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Homotopy groups of homogeneous space $Sp(n)/U(n)$

Dedicated to Professor A. Komatu on his 70th birthday

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§ 1. Introduction

Let $Sp(n)$, $U(n)$ denote the symplectic, unitary group respectively and Z_n the homogeneous space $Sp(n)/U(n)$.

The homotopy group $\pi_{2n+i}(Z_n)$ $(i \leqslant 0)$ is called stable and by Bott [1],

$$\pi_q(Sp/U) = \pi_{q+1}(Sp) \quad q=0,\ 1,\ 2,\ \cdots\cdots.$$

In this paper we compute the unstable homotopy groups of the homogeneous spaces Z_n. For $i \leqslant 7$, the group $\pi_{2n+i}(Z_n)$ are computed and the results are given by the following table :

Table of $\pi_{2n+i}(Z_n)$

$i \backslash n$	$4k$	$4k+1$	$4k+2$	$4k+3$
1	$Z_{n!}$	$Z+Z_2$	$Z_{2\times n!}$	Z
2	Z_2	Z_2	Z_2	0
3	$Z+Z_2+Z_2$	$Z_{(n+1)!}$	$Z+Z_2$	$Z_{(n+1)!/2}$
4	$Z_2+Z_{(24,n)}$	$Z_{(24,n+3)/2}$	$Z_{(24,n)}$	$Z_{(24,n+3)/2}$
5	$Z_{(n+2)!(24,n)/24}$	Z	$Z_{(n+2)!(24,n)/48}$	$Z+Z_2$
6	$Z_{(24,n+4)/2}$	$Z_{(24,n+1)}$	$Z_{(24,n+4)/2}$	$Z_2+Z_{(24,n+1)}$
7	$Z+Z_2$	$Z_{(n+3)!(24,n+1)/48}$	$Z+Z_2$	$Z_{(n+3)!(24,n+1)/24}$

where $(24, n)$ is the g. c. d. of 24 and n.

The computations will be done by use of the homotopy exact sequences (2. 1) and (2. 3).

§ 2. Preliminaries

Let $s_n : U(n) \longrightarrow Sp(n)$ be the inclusion and $p_n : Sp(n) \longrightarrow Z_n = Sp(n)/U(n)$ the projection.

Consider the commutative diagram

$$\begin{array}{ccc} \pi_{2n+i}(U(n)) & \xrightarrow{s_n} & \pi_{2n+i}(Sp(n)) \\ \downarrow i_n & & \downarrow i_n' \\ \pi_{2n+i}(U) & \longrightarrow & \pi_{2n+i}(Sp) \end{array}$$

induced by inclusion maps, where i'_n is an isomorphism for $i \leqslant 2n+1$. On the other hand, $\pi_{2n+i}(U(n))$ is finite group for $i \geqslant 0$ and $\pi_{2n+i}(U)$ is trivial or infinite cyclic group. Thus the homomorphism

$$s_n : \pi_{2n+i}(U(n)) \longrightarrow \pi_{2n+i}(Sp(n))$$

induced by the inclusion $s_n : U(n) \longrightarrow Sp(n)$ is trivial for $0 \leqslant i \leqslant 2n+1$.

From the homotopy exact sequence associated with the fibration $p_n : Sp(n) \longrightarrow Z_n$ with a fibre $U(n)$, it follows that the sequence

$$(2.1) \qquad 0 \longrightarrow \pi_{2n+i}(Sp(n)) \xrightarrow{p_n} \pi_{2n+i}(Z_n) \xrightarrow{\Delta} \pi_{2n+i-1}(U(n)) \longrightarrow 0$$

is exact for $1 \leqslant i \leqslant 2n+1$.

Consider the fibration $Sp(n+1)/U(n) \longrightarrow Sp(n+1)/Sp(n) = S^{4n+3}$ with a fibre $Z_n = Sp(n)/U(n)$. Then we have the isomorphism

$$(2.2) \qquad \pi_k(Z_n) \cong \pi_k(Sp(n)/U(n))$$

for $k \leqslant 4n+1$.

From the fibration

$$S^{2n+1} = U(n+1)/U(n) \longrightarrow Sp(n+1)/U(n) \longrightarrow Sp(n+1)/U(n+1) = Z_{n+1}$$

and (2.2), we have an exact sequence

$$(2.3) \qquad \cdots \longrightarrow \pi_k(S^{2n+1}) \xrightarrow{j_n} \pi_k(Z_n) \xrightarrow{r_n} \pi_k(Z_{n+1}) \xrightarrow{\bar{\partial}} \pi_{k-1}(S^{2n+1}) \longrightarrow \cdots$$

for $i \leqslant 4k+1$.

Further, we obtain the following commutative diagrams

$$(2.4) \qquad \begin{array}{ccccccccc} \longrightarrow & \pi_k(S^{2n+1}) & \xrightarrow{j_n} & \pi_k(Z_n) & \xrightarrow{r_n} & \pi_k(Z_{n+1}) & \xrightarrow{\bar{\partial}} & \pi_k(S^{2n+1}) & \longrightarrow \\ & \| & & \downarrow \Delta & & \downarrow \Delta & & \| & \\ \longrightarrow & \pi_k(S^{2n+1}) & \xrightarrow{\partial} & \pi_{k-1}(U(n)) & \xrightarrow{i_n} & \pi_{k-1}(U(n+1)) & \xrightarrow{q} & \pi_{k-1}(S^{2n+1}) & \longrightarrow \end{array}$$

with exact rows for $k \leqslant 4n+1$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_k(Sp(n)) & \xrightarrow{p_n} & \pi_k(Z_n) & \xrightarrow{\Delta} & \pi_{k-1}(U(n)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i'_n} & & \downarrow{\scriptstyle r_n} & & \downarrow{\scriptstyle i_n} & & \\ 0 & \longrightarrow & \pi_k(Sp(n+1)) & \xrightarrow{p_{n+1}} & \pi_k(Z_{n+1}) & \xrightarrow{\Delta} & \pi_{k-1}(U(n+1)) & \longrightarrow & 0 \end{array} \tag{2.5}$$

with exact rows for $2n+3 \leqslant k \leqslant 4n+5$.

From (2.4), we have the commutative diagram

$$\begin{array}{ccccc} \pi_{2n+i}(S^{2n+1}) & \xrightarrow{j_n} & \pi_{2n+i}(Z_n) & \xrightarrow{\bar{\partial}} & \pi_{2n+i-1}(S^{2n-1}) \\ & {\scriptstyle\partial}\searrow & \downarrow & \nearrow{\scriptstyle q} & \\ & & \pi_{2n+i-1}(U(n)). & & \end{array}$$

Then, from Lemma 1.1 of [3], $q: \pi_{2n}(U(n)) \longrightarrow \pi_{2n}(S^{2n-1})$ is given by

$$q(\partial\iota_{2n+1}) = 0 \quad \text{for } n \text{ odd}$$

$$q(\partial\iota_{2n+1}) = \eta_{2n-1} \quad \text{for } n \text{ even}$$

where $\partial\iota_{2n+1}$ is a generator of $\pi_{2n}(U(n))$. Then we obtain that

$$\bar{\partial}j_n(\iota_{2n+1}) = \eta_{2n-1} \quad \text{for } n \text{ even} \tag{2.6}$$

$$\bar{\partial}j_n(\iota_{2n+1}) = 0 \qquad \text{for } n \text{ odd}$$

and for the boundary homomorphism $\bar{\partial}$, we have the formula

$$\bar{\partial}j_n(\alpha\circ E\beta) = ((\bar{\partial}j_n)(\alpha))\circ\beta \tag{2.7}$$

where E ia a suspension homomorphism.

§ 3. Calculations.

Let $1 \leqslant i \leqslant 2n+1$. Then

$$\pi_{2n+i}(Sp(n)) = 0$$

for $2n+i \equiv 0, 1, 2, 6 \bmod 8$. Hence, from (2.1),

$$\pi_{2n+i}(Z_n) \cong \pi_{2n+i-1}(U(n)) \tag{3.1}$$

for $2n+i \equiv 0, 1, 2, 6 \bmod 8$ and $1 \leqslant i \leqslant 2n+1$.

From (2.5) it follows that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{8n+4}(Sp(4n+1)) & \longrightarrow & \pi_{8n+4}(Z_{4n+1}) & \longrightarrow & \pi_{8n+3}(U(4n+1)) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle i'} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \pi_{8n+4}(Sp(4n+1-k)) & \longrightarrow & \pi_{8n+4}(Z_{4n+1-k}) & \longrightarrow & \pi_{8n+3}(U(4n+1-k)) & \longrightarrow & 0 \end{array}$$

is commutative. i' is an isomorphism for $4k \leqslant 8n+1$. Because of commutativity in the above diagram, it follows that lower sequence is a split extension if the upper is. The sequence splits trivially, since $\pi_{8n+3}(U(4n+1))=0$. Thus

$$\pi_{8n+4}(Z_{4n+1-k}) \cong Z_2 + \pi_{8n+3}(U(4n+1-k)) \tag{3.2}$$

for $4k \leqslant 8n+1$.

Consider the exact sequence

$$\pi_{8n+6}(Z_{4n+3}) \longrightarrow \pi_{8n+5}(S^{8n+5}) \longrightarrow \pi_{8n+5}(Z_{4n+2}) \longrightarrow \pi_{8n+5}(Z_{4n+3})$$

of (2. 3) where $\pi_{8n+6}(Z_{4n+3}) \cong Z$, $\pi_{8n+5}(Z_{4n+3})=0$ and $\pi_{8n+5}(S^{8n+5}) \cong Z$. Thus, from the exactness of the sequence,

(3. 3) $\pi_{8n+5}(Z_{4n+2})$ is a cyclic group.

From (2. 5), we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{8n+5}(Sp(4n+2)) & \longrightarrow & \pi_{8n+5}(Z_{4n+2}) & \longrightarrow & \pi_{8n+4}(U(4n+2)) & \longrightarrow & 0 \\
 & & \uparrow i' & & \uparrow r_{4n+1} & & \uparrow i_{4n+1} & & \\
0 & \longrightarrow & \pi_{8n+5}(Sp(4n+1)) & \longrightarrow & \pi_{8n+5}(Z_{4n+1}) & \longrightarrow & \pi_{8n+4}(U(4n+1)) & \longrightarrow & 0 \\
 & & \uparrow i' & & \uparrow r_{4n} & & \uparrow i_{4n} & & \\
0 & \longrightarrow & \pi_{8n+5}(Sp(4n)) & \longrightarrow & \pi_{8n+5}(Z_{4n}) & \longrightarrow & \pi_{8n+4}(U(4n)) & \longrightarrow & 0 \\
 & & \uparrow i' & & \uparrow r_{4n-1} & & \uparrow i_{4n-1} & & \\
0 & \longrightarrow & \pi_{8n+5}(Sp(4n-1)) & \longrightarrow & \pi_{8n+5}(Z_{4n-1}) & \longrightarrow & \pi_{8n+4}(U(4n-1)) & \longrightarrow & 0
\end{array}$$

where i' are isomorphisms for $n \geqslant 1$. From [3], i_{4n+1} is a monomorphism and from [4], i_{4n}, i_{4n-1} are monomorphisms. Hence, from the five lemma, it follows that the homomorphism $r_{4n+i}\,;\ \pi_{4n+5}(Z_{4n+i}) \longrightarrow \pi_{4n+5}(Z_{4n+1+i})$ $(i=1,\,0,\,-1)$ is a monomorphism. Since a subgroup of a cyclic group is cyclic, we have that $\pi_{8n+5}(Z_{4n+2-i})$ $(i=0,\,1,\,2,\,3)$ is a cyclic group.

Now let $O(8n+4,\,4n+2-i)$ be the order of the cyclic group $\pi_{8n+4}(U(4n+2-i))$ for $0 \leqslant i \leqslant 3$. From the exact sequence

$$0 \longrightarrow \pi_{8n+5}(Sp(4n+i)) \longrightarrow \pi_{8n+5}(Z_{4n+i}) \longrightarrow \pi_{8n+4}(U(4n+i)) \longrightarrow 0$$

of (2. 1) and $\pi_{8n+5}(Sp(4n+i)) \cong Z_2$ for $-1 \leqslant i \leqslant 2$,

(3. 4) the group $\pi_{8n+5}(Z_{4n+2-i})$ is a cyclic group of order $2 \times O(8n+4, 4n+2-i)$ for $n \geqslant 1$, $0 \leqslant i \leqslant 3$.

Consider the exact sequence

$$\pi_{8n+8}(Z_{4n+4}) \longrightarrow \pi_{8n+7}(S^{8n+7}) \longrightarrow \pi_{8n+7}(Z_{4n+3}) \longrightarrow \pi_{8n+7}(Z_{4n+4})$$

of (2. 3) where $\pi_{8n+7}(Z_{4n+4}) = 0 = \pi_{8n+8}(Z_{4n+4})$. Thus

(3. 5) $$\pi_{8n+7}(Z_{4n+3}) \cong Z.$$

Consider the diagram

$$\begin{array}{ccccccc} & & \pi_{8n+6}(S^{8n+3}) & & & & \\ & & \uparrow \bar{\partial} & & & & \\ \pi_{8n+7}(S^{8n+5}) & \xrightarrow{j} & \pi_{8n+7}(Z_{4n+2}) & \longrightarrow & \pi_{8n+7}(Z_{4n+3}) & \longrightarrow & \pi_{8n+6}(S^{8n+5}) \end{array}$$

with exact row. From (2. 6) and (2. 7),

(3. 6) $$\bar{\partial} j(\eta^2{}_{8n+5}) = \bar{\partial} j(\iota_{8u+5})\eta^2{}_{8n+4} = \eta^3{}_{8n+3} = 12\nu_{8n+3} \neq 0.$$

Hence $j : \pi_{8n+7}(S^{8n+5}) \cong Z_2 \longrightarrow \pi_{8n+7}(Z_{4n+2})$ is a monomorphism. Thus, from the exactness of the above sequence,

(3. 7) $$\pi_{8n+7}(Z_{4n+2}) \cong Z + Z_2$$

where Z_2 is generated by $j(\eta^2{}_{8n+5})$.

From the exact sequence

$$0 = \pi_{8n+7}(S^{8n+3}) \longrightarrow \pi_{8n+7}(Z_{4n+1}) \longrightarrow \pi_{8n+7}(Z_{4n+2}) \xrightarrow{\bar{\partial}} \pi_{8n+6}(S^{8n+3})$$

and (3. 6), (3. 7), we obtain that

(3. 8) $$\pi_{8n+7}(Z_{4n+1}) \cong Z.$$

From (2. 5), it follows that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{8n+7}(Sp(4n+1)) & \longrightarrow & \pi_{8n+7}(Z_{4n+1}) & \longrightarrow & \pi_{8n+6}(U(4n+1)) & \longrightarrow & 0 \\ & & \uparrow i' & & \uparrow r_{4n} & & \uparrow i_{4n} & & \\ 0 & \longrightarrow & \pi_{8n+7}(Sp(4n)) & \longrightarrow & \pi_{8n+7}(Z_{4n}) & \longrightarrow & \pi_{8n+6}(U(4n)) & \longrightarrow & 0 \end{array}$$

is commutative. i' is an isomorphism for $i \geqslant 1$. From [5], i_{4n} is the split epimorphism and a kernel of i_{4n} is isomorphic to Z_2. From lemma 3. 6 of [7], r_{4n} is the split epimorphism and the kernel of r_{4n} is isomorphic to Z_2. Thus

(3. 9) $$\pi_{8n+7}(Z_{4n}) \cong Z + Z_2.$$

Consider the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \pi_{8n+3}(U) & \longrightarrow & \pi_{8n+3}(Sp) & \longrightarrow & \pi_{8n+3}(Z) & \longrightarrow & 0 \\
 & & & & \uparrow i' & & \uparrow r & & \\
 & & 0 & \longrightarrow & \pi_{8n+3}(Sp(4n+1)) & \xrightarrow{p} & \pi_{8n+3}(Z_{4n+1}) & \longrightarrow & \pi_{8n+2}(U(4n+1)) \longrightarrow 0 \\
 & & & & & & \uparrow j & & \\
 & & & & & & \pi_{8n+3}(S^{8n+3}) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

where rows, column are exact and i' is an isomorphism. From the exactness of the column sequence, the group $\pi_{8n+3}(Z_{4n+1})$ is either Z or $Z+Z_2$. From the commutativity of the above diagram, $\pi_{8n+3}(Z_{4n+1})$ must be $Z+Z_2$. Hence

$$\pi_{8n+3}(Z_{4n+1}) \cong Z+Z_2. \tag{3.10}$$

Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{8n+3}(Sp(4n+1)) & \longrightarrow & \pi_{8n+3}(Z_{4n+1}) & \longrightarrow & \pi_{8n+2}(U(4n+1)) & \longrightarrow & 0 \\
 & & \uparrow i' & & \uparrow r_{4n} & & \uparrow i_{4n} & & \\
0 & \longrightarrow & \pi_{8n+3}(Sp(4n)) & \longrightarrow & \pi_{8n+3}(Z_{4n}) & \longrightarrow & \pi_{8n+2}(U(4n)) & \longrightarrow & 0
\end{array}
$$

of (2.5) where i' is an isomorphism for $n \geqslant 1$. i_{4n} is the split epimorphism and its kernel is isomorphic to Z_2. Thus from lemma 3.6 of [7], r_{4n} is the split epimorphism and its kernel is isomorphic to Z_2. Hence

$$\pi_{8n+3}(Z_{4n}) \cong Z+Z_2+Z_2. \tag{3.11}$$

Consider the exact sequence

$$
\begin{array}{ccccccc}
0=\pi_{8n+3}(S^{8n-1}) & \longrightarrow & \pi_{8n+3}(Z_{4n-1}) & \longrightarrow & \pi_{8n+3}(Z_{4n}) & \xrightarrow{\bar{\partial}} & \pi_{8n+2}(S^{8n-1}) \\
 & & & & \uparrow j & & \\
 & & & & \pi_{8n+3}(S^{8n+1}). & &
\end{array}
$$

From (2.6) and (2.7),

$$\bar{\partial}\, j(\eta^2_{8n+1}) = (\bar{\partial} j)(\iota_{8n+1})\eta^2_{8n+1} = \eta^3_{8n-1} = 12\nu_{8n-1} \neq 0.$$

Hence from the exactness, we have

$$\pi_{8n+3}(Z_{4n-1}) \cong Z+Z_2. \tag{3.12}$$

From (2. 5), the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{8n+3}(Sp(4n-1)) & \longrightarrow & \pi_{8n+3}(Z_{4n-1}) & \longrightarrow & \pi_{8n+2}(U(4n-1)) & \longrightarrow & 0 \\ & & \uparrow i' & & \uparrow r_{4n-2} & & \uparrow i_{4n-2} & & \\ 0 & \longrightarrow & \pi_{8n+3}(Sp(4n-2)) & \longrightarrow & \pi_{8n+3}(Z_{4n-2}) & \longrightarrow & \pi_{8n+3}(U(4n-2)) & \longrightarrow & 0 \end{array}$$

is commutative where i' is an isomorphism for $n \geqslant 2$. Since i_{4n-2} is an isomorphism, r_{4n-2} is so. Thus

(3. 13) $$\pi_{8n+3}(Z_{4n-2}) \cong Z + Z_2$$

for $n \geqslant 2$.

References

1. R. BOTT, The stable homotopy of the classical groups, Ann. of Math., 70 (1956) 313–337.
2. B. HARRIS, On the homotopy groups of the classical groups, Ann. of Math., 74 (1961) 407–413.
3. M. KERVAIRE, Some non-stable homotopy groups of Lie groups, III. J. Math., 4 (1960) 161–169.
4. H. MATSUNAGA, The homotopy groups $\pi_{2n+i}(U(n))$ for $i = 3$, 4 and 5, Mem. Fac. Sci. Kyushu Univ., 15 (1961) 72–80.
5. ———, Applications of functional cohomology operations to the calculus of $\pi_{2n+i}(U(n))$ for $i=6$ and 7, Mem. Fac. Sci. Kyushu Univ., 17 (1963) 29–62.
6. ———, Corrections of the preceding paper and note on the James number, Memo. Fac. Sci. Kyushu Univ., 16 (1962) 60–61
7. H. KACHI, Homotopy groups of homogeneous space $SU(n)/SO(n)$, J. Fac. Sci. Shinshu Univ., 13 (1978) 27–23