n) & \xrightarrow{k_n} & SU(n) \\ \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle r_n} \\ SO & \xrightarrow{k} & SU \end{array}$$

and the fibration

(2. 2)
$$SO(n) \xrightarrow{k_n} SU(n) \xrightarrow{p_n} Y_n.$$

The following two lemmas are well known ;

Lemma 2. 1. *For the homomorphism* $k : \pi_n(SO) \longrightarrow \pi_n(SU)$ *induced by the inclusion map* $k : SO \longrightarrow SU$, *we have that*

(1) *k is a trivial homomorphism for* $n \not\equiv -1$, 3 mod 8,

(2) *k is an epimorphism for* $n \equiv -1$ mod 8

and

(3) *k map a generator onto 2 time generator for* $n \equiv 3$ *mod* 8.

Lemma 2. 2. *(See [4]) Consider the homomorphism* $j_n^m : \pi_m(SO(n)) \longrightarrow \pi_m(SO)$ *induced by the inclusion map* $j_n : SO(n) \longrightarrow SO$. *Then,*

(1) $j_{8s-i}^{8s} : \pi_{8s}(SO(8s-i)) \longrightarrow \pi_{8s}(SO)$ *is a split epimorphism for* $s \geqslant 2$, $i \leqslant 4$ *or* $s=1$, $i \leqslant 1$.

(2) $j_{8s-i}^{8s+1} : \pi_{8s+1}(SO(8s-i)) \longrightarrow \pi_{8s+1}(SO)$ *is an epimorphism for* $s \geqslant 2$, $i \leqslant 4$ *or* $s=1$, $i \leqslant 2$.

(3) $j^{8s+3}_{8s-i}: \pi_{8s+3}(SO(8s-i)) \longrightarrow \pi_{8s+3}(SO)$ *is an epimorphism for* $s \geqslant 2$, $i \leqslant 2$ *or* $s=1$, $i \leqslant 0$.

(4) $j^{8s-1}_{8s-i}: \pi_{8s-1}(SO(8s-i)) \longrightarrow \pi_{8s-1}(SO)$ *is an epimorphism for* $s \geqslant 2$, $i \leqslant 5$.

(5) $j^{n}_{n-i}: \pi_n(SO(n-i)) \longrightarrow \pi_n(SO)$ *is trivial homomorphism for* $n \neq 8s$, $8s+1$, $8s+3$, $8s-1$ *and all* i, $s \geqslant 1$.

Consider the commutative diagram

$$\begin{array}{ccc} \pi_n(SO(n-i)) & \xrightarrow{k^n_{n-i}} & \pi_n(SU(n-i)) \\ \downarrow j^n_{n-i} & & \downarrow r_n \\ \pi_n(SO) & \xrightarrow{k} & \pi_n(SU) \end{array}$$

induced by (2. 1). If $n \geqslant 2i+1$, then r_n is an isomorphism.
Thus from Lemma 2. 1 and Lemma 2. 2, we have

Lemma 2. 3. *If* $n \geqslant 2i+1$, *then*

(1) $k^{n}_{n-i}: \pi_n(SO(n-i)) \longrightarrow \pi_n(SU(n-i))$ *is trivial homomorphism for* $n \not\equiv -1$, 3 *mod* 8,

(2) $k^{8s-1}_{8s-1-i}: \pi_{8s-1}(SO(8s-1-i)) \longrightarrow \pi_{8s-1}(SU(8s-1-i))$ *is an epimorphism for* $s \geqslant 2$ *and* $i \leqslant 4$.

§ 3. Calculations

If n is even and $n \geqslant 2i+1$, then $\pi_n(SU(n-i)) \cong \pi_n(SU) = 0$. Thus, the homotopy exact sequence associated with the fibration (2. 2) breaks into the following exact sequence

$$(3.1) \qquad 0 \longrightarrow \pi_{n+1}(Y_{n-i}) \longrightarrow \pi_n(SO(n-i)) \xrightarrow{k^n_{n-i}} \pi_n(SU(n-i)) \longrightarrow \pi_n(Y_{n-i}) \longrightarrow \pi_{n-1}(SO(n-i)) \longrightarrow 0$$

where $n \geqslant 2i+2$ and n odd.

Proposition 3. 1. *Let* $n = 8s+1$ *or* $8s+5$ $(s \geqslant 1)$ *and* $n \geqslant 2i+2$. *Then,*

(*i*) $\pi_{n+1}(Y_{n-i}) \cong \pi_n(SO(n-i))$,

(*ii*) *The sequence*

$$0 \longrightarrow \pi_n(SU(n-i)) \cong Z \longrightarrow \pi_n(Y_{n-i}) \longrightarrow \pi_{n-1}(SO(n-i)) \longrightarrow 0$$

is exact.

Proof. From (3.1) and (1) of Lemma 2.3, we obtain the results.

Proposition 3.2. *Let* $n=8s-1$, $s\geqslant 2$ *and* $i\leqslant 4$. *Then*
(*i*) $\pi_n(Y_{n-i})\cong\pi_{n-1}(SO(n-i))$,
(*ii*) *The sequence*

$$0\longrightarrow\pi_{n+1}(Y_{n-i})\longrightarrow\pi_n(SO(n-i))\longrightarrow\pi_n(SU(n-i))\cong Z\longrightarrow 0$$

is split exact.

Proof. From (3.1) and (2) of Lemma 2.3, we obtain the lemma.

Lemma 3.3. *If the diagram of groups and homomorphisms*

$$\begin{array}{ccccccccccc}
\longrightarrow & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \xrightarrow{h_i} & A_{i+1} & \xrightarrow{f_{i+1}} & B_{i+1} & \longrightarrow \\
& \downarrow{\scriptstyle\alpha_i} & & \downarrow{\scriptstyle\beta_i} & & \downarrow{\scriptstyle\gamma_i} & & \downarrow{\scriptstyle\alpha_{i+1}} & & \downarrow{\scriptstyle\beta_{i+1}} & \\
\longrightarrow & A'_i & \xrightarrow{f'_i} & B'_i & \xrightarrow{g'_i} & C'_i & \xrightarrow{h'_i} & A'_{i+1} & \xrightarrow{f'_{i+1}} & B'_{i+1} & \longrightarrow
\end{array}$$

is commutative, horizontal sequences are exact and γ_i *are isomorphism, then the sequence*

$$\cdots\cdots\longrightarrow A'_i\xrightarrow{(\alpha_i,f_i)}A'_i+B_i\xrightarrow{f_i'-\beta_i}B'_i\xrightarrow{h_i\gamma_i^{-1}g'_i}A_{i+1}\longrightarrow\cdots\cdots$$

is exact.

Proposition 3.4. *Assume that* $s\geqslant 2$, $i\leqslant 2$ *or* $s=1$, $i\leqslant 0$.
Then,

(*i*)
$$\begin{aligned}\pi_{8s+3}(Y_{8s-i})&\cong\pi_{8s+3}(Y)\oplus\pi_{8s+2}(SO(8s-i))\\&\cong Z_2\oplus\pi_{8s+2}(SO(8s-i))\end{aligned}$$

where $Y=SU/SO$,
(*ii*) *The sequence*

$$0\longrightarrow\pi_{8s+4}(Y_{8s-i})\longrightarrow\pi_{8s+3}(SO(8s-i))\longrightarrow\pi_{8s+3}(SO)\cong Z\longrightarrow 0$$

is split exact.

Proof. From (2.1) and (3.1), it follows that the diagram

$$\begin{array}{ccccc}
0\longrightarrow\pi_{8s+4}(Y_{8s-i}) & \longrightarrow & \pi_{8s+3}(SO(8s-i)) & \longrightarrow & \pi_{8s+3}(SU(8s-i)) \\
\downarrow & & \downarrow{\scriptstyle j^{8s+3}_{8s-i}} & & \downarrow{\scriptstyle r_{8s-i}} \\
0 & \longrightarrow & \pi_{8s+3}(SO) & \longrightarrow & \pi_{8s+3}(SU)
\end{array}$$

$$\begin{array}{ccccc} \longrightarrow \pi_{8s+3}(Y_{8s-i}) & \longrightarrow & \pi_{8s+2}(SO(8s-i)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \longrightarrow \pi_{8s+3}(Y) & \longrightarrow & 0 & & \end{array}$$

is commutative with rows exact.

If $4s \geqslant i+2$, then r_{8s-i} is an isomorphism. Thus, from Lemma 3.3, we have the following exact sequence ;

$$0 \longrightarrow \pi_{8s+4}(Y_{8s-i}) \longrightarrow \pi_{8s+3}(SO(8s-i)) \xrightarrow{j^{8s+3}_{8s-i}} \pi_{8s+3}(SO) \cong Z$$
$$\longrightarrow \pi_{8s+3}(Y_{8s-i}) \longrightarrow \pi_{8s+3}(Y) \oplus \pi_{8s+3}(SO(8s-i)) \longrightarrow 0.$$

By (3) of Lemma 2.2, j^{8s+3}_{8s-i} is an epimorphism for $s \geqslant 2$, $i \leqslant 2$ or $s=1$, $i \leqslant 0$. Thus we obtain the results.

Put $n = 8s+5 \geqslant 2i+2$. Then, from (2.1) and (ii) of Proposition 3.1, it follows that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_n(SU) & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_{n-1}(SO) & \longrightarrow & 0 \\ & & \uparrow r_{n-i} & & \uparrow & & \uparrow j^{n-1}_{n-i} & & \\ 0 & \longrightarrow & \pi_n(SU(n-i)) & \longrightarrow & \pi_n(Y_{n-i}) & \longrightarrow & \pi_{n-1}(SO(n-i)) & \longrightarrow & 0 \end{array}$$

is commutative and r_{n-i} is an isomorphism. It follows that the lower sequence is a split extension if the upper is. But the upper sequence splits trivially, since $\pi_{8s+4}(SO)=0$. Thus we have

Proposition 3.5. *Let* $n = 8s+5 \geqslant 2i+2$. *Then*

$$\pi_n(Y_{n-i}) \cong \pi_n(SU(n-i)) \oplus \pi_{n-1}(SO(n-i))$$
$$\cong Z \oplus \pi_{n-1}(SO(n-i)).$$

Lemma 3.6. *If the diagram of groups and homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 0 \end{array}$$

is commutative, horizontal sequences are exact, h *is a split epimorphism and* f *is an isomorphism, then*

(i) *ker.* $g \cong$ *ker.* h
(ii) *The sequence*

$$0 \longrightarrow ker.\ g \longrightarrow G_2 \xrightarrow{g} H_2 \longrightarrow 0$$

is split exact.

Proposition 3.7. *Put* $n = 8s+1$. *Assume that* $s \geqslant 2$, $i \leqslant 5$ *or* $s=1$, $i \leqslant 2$. *Then*

$$\begin{aligned}\pi_n(Y_{n-i}) &\cong \pi_n(Y) \oplus kernel\ of\ j^{n-1}_{n-i} \\ &\cong Z \oplus \pi_n(V_{m,m-n+i})\end{aligned}$$

where m *is to be large.*

Proof. From (ii) of Proposition 3.1 and (2.1), it follows that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_n(SU(n-i)) & \longrightarrow & \pi_n(Y_{n-i}) & \longrightarrow & \pi_{n-1}(SO(n-i)) & \longrightarrow & 0 \\ & & \downarrow r_{n-i} & & \downarrow & & \downarrow j^{n-1}_{n-i} & & \\ 0 & \longrightarrow & \pi_n(SU) & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_{n-1}(SO) & \longrightarrow & 0 \end{array}$$

is commutative and r_{n-i} is an isomorphism. From (1) of Lemma 2.2, j^{n-1}_{n-i} is a split epimorphism. Thus from Lemma 3.6, we obtain the result.

§ 4 The homotopy groups of Y_n for low values of n.

Proposition 4.1. $\pi_3(Y_3) \cong Z_4$, $\pi_3(Y_4) \cong Z_2$, $\pi_4(Y_3) = 0$ *and* $\pi_4(Y_4) \cong Z$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccccccc} & & & & 0 & \longrightarrow & \pi_3(SO(5)) & \xrightarrow{k_5} & \pi_3(SU(5)) & \longrightarrow & \pi_3(Y_5) & \longrightarrow & 0 \\ & & & & & & \uparrow j_4 & & \uparrow r_4 & & \uparrow & & \\ 0 & \longrightarrow & \pi_4(Y_4) & \longrightarrow & & & \pi_3(SO(4)) & \xrightarrow{k_4} & \pi_3(SU(4)) & \longrightarrow & \pi_3(Y_4) & \longrightarrow & \pi_2(SO(4)) \longrightarrow 0 \\ & & \uparrow & & & & \uparrow j_3 & & \uparrow r_3 & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \pi_4(Y_3) & \longrightarrow & & & \pi_3(SO(3)) & \xrightarrow{k_3} & \pi_3(SU(3)) & \longrightarrow & \pi_3(Y_3) & \longrightarrow & \pi_2(SO(3)) \longrightarrow 0 \end{array}$$

where r_3, r_4 are isomorphism. Now $j_4 : \pi_3(SO(4)) \cong Z+Z \longrightarrow \pi_3(SO(5)) \cong Z$ is an epimorphism and $k_5 : \pi_3(SO(5)) \cong Z \longrightarrow \pi_3(SU(5)) \cong Z$ maps a generator of $\pi_3(SO(5))$ onto 2-time generator of $\pi_3(SU(5))$. Thus, from the commutativity of diagram, $k_4(\pi_3(SO(4))) = 2Z \subset \pi_3(SU(4))$. Since $\pi_2(SO(4)) = 0$, we obtain that $\pi_3(Y_4) \cong Z_2$ and $\pi_4(Y_4) \cong Z$.

From [6], $j_4 j_3 : \pi_3(SO(3)) \cong Z \longrightarrow \pi_3(SO(5)) \cong Z$ maps a generator of $\pi_3(SO(3))$ onto 2-time generator of $\pi_3(SO(5))$. Thus, from (3) of Lemma 2.1, k_3 maps a generator onto 4-time generator of $\pi_3\,(SU(3))$. Since $\pi_2\,(SO(3)) = 0$, we obtain that $\pi_3(Y_3) \cong Z_4$ and $\pi_4(Y_3) = 0$.

Proposition 4.2. *(i)* $\pi_6(Y_{6-i}) \cong \pi_5(SO(6-i))$ *for* $i = 0,\ 1,\ 2$ *and* 3.
(ii) The sequence

$$0 \longrightarrow \pi_5(SU(6-i)) \cong Z \longrightarrow \pi_5(Y_{6-i}) \longrightarrow \pi_4(SO(6-i)) \longrightarrow 0$$

is split exact for $0 \leqslant i \leqslant 3$.

Proof. Since $j^5_{6-i} : \pi_5(SO(6-i)) \longrightarrow \pi_5(SO)$ is trivial we obtain the results by the same calculations in Propositions 3.1 and 3.5.

Proposition 4.3. *(i)* $\pi_7(Y_7) \cong Z_2$, $\pi_7(Y_6) \cong Z_4$ *and* $\pi_7(Y_5) \cong Z_8$.
(ii) $\pi_8(X_8) \cong Z$ *and* $\pi_8(Y_{8-i}) = 0$ *for* $1 \leqslant i \leqslant 3$.

Proof. $\pi_6(SO(m)) = 0$ for $m \geqslant 5$. Thus, from (3.1), we obtain the following commutative diagram ;

$$\begin{array}{ccccccccc}
 & & \pi_7(SO) & \xrightarrow{k} & \pi_7(SU) & & & & \\
 & & \uparrow j_8 & & \uparrow r_8 & & & & \\
0 \longrightarrow & \pi_8(Y_8) & \longrightarrow \pi_7(SO(8)) & \xrightarrow{k_8} & \pi_7(SU(8)) & \longrightarrow & \pi_7(Y_8) & \longrightarrow & 0 \\
 & \uparrow & \uparrow j_7 & & \uparrow r_7 & & \uparrow & & \\
0 \longrightarrow & \pi_8(Y_7) & \longrightarrow \pi_7(SO(7)) & \xrightarrow{k_7} & \pi_7(SU(7)) & \longrightarrow & \pi_7(Y_7) & \longrightarrow & 0 \\
 & \uparrow & \uparrow j_6 & & \uparrow r_6 & & \uparrow & & \\
0 \longrightarrow & \pi_8(Y_6) & \longrightarrow \pi_7(SO(6)) & \xrightarrow{k_6} & \pi_7(SU(6)) & \longrightarrow & \pi_7(Y_6) & \longrightarrow & 0 \\
 & \uparrow & \uparrow j_5 & & \uparrow r_5 & & \uparrow & & \\
0 \longrightarrow & \pi_8(Y_5) & \longrightarrow \pi_7(SO(5)) & \xrightarrow{k_5} & \pi_7(SU(5)) & \longrightarrow & \pi_7(Y_5) & \longrightarrow & 0
\end{array}$$

where r_i is an isomorphism for $5 \leqslant i \leqslant 8$ and $\pi_7(SO(8)) \cong Z + Z$ and $\pi_7(SO(m)) \cong Z$ for $5 \leqslant m \leqslant 7$. Now $k : \pi_7(SO) \longrightarrow \pi_7(SU)$ is an isomorphism and j_8 is an epimorphism. Thus $k_8 : \pi_7(SO(8) \longrightarrow \pi_7(SU(8))$ is an epimorphism. Therefore, form the exactness, we have $\pi_8(Y_8) \cong Z$ and $\pi_7(Y_8) = 0$.

$j_8 j_7(\pi_7(SO(7))) = 2Z \subset \pi_7(SO)$. From the commutativity of diagram, $k_7(\pi_7(SO(7))) = 2Z \subset \pi_7(SU(7))$. Thus, from the exactness, it follows that $\pi_7(Y_7) \cong Z_2$ *and* $\pi_8(Y_7) = 0$.

$j_m(m = 5,\ 6)$ maps a generator of $\pi_7(SO(m))$ onto 2-time generator of $\pi_7(SO(m+1))$. From the commutativity of the diagram, $k_6(\pi_7(SO(6))) = 4Z \subset \pi_7(SU(6))$ and $k_5(\pi_7(SO(5))) = 8Z \subset \pi_7(SU(5))$. Then, from the exactness of the horizontal sequences, we have

$$\pi_7(Y_6) \cong Z_4, \quad \pi_7(Y_5) \cong Z_8 \text{ and } \pi_8(Y_m) = 0$$

for $m = 5, 6$.

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