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On the $\mathfrak{S}_n$-Equivariant Self Homotopy Equivalences of Spheres

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§ 0. Introduction.

Let G be a finite group, V its complex representation with $dim_{\mathbf{R}}V^G \geqq 2$, and $S(V)$ the unit sphere in V. Let $[S(V), S(V)]_G$ be the ring of all G-equivariant homotopy classes of G-equivariant maps of $S(V)$ into itself. Let $\boldsymbol{E}_G(S(V))$ be the multiplicative group of the ring $[S(V), S(V)]_G$.

In the previous paper [1], we have determined the order of the group $\boldsymbol{E}_G(S(V))$ for G to be any finite abelian group and the dihedoral group D_n, in conformity with the set of orbit types on $S(V)$. Let $\mathfrak{S}_n$ be the group of all permutations of the n elements set $F(n)=\{1, \cdots, n\}$. Let us abbreviate $\boldsymbol{E}\mathfrak{S}_n(S(V))$ to $\boldsymbol{E}_n(V)$. In this paper we shall prove the following

Theorem A. *We have*

$$|\boldsymbol{E}_n(V)| = 4 \quad if \quad \boldsymbol{C}_n = \boldsymbol{O}(V)_n \ (n \geqq 2),$$

where $\boldsymbol{C}_n$ and $\boldsymbol{O}(V)_n$ are defined in § 1.

§ 1. Notations.

Throughout this paper, we use the following notations:

$(\mathfrak{S}_n)z$	the isotropy subgroup at a point z of an $\mathfrak{S}_n$-set X,
(H)	the conjugacy class of a subgroup H of a group G,
$Car. X$	the cardinal number of a set X,
X^G	the set of the fixed points of a G-set X,
$F(n)$	the set $\{1, \cdots, n\}$ for an integer n,
e	the unit element of $\mathfrak{S}_n$,
Ω_n	the set $\{\boldsymbol{F} \mid \boldsymbol{F}=\{M_i\},\ M_i \subset F(n),\ Car. M_i \geqq 2$ *and* $M_i \cap M_j = \phi$ if $i \neq j\} \cup \{F(1)\}$
$\mathfrak{S}_n(M)$	the subgroup $\{\sigma \mid \sigma \in \mathfrak{S}_n$ and $\sigma(i)=i$ if $i \notin M\}$ for a subset M of $F(n)$, and $\mathfrak{S}_n(M)=e$ if $Car. M \leqq 1$,

$\mathfrak{S}_n(\boldsymbol{F})$	the subgroup $\prod_{M\in F} \mathfrak{S}_n(M)$ for a $\boldsymbol{F} \in \Omega_n$, where we denote by Π the direct product of the groups $\mathfrak{S}_n(M)$,
$\boldsymbol{C}_n$	the set $\{(\mathfrak{S}_n(\boldsymbol{F}))\,\vert\,\boldsymbol{F} \in \Omega_n\}$,
$\boldsymbol{O}(V)_n$	the set $\{(H)\,\vert\,H$ is a subgroup of $\mathfrak{S}_n$ such that $(H)=((\mathfrak{S}_n)z)$ for an element z of $S(V)\}$,
$\mathbf{Z}$	the ring of rational integers,
R*	the multiplicative group of a ring R,
$A(\mathfrak{S}_n)$	the Burnside ring of $\mathfrak{S}_n$,
$\langle X\rangle$	the element of $A(\mathfrak{S}_n)$ represented by an $\mathfrak{S}_n$-set X,
$X(H)_n$	the left $\mathfrak{S}_n$-set $\mathfrak{S}_n/H$ of left cosets for the subgroup H of $\mathfrak{S}_n$,
$A(C_n)$	the submodule of $A(\mathfrak{S}_n)$ generated by the set $\{\langle X(\mathfrak{S}_n(\boldsymbol{F}))_n\rangle\,\vert\,\boldsymbol{F} \in \Omega_n\}$ (let us abbeviate $X(\mathfrak{S}_n(\boldsymbol{F}))_n$ to $X(\boldsymbol{F})_n$ or $X(\boldsymbol{F})$ and $\langle X(F(n)\rangle$ to 1_n),

and

$A(V)_n$	the subring of $A(\mathfrak{S}_n)$ generated by the set $\{\langle X(H)_n\rangle\,\vert\,(H) \in \boldsymbol{O}(V)_n\}$ (c. f. 2. 3 [1]).

§ 2. Classification of $\mathfrak{S}_n$-equivariant maps.

Theorem 2. 1. *(Theorem 2. 4, Corollary 2. 6 [1] and Theorem 8. 4 [2]) For each $n \geqq 2$, there is a group isomorphism*

$$h_n : \boldsymbol{E}_n(V) \longrightarrow A(V)_n^*$$

such that the diagram

$$\begin{array}{ccc} \boldsymbol{E}_n(V) & \xrightarrow{\quad h_n \quad} & A(V)_n^* \\ & \searrow^{Deg^H} \quad \swarrow_{\chi_H^n} & \\ & Z_2 = \{\pm 1\} & \end{array}$$

commutes for every subgroup H of $\mathfrak{S}_n$, where Deg^H and χ_H^n are the homomorphisms defined by

$$Deg^H([f]) = \text{degree of } f^H(=f\,\vert\,S(V)^H) : S(V)^H \longrightarrow S(V)^H$$

and

$$\chi_H^n(\langle X\rangle) = Car.\ X^H.$$

Moreover, two $\mathfrak{S}_n$-equivariant maps

(2. 1. 1) *$f_0,\ f_1 : S(V) \longrightarrow S(V)$ are $\mathfrak{S}_n$-homotopic if and only if $Deg^H([f_0]) = Deg^H([f_1])$ for all $(H) \in \boldsymbol{O}(V)_n$.*

2. 2. We define an $\mathfrak{S}_n$-action on $\mathbf{C}^n$ (which is a complex representation V_0 of $\mathfrak{S}_n$ with $\dim_{\mathbf{R}} V_0{}^G = 2$) by permutation of coordinates. For a cyclic permutation $\sigma = (i_1, \cdots, i_p)$, $\sigma(z_1, \cdots, z_n) = (z_1, \cdots, z_n)$ implies $z_{i_1} = \cdots = z_{i_p}$. So we have

$$O(V_0)_n = C_n, \quad A(V_0)_n = A(C_n) \tag{2. 2. 1}$$

and

$$A(C_n) \text{ is a subring of } A(\mathfrak{S}_n).$$

Natural inclusion map $F(n-1) \subset F(n)$ induce the group homomorphisms

$$\Theta : \mathfrak{S}_{n-1} \longrightarrow \mathfrak{S}_n,$$
$$\Theta : \boldsymbol{E}_n(V_0) \longrightarrow \boldsymbol{E}_{n-1}(V_0),$$

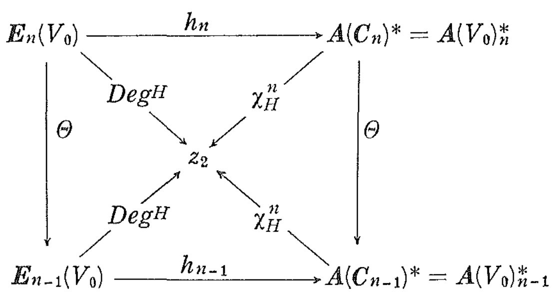
and

$$\Theta : A(\mathfrak{S}_n)^* \longrightarrow A(\mathfrak{S}_{n-1})^*$$

(we use the same notation Θ when there arises no conffusion). Then we have

$$\Theta(A(C_n)^*) \subset A(C_{n-1})^*, \tag{2. 2. 2}$$

and the diagram



commutes for every subgroup H of $\mathfrak{S}_{n-1}$. Therefore, to prove the Theorem A, we can assume that $V = V_0$.

§ 3. Conjugate subgroups of $\mathfrak{S}_n$.

3. 1. We denote by

Ω^r_n the set $\{F \mid F \in \Omega_n \text{ and } \sum_{M \in F} Car.\, M = r\}$ for $2 \leqq r \leqq n-1$,

Ω^n_n the set $\{F \mid F \in \Omega_n,\ \sum_{M \in F} Car.\, M = n \text{ and } F \neq F(n)\}$

$\tilde{\Omega}^r_n(p)$ the set $\{F \mid F \in \Omega^r_n,\ F = \{M_i\}_{i=1,.,p},\ s < t$ for any $s \in M_i$ and $t \in M_j$ if $i < j$, $2 \leqq Car.\, M_1 \cdots \leqq Car.\, M_p$ and $\bigcup_i M_i = F(r)\}$ for $2 \leqq r \leqq n$,

$\widetilde{\Omega}_n^1$ the set $\{F(1)\}$,

$\widetilde{\Omega}_n^r$ the set $\bigcup_p \widetilde{\Omega}_n^r(p)$,

$\widetilde{\Omega}_n$ the set $\bigcup_r \widetilde{\Omega}_n^r \cup \{F(n)\}$,

and

$\widetilde{C}_n$ the set $\{(\mathfrak{S}_n(F)) \mid F \in \widetilde{\Omega}_n\}$.

Lemma 3.2. *For each element $F \in \Omega_n$,* we put

$$L(F) = (Car.\ M_1,\ \cdots,\ Car.\ M_p),$$

where $F = \{M_i\}_{i=1,\cdots,p}$ and $Car.\ M_1 \leqq \cdots \leqq Car.\ M_p$. Then we have

$$\sigma\mathfrak{S}_n(M)\sigma^{-1} = \mathfrak{S}_n(\sigma(M)) \text{ for each } \sigma \in \mathfrak{S}_n \text{ and } M \subset F(n), \tag{3.2.1}$$

and

(3.2.2) $(\mathfrak{S}_n(F_1)) = (\mathfrak{S}_n(F_2))$ *if and only if*
$L(F_1) = L(F_2)$ *for arbitrary two elements*
$F_i \in \Omega_n$ $(i = 1, 2)$.

Proof. For each $j \notin \sigma(M)$ and $\tau \in \mathfrak{S}_n(M)$, $\sigma\tau\sigma^{-1}(j) = j$. So we have (3.2.1). Let $F_i = \{M_j(i)\}$ $(i = 1, 2)$. If $L(F_1) = L(F_2)$, then there is an integer $a(j)$ for each j such that $a(j) \neq a(j')$ if $j \neq j'$, and $\sigma(M_j(1)) = M_{a(j)}(2)$ for some $\sigma \in \mathfrak{S}_n$. Therefore we have

$$\sigma\mathfrak{S}_n(F_1)\sigma^{-1} = \mathfrak{S}_n(F_2)$$

by (3.2.1). Conversely if $(\mathfrak{S}_n(F_1)) = (\mathfrak{S}_n(F_2))$, then $\mathfrak{S}_n(\sigma(F_1)) = \mathfrak{S}_n(F_2)$ for some $\sigma \in \mathfrak{S}_n$, and $L(F_1) = L(F_2)$. *Q.E.D.*

Lemma 3.3. *We have*

$$\mathfrak{S}_n(F_1) \cap \mathfrak{S}_n(F_2) = \mathfrak{S}_n(F_1 \cap F_2),$$

where $F_i \in \Omega_n$ $(i = 1, 2)$ and $F_1 \cap F_2 = \{M \cap M' \mid Car.\,(M \cap M') \geqq 2,\ M \in F_1$ and $M' \in F_2\}$.

Proof. We have

$$\mathfrak{S}_n(M \cap M') = \begin{cases} e & \text{if } Car.\,(M \cap M') \leqq 1 \\ \mathfrak{S}_n(M) \cap \mathfrak{S}_n(M') & \text{otherwise,} \end{cases}$$

so the desired result follows at once. *Q.E.D.*

Corollary 3.4. *We have*

$$C_n = \widetilde{C}_n$$

and

$$(\mathfrak{S}_n(F_1)) \neq (\mathfrak{S}_n(F_2))$$

for distinct $F_1, F_2 \in \widetilde{\Omega}_n$.

Proof. It is trivial from Lemma 3.2 and the definition of $\widetilde{\Omega}_n$. *Q. E. D.*

We define a partial order on C_n by setting

$$(\mathfrak{S}_n(F_1)) \leqq (\mathfrak{S}_n(F_2))$$

if and only if $\mathfrak{S}_n(F_1)$ is conjugate to a subgroup of $\mathfrak{S}_n(F_2)$. Then we have the following Lemmas 3.5–3.6.

Lemma 3.5. *Let* $F_i = \{M_j(i)\}_{j=1,\cdots,p_i} \in \widetilde{\Omega}^r_n(p_i)$ $(i = 1, 2)$, *then*

$$(\mathfrak{S}_n(F_1)) \leqq (\mathfrak{S}_n(F_2))$$

if and only if there is a subset $j^{\#} \subset F(p_1)$ for each $j \in F(p_2)$ such that

$$\sum_{i \in j^{\#}} Car.\ M_i(1) = Car.\ M_j(2),$$

$$\bigcup_{j \in F(p_2)} j^{\#} = F(p_1),$$

and

$$j_1^{\#} \cap j_2^{\#} = \phi \qquad if \quad j_1 \neq j_2.$$

Proof. It is trivial from (3.2.1) *Q. E. D.*

Lemma 3.6. *Let* $F_i = \{M_j(i)\}_{j=1,\cdots,p_i} \in \widetilde{\Omega}^{r_i}_n(p_i)$ $(i = 1, 2)$, then

$$(\mathfrak{S}_n(F_1)) \not\leqq (\mathfrak{S}_n(F_2))$$

if one of the following three conditions is satisfied:

(3.6.1) $\qquad r_1 > r_2,$

(3.6.2) $\qquad r_1 = r_2 \quad and \quad p_1 < p_2,$

and

(3.6.3) $\qquad r_1 = r_2, \ p_1 = p_2 \quad and \quad F_1 \neq F_2.$

Proof.

: ***In the case*** (3.6.1) : *If* $\sigma^{-1}\mathfrak{S}_n(F_1)\sigma \subset \mathfrak{S}_n(F_2)$ for some $\sigma \in \mathfrak{S}_n$, then $(\sigma^{-1}\mathfrak{S}_n(F_1)\sigma)(k) = k$ for each $k \in \{r_2+1, \cdots, n\}$. *If* $(\mathfrak{S}_n(F_1))(k) = k$, then $k \in \{r_1+1, \cdots, n\}$. *So*

$\sigma(\{r_2+1, \cdots, n\}) \subset \{r_1+1, \cdots, n\}$ and $r_2 \geqq r_1$. Therefore this contradiction establishes the result.

In the cases (3. 6. 2)–(3. 6. 3). It is trivial from *Lemma* 3. 5. *Q. E. D.*

§ 4. On the groups $A(C_n)^*$.

4. 1. For an $\mathfrak{S}_n$-set X, we have the following formula:

$$\langle X\rangle = \Sigma\lambda_i\langle X(H_i)_n\rangle, \tag{4.1.1}$$

where $\lambda_i = Car.\ \{a \mid z \in a \in X/\mathfrak{S}_n$ and $((\mathfrak{S}_n)z) = (H_i)\}$, and

$$\Sigma\lambda_i\, Car.\, X(H_i)_n = Car.\, X \text{ (c. f., (2. 1. 1) [1]).}$$

From Lemma 2. 7 [1], we have

$$\Delta^2 = 1_n \quad \text{if } \Delta \text{ is an element of } A(C_n)^*. \tag{4.1.2}$$

4. 2. We denote by

$$F(Car.\, M_1, \cdots, Car.\, M_p)$$

the element of $\boldsymbol{F} = \{M_i\}_{i=1,\cdots,p} \in \widetilde{\Omega}_n^r(p)$, *and* $m_{(i,k)}$ the element $(m_i, m_i, \cdots, m_i) \in (\mathbf{Z}^+)^k$.

Let $\boldsymbol{F} = F(m_{(1,k_1)}, \cdots, m_{(q,k_q)})$ be an element of $\widetilde{\Omega}_n^r(p)$ such that

$$\sum_{i=1}^{q} m_i k_i = r \leqq n, \quad \sum_{i=1}^{q} k_i = p \quad \text{and} \quad m_i < m_j \quad \text{if} \quad i<j,$$

then we define the following maps

$$\phi_j : \widetilde{\Omega}_n^r \longrightarrow \widetilde{\Omega}_n^r \cup \widetilde{\Omega}_n^{r-1} \cup \widetilde{\Omega}_n^{r-2}$$
$$(1 \leqq j \leqq p+n-r)$$

by setting

$$\phi_j(\boldsymbol{F}) = \begin{cases} F(m_{(1,k_1-1)}, m_{(2,k_2)}, \cdots, m_{(q,k_q)}) \text{ if } j \leqq k_1 \text{ and } m_1 = 2, \\ F(m_1-1,\ m_{(1,k_1-1)},\ m_{(2,k_2)}\cdots,\ m_{(q,k_q)}) \text{ if } j \leqq k_1 \text{ and } m_1 > 2, \\ F(m_{(1,k_1)}, \cdots, m_s - 1, m_{(s,k_s-1)}, \cdots, m_{(q,k_q)}) \\ \qquad\qquad \text{if } \sum_{i=1}^{s-1} k_i < j \leqq \sum_{i=1}^{s} k_i \text{ and } s > 1, \\ \boldsymbol{F} \qquad\qquad \text{if } p < j \leqq (p+n-r). \end{cases}$$

For example, we have

$$\phi_j(F(2,2,2,3,3,4)=\begin{cases}F(2,2,3,3,4) & \text{if } 1\leqq j\leqq 3,\\ F(2,2,2,2,3,4) & \text{if } 4\leqq j\leqq 5,\\ F(2,2,2,3,3,3) & \text{if } j=6,\\ F(2,2,2,3,3,4) & \text{if } j>6.\end{cases}$$

Lemma 4.3. *We have the following formula :*

$$\Theta(\langle X(F)_n\rangle)=\begin{cases}(n-r)\langle X(F)_{n-1}\rangle+\sum_{i=1}^{q}k_i\langle X(\phi_{j(i)}(F))_{n-1}\rangle & if\ r\geqq 2\ ;\\ n\langle X(F(1))_{n-1}\rangle & if\ \mathrm{r}=1,\end{cases}$$

where $F_i=\{M_i\}_{i=1,\cdots,p}=F(m_{(1,k_1)},\ \cdots,\ m_{(q,k_q)})\in\tilde{\Omega}_n^r(p)$ and $j(i)=\sum_{s=1}^{i}k_s$.

Proof. Let σ_j $(j=1,\cdots,p)$ be the elements of $\mathfrak{S}_n$ with $\sigma_j^{-1}(n)\in M_j$ for each j. We put

$$z_j=\sigma_j\mathfrak{S}_n(F)\in X(F)_n,$$

then we have

$$(\mathfrak{S}_{n-1})z_j=\sigma_j\mathfrak{S}_n(F)\sigma_j^{-1}\cap\mathfrak{S}_{n-1}$$

and

$$((\mathfrak{S}_{n-1})_{z_j})=(\mathfrak{S}_{n-1}(\phi_j(F)))$$

by Lemmas 3.2–3.3. Let τ_j $(j=1,\cdots,p)$ be other elements of $\mathfrak{S}_n$ with $\tau_j^{-1}(n)\in M_j$. Then

$$\sigma_j\alpha_j\tau_j^{-1}\in\mathfrak{S}_{n-1}$$

for some $\alpha_j^{-1}\in\mathfrak{S}_n(M_j)$, so we have

$$(4.3.1)\qquad [\sigma_j\mathfrak{S}_n(F)]=[\sigma_j\alpha_j\mathfrak{S}_n(F)]=[\tau_j\mathfrak{S}_n(F)]\quad \text{in } X(F)/\mathfrak{S}_{n-1}.$$

If $\alpha\sigma_j\beta=\sigma_i\gamma$ for some $\alpha\in\mathfrak{S}_{n-1}$ and $\beta,\ \gamma\in\mathfrak{S}_n(F)$, then $n=\alpha\sigma_j\beta\gamma^{-1}\sigma_i{}^{-1}(n)\in\alpha\sigma_j(M_i)$. If $i\neq j$, then $n\notin\sigma_j(M_i)$ and $n\notin\alpha\sigma_j(M_i)$. Therefore we have

$$(4.3.2)\qquad [\sigma_j\mathfrak{S}_n(F)]\neq[\sigma_i\mathfrak{S}_n(F)]\quad \text{in } X(F)_n/\mathfrak{S}_{n-1}\ \text{if } i\neq j.$$

Let σ and τ be the elements of $\mathfrak{S}_n$ such that $\sigma^{-1}(n)>r$ and $\tau^{-1}(n)>r$, then by the same way as in the proofs of (4.3.1) and (4.3.2), we have

$$(4.3.3)\qquad [\sigma\mathfrak{S}_n(F)]=[\tau\mathfrak{S}_n(F)]\ \text{in } X(F)_n/\mathfrak{S}_{n-1}\ \text{if and only if } \sigma^{-1}(n)=\tau^{-1}(n).$$

From definition, we have

$$\phi_i(F) = \phi_j(F) \quad \text{if and only if} \quad Car.\, M_i = Car.\, M_j,$$

so the desired result follows from (4. 3. 1)–(4. 3. 3) and (4. 1. 1). *Q. E. D.*

For example, we have

$$\begin{aligned}\Theta(\langle X(F(2,2,2,3,3,4)_{20}\rangle) = & \ 4\langle X(F(2,2,2,3,3,4))_{19}\rangle \\ & + 3\langle X(F(2,2,3,3,4))_{19}\rangle \\ & + 2\langle X(F(2,2,2,2,3,4))_{19}\rangle \\ & + \langle X(F(2,2,2,3,3,3))_{19}\rangle.\end{aligned}$$

4. 4. We denote by

$$X(F_1,\ F_2)_n$$

the $\mathfrak{S}_n$–set $X(F_1)_n \times X(F_2)_n$ with diagonal $\mathfrak{S}_n$–action. We denote by

$$\lambda(F\ ;\ \Delta)$$

the coefficient of $\langle X(F)_n\rangle$ in Δ for each $F \in \tilde{\Omega}_n$ and $\Delta \in A(C_n)$. Let us abbreviate $\lambda(F\ ;\ \langle X(F_1, F_2)_n\rangle)$ to $\lambda(F : F_1, F_2)_n$ or $\lambda(F\ ;\ F_1,\ F_2)$, when there arises no confussion.

For each $F = \{M_i\}_{i=1,\cdots,p} \in \tilde{\Omega}_n^r(p)$, we consider the following five cases :

(4. 4. 1) $Car.\, M_i \neq Car.\, M_j$ if $i \neq j$,

(4. 4. 2) there exists only one integer i_0 such that $Car.\, M_{i_0} = Car.\, M_{i_0+1}$,

(4. 4. 3) there exist only two integers i_0 and i_1 such that $Car.\, M_{i_0} = Car.\, M_{i_0+1} < Car.\, M_{i_1} = Car.\, M_{i_1+1}$,

(4. 4. 4) there exists only one integer i_0 such that $Car.\, M_{i_0} = Car.\, M_{i_0+1} = Car.\, M_{i_0+2}$,

and

(4. 4. 5) other cases.

Lemma 4. 5. *For each $F \in \tilde{\Omega}_n^r$, we have*

$$\lambda(F\ ;\ F, F) = \begin{cases} (n-r)! & if\ (4.4.1), \\ 2(n-r)! & if\ (4.4.2), \\ 4(n-r)! & if\ (4.4.3), \\ 6(n-r)! & if\ (4.4.4), \\ \geqq 8(n-r)! & if\ (4.4.5). \end{cases}$$

Proof. For $\sigma \in \mathfrak{S}_n$, let $Z(\sigma)$ be an element $(\sigma\mathfrak{S}_n(F),\ \mathfrak{S}_n(F))$ of $X(F, F)_n$ such that

$$((\mathfrak{S}_n)_{Z(\sigma)}) = (\mathfrak{S}_n(F)).$$

Since $(\mathfrak{S}_n)_{z(\sigma)} = \mathfrak{S}_n(\sigma(\boldsymbol{F}) \cap \boldsymbol{F}) \subset \mathfrak{S}_n(\boldsymbol{F})$, so we have $\mathfrak{S}_n(\sigma(\boldsymbol{F})) = \mathfrak{S}_n(\boldsymbol{F})$. Then there exists an integer $a(i, \sigma)$ for each i such that

$$\sigma(M_i) = M_{a(i,\sigma)},$$

where $\boldsymbol{F} = \{M_i\}_{i=1,\cdots,p}$. Let τ be another element of $\mathfrak{S}_n$ such that

$$((\mathfrak{S}_n)_{z(\tau)}) = (\mathfrak{S}_n(\boldsymbol{F})),$$

then we have

(4. 5. 1) $$\begin{aligned}&[z(\sigma)] \neq [z(\tau)] \quad \text{in} \quad X(\boldsymbol{F}, \boldsymbol{F})_n/\mathfrak{S}_n\\ &\text{if either } a(i, \sigma) \neq a(i, \tau) \quad \text{for some } i\\ &\text{or} \quad \sigma(j) \neq \tau(j) \quad \text{for some} \quad j > r.\end{aligned}$$

Therfore the desired result follows from (4. 1. 1) and (4. 5. 1). *Q. E. D.*

Lemma 4. 6. *For three elements* $\boldsymbol{F}_i \in \tilde{\Omega}^r_n(p_i)$ $(i = 1, 2, 3)$, *we have*

$$\lambda(\boldsymbol{F}_1 ; \boldsymbol{F}_2, \boldsymbol{F}_3) = 0$$

if either $(\mathfrak{S}_n(\boldsymbol{F}_1)) \not\leqq (\mathfrak{S}_n(\boldsymbol{F}_2))$ or $(\mathfrak{S}_n(\boldsymbol{F}_1)) \not\leqq (\mathfrak{S}_n(\boldsymbol{F}_3))$.

Proof. Let H be the isotropy subgroup at a point z of $X(\boldsymbol{F}_2, \boldsymbol{F}_3)_{n'}$ then we have

$$(H) \leqq (\mathfrak{S}_n(\boldsymbol{F}_2)) \quad \text{and} \quad (H) \leqq (\mathfrak{S}_n(\boldsymbol{F}_3)).$$

So the desired result follows from (4. 1. 1). *Q. E. D.*

Lemma 4. 7. *We have the following equalites :*

(4. 7. 1)
$$\begin{aligned}
&\lambda(F(2,2,n-4) ; F(2,2,n-4),\ F(2,n-2)) &&= \begin{cases}3 & \text{if } n=6\\ 2 & \text{if } n>6,\end{cases}\\
&\lambda(F(2,2,n-4) ; F(2,2,n-4),\ F(3,n-3)) &&= \begin{cases}1 & \text{if } n=7\\ 0 & \text{if } n>7,\end{cases}\\
&\lambda(F(2,2,n-4) ; F(2,2,n-4),\ F(4,n-4)) &&= \begin{cases}2 & \text{if } n=8\\ 1 & \text{if } n>8,\end{cases}\\
&\lambda(F(2,2,n-4) ; F(2,n-2),\ F(3,n-3)) &&= \begin{cases}1 & \text{if } n=7\\ 0 & \text{if } n \geqq 6 \ (n \neq 7),\end{cases}\\
&\lambda(F(2,2,n-4) ; F(2,n-2),\ F(4,n-4)) &&= \begin{cases}2 & \text{if } n=8\\ 1 & \text{if } n>8,\end{cases}\\
&\lambda(F(2,2,n-4) ; F(3,n-3),\ F(4,n-4)) &&= 0 \quad \text{if } n \geqq 8,\\
&\lambda(F(2,2,n-4) ; F(2,2,n-4),\ F(2,2,n-4)) &&= \begin{cases}6 & \text{if } n=6\\ 2 & \text{if } n>6,\end{cases}\\
&\lambda(F(2,2,n-4) ; F(2,n-2),\ F(2,n-2)) &&= 1 \quad \text{if } n \geqq 6,
\end{aligned}$$

$$\lambda(F(2,2,n-4)\,;\,F(3,n-3),\ F(3,n-3)) = 0 \quad if\ n\geqq 6,$$

$$\lambda(F(2,2,n-4)\,;\,F(4,n-4),\ F(4,n-4)) = 0 \quad if\ n\geqq 8,$$

(4.7.2) $$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(2,n-2)) = \begin{cases}4 & if\ n=8\\ 3 & if\ n>8,\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(3,n-3)) = \begin{cases}1 & if\ n=9\\ 0 & if\ n\geqq 8\ \ (n\neq 9),\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(4,n-4)) = \begin{cases}6 & if\ n=8\\ 3 & if\ n>8,\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(2,2,n-4)) = \begin{cases}12 & if\ n=8\\ 6 & if\ n>8,\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(5,n-5)) = \begin{cases}1 & if\ n=11\\ 0 & if\ n\geqq 10\ (n\neq 11),\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(2,3,n-5)) = \begin{cases}3 & if\ n=9\\ 0 & if\ n\geqq 8\ \ (n\neq 9),\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(6,n-6)) = \begin{cases}2 & if\ n=12\\ 1 & if\ n>12,\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(2,4,n-6)) = \begin{cases}6 & if\ n=10\\ 3 & if\ n>10,\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(3,3,n-6)) = 0 \quad if\ n\geqq 9,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,2,2,n-6),\ F(2,2,2,n-6)) = \begin{cases}24 & if\ n=8\\ 6 & if\ n>8,\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(3,n-3)) = 0 \quad if\ n\geqq 8,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(4,n-4)) = 0 \quad if\ n\geqq 8,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(2,2,n-4)) = 1 \quad if\ n\geqq 8,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(5,n-5)) = 0 \quad if\ n\geqq 10,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(2,3,n-5)) = \begin{cases}1 & if\ n=9\\ 0 & if\ n\geqq 8\ \ (n\neq 9),\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(6,n-6)) = 0 \quad if\ n\geqq 12,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(2,4,n-6)) = \begin{cases}2 & if\ n=10\\ 1 & if\ n>10,\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(2,n-2),\ F(3,3,n-6)) = 0 \quad if\ n\geqq 9,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(3,n-3),\ F(4,n-4)) = 0 \quad if\ n\geqq 8,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(3,n-3),\ F(2,2,n-4)) = \begin{cases}1 & if\ n=9\\ 0 & if\ n\geqq 8\ \ (n\neq 9),\end{cases}$$

$$\lambda(F(2,2,2,n-6)\,;\,F(3,n-3),\ F(5,n-5)) = 0 \quad if\ n\geqq 10,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(3,n-3),\ F(2,3,n-5)) = 0 \quad if\ n\geqq 9,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(3,n-3),\ F(6,n-6)) = 0 \quad if\ n\geqq 12,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(3,n-3),\ F(3,3,n-6)) = 0 \quad if\ n\geqq 9,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(3,n-3),\ F(2,4,n-6)) = 0 \quad if\ n\geqq 10,$$

$$\lambda(F(2,2,2,n-6)\,;\,F(4,n-4),\ F(2,2,n-4)) = \begin{cases}3 & if\ n=10\\ 2 & if\ n\geqq 8\ (n\neq 10),\end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(4,n-4),\ \ F(5,n-5)) = 0 \qquad if\ n \geqq 10,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(4,n-4),\ \ F(2,3,n-5)) = \begin{cases} 1 & if\ n=9 \\ 0 & if\ n\geqq 8\ \ (n\neq 9), \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(4,n-4),\ \ F(6,n-6)) = 0 \qquad if\ n \geqq 12,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(4,n-4),\ \ F(2,4,n-6)) = \begin{cases} 2 & if\ n=10 \\ 1 & if\ n> 10, \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(4,n-4),\ \ F(3,3,n-6)) = 0 \qquad if\ n \geqq 9,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,2,n-4),\ \ F(5,n-5)) = \begin{cases} 1 & if\ n=11 \\ 0 & if\ n\geqq 10\ (n\neq 11), \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,2,n-4),\ \ F(2,3,n-5)) = \begin{cases} 3 & if\ n=9 \\ 0 & if\ n\geqq 8\ \ (n\neq 9), \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,2,n-4),\ \ F(6,n-6)) = \begin{cases} 2 & if\ n=12 \\ 1 & if\ n> 12, \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,2,n-4),\ \ F(2,4,n-6)) = \begin{cases} 6 & if\ n=10 \\ 2 & if\ n> 10, \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,2,n-4),\ \ F(3,3,n-6)) = 0 \qquad if\ n \geqq 9,$$

$$\lambda(F(2,2,2,n-6)\ ;\ P(5,n-5),\ \ F(6,n-6)) = 0 \qquad if\ n \geqq 12,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(5,n-5),\ \ F(2,4,n-6)) = 0 \qquad if\ n \geqq 10,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(5,n-5),\ \ F(3,3,n-6)) = 0 \qquad if\ n \geqq 10,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(6,n-6),\ \ F(2,4,n-6)) = 0 \qquad if\ n \geqq 12,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(6,n-6),\ \ F(3,3,n-6)) = 0 \qquad if\ n \geqq 12,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,n-2),\ \ F(2,n-2)) = 0 \qquad if\ n \geqq 8,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(4,n-4),\ \ F(4,n-4)) = 0 \qquad if\ n \geqq 8,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,2,n-4),\ \ F(2,2,n-4)) = \begin{cases} 5 & if\ n=8 \\ 4 & if\ n> 8, \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(5,n-5),\ \ F(5,n-5)) = 0 \qquad if\ n \geqq 10,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,3,n-5),\ \ F(2,3,n-5)) = \begin{cases} 1 & if\ n=9 \\ 0 & if\ n\geqq 8\ \ (n\neq 9), \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(6,n-6),\ \ F(6,n-6)) = 0 \qquad if\ n \geqq 12,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,4,n-6),\ \ F(2,4,n-6)) = \begin{cases} 2 & if\ n=10 \\ 1 & if\ n> 10, \end{cases}$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(3,3,n-6),\ \ F(3,3,n-6)) = 0 \qquad if\ n \geqq 9,$$

$$\lambda(F(2,2,2,n-6)\ ;\ F(2,2,2,n-6),\ \ F(2,2,2,n-6)) = \begin{cases} 24 & if\ n=8 \\ 6 & if\ n> 8, \end{cases}$$

(4.7.3) $$\lambda(F(2,4,n-6)\ ;\ F(2,4,n-6),\ \ F(2,n-2)) = 1 \qquad if\ n \geqq 10,$$

$$\lambda(F(2,4,n-6)\ ;\ F(2,4,n-6),\ \ F(4,n-4)) = 1 \qquad if\ n \geqq 10,$$

$$\lambda(F(2,4,n-6)\ ;\ F(2,4,n-6),\ \ F(5,n-5)) = \begin{cases} 1 & if\ n=11 \\ 0 & if\ n\geqq 10\ (n\neq 11), \end{cases}$$

$$\lambda(F(2,4,n-6)\ ;\ F(2,4,n-6),\ \ F(6,n-6)) = \begin{cases} 2 & if\ n=12 \\ 1 & if\ n> 12, \end{cases}$$

$$\lambda(F(2,4,n-6)\ ;\ F(2,n-2),\ \ F(4,n-4)) = 1 \qquad if\ n \geqq 10,$$

$$\lambda(F(2,4,n-6)\,;\,F(2,n-2),\ F(5,n-5)) = \begin{cases} 1 & if\ n=11 \\ 0 & if\ n \geqq 10\ (n \neq 11), \end{cases}$$

$$\lambda(F(2,4,n-6)\,;\,F(2,n-2),\ F(6,n-6)) = \begin{cases} 2 & if\ n=12 \\ 1 & if\ n > 12, \end{cases}$$

$$\lambda(F(2,4,n-6)\,;\,F(4,n-4),\ F(5,n-5)) = \begin{cases} 1 & if\ n=11 \\ 0 & if\ n \geqq 10\ (n \neq 11), \end{cases}$$

$$\lambda(F(2,4,n-6)\,;\,F(4,n-4),\ F(6,n-6)) = \begin{cases} 2 & if\ n=12 \\ 1 & if\ n > 12, \end{cases}$$

$$\lambda(F(2,4,n-6)\,;\,F(5,n-5),\ F(6,n-6)) = 0 \quad if\ n \geqq 12,$$

$$\lambda(F(2,4,n-6)\,;\,F(2,n-2),\ F(2,n-2)) = 0 \quad if\ n = 10,$$

$$\lambda(F(2,4,n-6)\,;\,F(4,n-4),\ F(4,n-4)) = \begin{cases} 1 & if\ n=10 \\ 0 & if\ n > 10, \end{cases}$$

$$\lambda(F(2,4,n-6)\,;\,F(5,n-5),\ F(5,n-5)) = 0 \quad if\ n \geqq 10,$$

$$\lambda(F(2,4,n-6)\,;\,F(6,n-6),\ F(6,n-6)) = 0 \quad if\ n \geqq 12,$$

$$\lambda(F(2,4,n-6)\,;\,F(2,4,n-6),\ F(2,4,n-6)) = \begin{cases} 2 & if\ n=10 \\ 1 & if\ n > 10, \end{cases}$$

(4. 7. 4)

$$\lambda(F(3,3,n-6)\,;\,F(3,3,n-6),\ F(3,n-3)) = \begin{cases} 3 & if\ n=9 \\ 2 & if\ n > 9, \end{cases}$$

$$\lambda(F(3,3,n-6)\,;\,F(3,3,n-6),\ F(4,n-4)) = \begin{cases} 1 & if\ n=10 \\ 0 & if\ n \geqq 9\ (n \neq 10), \end{cases}$$

$$\lambda(F(3,3,n-6)\,;\,F(3,3,n-6),\ F(5,n-5)) = \begin{cases} 1 & if\ n \geqq 11 \\ 0 & if\ n \geqq 10\ (n \neq 11), \end{cases}$$

$$\lambda(F(3,3,n-6)\,;\,F(3,3,n-6),\ F(6,n-6)) = \begin{cases} 2 & if\ n=12 \\ 1 & if\ n > 12, \end{cases}$$

$$\lambda(F(3,3,n-6)\,;\,F(3,n-3),\ F(4,n-4)) = \begin{cases} 1 & if\ n=10 \\ 0 & if\ n \geqq 9\ (n \neq 10), \end{cases}$$

$$\lambda(F(3,3,n-6)\,;\,F(3,n-3),\ F(5,n-5)) = \begin{cases} 1 & if\ n=11 \\ 0 & if\ n \geqq 10\ (n \neq 11), \end{cases}$$

$$\lambda(F(3,3,n-6)\,;\,F(3,n-3),\ F(6,n-6)) = \begin{cases} 2 & if\ n=12 \\ 1 & if\ n > 12, \end{cases}$$

$$\lambda(F(3,3,n-6)\,;\,F(4,n-4),\ F(5,n-5)) = 0 \quad if\ n \geqq 10,$$

$$\lambda(F(3,3,n-6)\,;\,F(4,n-4),\ F(6,n-6)) = 0 \quad if\ n \geqq 12,$$

$$\lambda(F(3,3,n-6)\,;\,F(5,n-5),\ F(6,n-6)) = 0 \quad if\ n \geqq 12,$$

$$\lambda(F(3,3,n-6)\,;\,F(3,n-3),\ F(3,n-3)) = 1 \quad if\ n \geqq 9,$$

$$\lambda(F(3,3,n-6)\,;\,F(4,n-4),\ F(4,n-4)) = 0 \quad if\ n \geqq 9,$$

$$\lambda(F(3,3,n-6)\,;\,F(5,n-5),\ F(5,n-5)) = 0 \quad if\ n \geqq 10,$$

$$\lambda(F(3,3,n-6)\,;\,F(6,n-6),\ F(6,n-6)) = 0 \quad if\ n \geqq 12,$$

(4. 7. 5)

$$\lambda(F(2,3,n-5)\,;\,F(2,3,n-5),\ F(2,n-2)) = 1 \quad if\ n \geqq 8,$$

$$\lambda(F(2,3,n-5)\,;\,F(2,3,n-5),\ F(3,n-3)) = \begin{cases} 2 & if\ n=8 \\ 1 & if\ n > 8, \end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(2,3,n-5),\ F(4,n-4)) = \begin{cases} 1 & if\ n=9 \\ 0 & if\ n \geqq 8\ \ (n \neq 9), \end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(2,3,n-5),\ F(5,n-5)) = \begin{cases}2 & if\ n=10\\ 1 & if\ n>10,\end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(2,n-2),\ F(3,n-3)) = 1 \quad if\ n\geqq 8,$$

$$\lambda(F(2,3,n-5)\,;\,F(2,n-2),\ F(4,n-4)) = \begin{cases}1 & if\ n=9\\ 0 & if\ n\geqq 8\ \ (n\neq 9),\end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(2,n-2),\ F(5,n-5)) = \begin{cases}2 & if\ n=10\\ 1 & if\ n>10,\end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(3,n-3),\ F(4,n-4)) = \begin{cases}1 & if\ n=9\\ 0 & if\ n\geqq 8\ \ (n\neq 9),\end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(3,n-3),\ F(5,n-5)) = \begin{cases}2 & if\ n=10\\ 1 & if\ n>10,\end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(4,n-4),\ F(5,n-5)) = 0 \quad if\ n\geqq 10,$$

$$\lambda(F(2,3,n-5)\,;\,F(2,n-2),\ F(2,n-2)) = 0 \quad if\ n\geqq 8,$$

$$\lambda(F(2,3,n-5)\,;\,F(3,n-3),\ F(3,n-3)) = \begin{cases}1 & if\ n=8\\ 0 & if\ n>8,\end{cases}$$

$$\lambda(F(2,3,n-5)\,;\,F(4,n-4),\ F(4,n-4)) = 0 \quad if\ n\geqq 8,$$

$$\gamma(F(2,3,n-5)\,;\,F(5,n-5),\ F(5,n-5)) = 0 \quad if\ n\geqq 10,$$

and

$$\lambda(F(2,3,n-5)\,;\,F(2,3,n-5),\ F(2,3,n-5)) = \begin{cases}2 & if\ n=8\\ 1 & if\ n>8.\end{cases}$$

Proof. For $\boldsymbol{F}_0=\{M_i(0)\}_{i=1,2,3}=F(m_1,m_2,m_3)\in\widetilde{\Omega}^n_n(3)$, we put

$$\boldsymbol{F}_1=\{M_i(1)\}_{i=1,2}=F(m_1,m_2+m_3),$$

$$\boldsymbol{F}_2=\{M_i(2)\}_{i=1,2}=\begin{cases}F(m_1+m_2,m_3) & \text{if}\quad m_1+m_2\leqq m_3\\ F(m_3,m_1+m_2) & \text{if}\quad m_1+m_2\geqq m_3,\end{cases}$$

and

$$\boldsymbol{F}_3=\{M_i(3)\}_{i=1,2}=F(m_2,\ m_1+m_3).$$

For each $\sigma\in\mathfrak{S}_n$, let $z(\sigma)$ be the element $(\sigma\mathfrak{S}_n(\boldsymbol{F}_s),\ \mathfrak{S}_n(\boldsymbol{F}_t))$ of $X(\boldsymbol{F}_s,\ \boldsymbol{F}_t)_n$. If $((\mathfrak{S}_n)_{z(\sigma)})=(\mathfrak{S}_n(\boldsymbol{F}_0))$, then we have

(4.7.6) $\qquad Car.\,(\sigma(M_i(s))\cap M_j(t))\neq 1$ for any $i,\ j,\ s$ and t

by Lemmas 3.5 and 4.6. Let $z(\tau)$ be another element of $X(\boldsymbol{F}_s,\ \boldsymbol{F}_t)$ such that $((\mathfrak{S}_n)_{z(\tau)})=(\mathfrak{S}_n(\boldsymbol{F}_0))$. If

$$\tau(M_i(s))\subset M_j(t)\quad\text{and}\quad\sigma(M_j(s))\subset M_j(t)$$

for some i and j, then there is an element $\alpha\in\mathfrak{S}_n(M_j(t))\subset\mathfrak{S}_n(\boldsymbol{F}t)$ such that

(4.7.7) $\qquad \alpha\tau(r)=\sigma(r)$ for any $r\in M_i(s)$.

If

$$\tau(M_i(s)) = \sigma(M_i(s))$$

for some i, then there is an element $\beta \in \mathfrak{S}_n(M_i(s)) \subset \mathfrak{S}_n(F_s)$ such that

$$\tau\beta(r) = \sigma(r) \quad \text{for any} \quad r \in M_i(s). \tag{4.7.8}$$

If there is an integer i such that

$$\tau(M_i(s)) \neq \sigma(M_i(s)) \quad \text{and} \quad \tau(M_i(s)) \cup \sigma(M_i(s)) \not\subset M_j(t)$$
for any j,

then we have

$$[z(\sigma)] \neq [z(\tau)] \quad \text{in} \quad X(F_s, F_t)_n/\mathfrak{S}_n. \tag{4.7.9}$$

If either

$$\tau(M_i(s)) = \sigma(M_i(s)) \quad \text{for any } i,$$

or

there is an integer j for each i such that
$\tau M_i(s)) \cup \sigma(M_i(s)) \subset M_j(t)$,

then we have

$$[z(\sigma)] = [z(\tau)] \quad \text{in} \quad X(F_s, F_t)_n/\mathfrak{S}_n. \tag{4.7.10}$$

Therefore we have (4.7.1) and (4.7.3)–(4.7.5) from (4.7.6)–(4.7.10), Lemmas 3.2–3.3 and (4.1.1) by the simple caluculations. Moreover, (4.7.2) is proved by the same way. *Q.E.D.*

Lemma 4.8. *Let*

$$F_0 = F(m_1, \cdots, m_p),$$
$$F_1 = F(r_1, \cdots, r_a, m_2, \cdots, m_p),$$
$$F_2 = F(s_1, \cdots, s_b, m_2, \cdots, m_p),$$

and

$$F_3 = F(t_1, \cdots, t_c, m_2, \cdots, m_p)$$

be the elements of $\tilde{\Omega}_n^n(p)$. *Let* Q_i $(i = 1, \cdots, 5)$ *be the subsets*

$$\{F \mid F \text{ is an element of } \tilde{\Omega}_n^n \text{ satisfing the condition } (4.4.i)\}$$

of $\tilde{\Omega}_n^n$. *Then we have the following equalities :*

$$\lambda(F_1\,;\,F_2,F_0)_n=\begin{cases}0 & \text{if } F_1\neq F_2,\\ 1 & \text{if } F_1=F_2 \text{ and } F_0\in Q_1,\\ 2 & \text{if } F_1=F_2 \text{ and } F_0\in Q_2,\end{cases}$$

$$\lambda(F_1\,;\,F_2,F_3)_n=\begin{cases}\lambda(F_1'\,;\,F_2',F_3')_{m_1} & \text{if either } F_0\in Q_1 \text{ or}\\ & F_0\in Q_2,\ b\geqq 2,\ c\geqq 2, \text{ and } m_1=m_2,\\ 2\lambda(F_1'\,;\,F_2',F_3')_{m_1} & \text{if } F_0\in Q_2 \text{ and } m_1<m_2,\end{cases}$$

where $F_1'=F(r_1,\cdots,r_a)$, $F_2'=F(s_1,\cdots,s_b)$ and $F_3'=F(t_1,\cdots,t_c)$.

Proof. We denote by $N(x,y)$ the set $\{i\,|\,x<i\leqq y\}$ for two integers x and y. For each $\sigma\in\mathfrak{S}_n$, let $z(\sigma)$ be the element $(\sigma\mathfrak{S}_n(F_2),\ \mathfrak{S}_n(F_3))$ of $X(F_2,F_3)_n$ such that $((\mathfrak{S}_n)_{z(\sigma)})=(\mathfrak{S}_n(F_1))$. If either $F_0\in Q_1$ or $F_0\in Q_2$, $b\geqq 2$, $c\geqq 2$ and $m_1=m_2$, then we have

$$\sigma(M_j)=M_j \quad \text{for each} \quad 1\leqq j\leqq p, \tag{4.8.1}$$

where $M_j=N\left(\sum_{i=1}^{j-1}m_i,\ \sum_{i=1}^{j}m_i\right)(2\leqq j\leqq p)$ and $M_1=N(0,m_1)$, by Lemma 3.5 and Lemma 4.6. If $F_0\in Q_2$ and $m_1<m_2$ (i.e., there exists only one integer $i_0\geqq 2$ such that $m_{i_0}=m_{i_0+1}$), then we have

$$\sigma(M_j)=M_j \quad \text{for each} \quad 1\leqq j\leqq p$$

or

$$\sigma(M_j)=\begin{cases}M_{i_0+1} & \text{for } j=i_0,\\ M_{i_0} & \text{for } j=i_0+1,\\ M_j & \text{otherwise,}\end{cases} \tag{4.8.2}$$

by Lemma 3.5 and Lemma 4.6. If σ_1 is an element of $\mathfrak{S}_n$ satisfing (4.8.1) and σ_2 an element of $\mathfrak{S}_n$ satisfing (4.8.2), then

$$[z(\sigma_1)]\neq[z(\sigma_2)] \quad \text{in } X(F_2,F_3)/\mathfrak{S}_n.$$

If σ_i $(i=1,2)$ are the elements of $\mathfrak{S}_n$ satisfing

$$\sigma_1(M_j)=\sigma_2(M_i) \quad \text{for each } 1\leqq j\leqq p,$$

then there exist p-1 elements $\tau_j\in\mathfrak{S}_n(M_j)$ $(2\leqq j\leqq p)$ such that

$$(\sigma_1\tau_2\cdots\tau_p)(r)=\sigma_2(r) \quad \text{for each} \quad r>m_1.$$

Since $\tau_j\in\mathfrak{S}_n(F_2)$, so

$$z(\sigma_1)=z(\sigma_1\tau_2\cdots\tau_p) \quad \text{in } X(F_2,F_3)_n.$$

Therefore the desired result follows from (4. 1. 1). *Q. E. D.*

Lemma 4. 9. *Let Δ be an element of $A(C_n)^*$ such that*

$$\lambda(F\,;\Delta)=0 \quad if \quad F\in\bigcup_{i=2}^{p}\tilde{\Omega}_n^n(i),$$

then we have

$$\lambda(F\,;\Delta)=\begin{cases}0 & if \quad F\in\tilde{\Omega}_n^n(p+1)\cap\{Q_3\cup Q_4\cup Q_5\},\\ 0 \ or \ -2\delta & if \quad F\in\tilde{\Omega}_n^n(p+1)\cap Q_1,\\ 0 \ or \ -\delta & if \quad F\in\tilde{\Omega}_n^n(p+1)\cap Q_2,\end{cases}$$

where $\delta=\pm 1$ and Q_i $(1\leqq i\leqq 5)$ are the sets defined in Lemma 4. 8.

Proof. Since $\Delta^2=1_n$, so $\lambda(F(n)\,;\Delta)=\delta$. From the assumption, Lemma 3. 6 and Lemma 4. 6, we can deduce

$$\lambda(F\,;\Delta^2)=\lambda(F\,;\Delta)^2\lambda(F\,;F,\,F)+2\lambda(F\,;\Delta)\delta=0$$

for each $F\in\tilde{\Omega}_n^n(p+1)$. Since $\lambda(F\,;\Delta)\in\mathbf{Z}$, so the desired result follows from Lemma 4. 5. *Q. E. D.*

§ 5. The group homomorphisms $\Theta: A(C_n)^* \longrightarrow A(C_n)^*$.

5. 1. For each $F\in\tilde{\Omega}_n$, let $\Phi(F)$, $\tilde{\Phi}(F)$, $\tilde{\Phi}_n(F)$ and $\Gamma(F)$ be the subsets of $\tilde{\Omega}_n$ defined as follows :

$$\Phi(F)=\{F'\,|\,F'\in\tilde{\Omega}_n,\ F\neq F' \text{ and } \phi_i(F')=F \text{ for some } i\},$$

$$\tilde{\Phi}(F)=\{F'\,|\,F'\in\tilde{\Omega}_n,\ F\neq F' \text{ and } \phi_{i_1}\cdots\phi_{i_p}(F')=F \text{ for some } i_1,\cdots,i_p\},$$

$$\tilde{\Phi}_n(F)=\tilde{\Phi}(F)\cap\tilde{\Omega}_n^n,$$

and

$$\Gamma(F)=\{F'\,|\,F'\in\tilde{\Omega}_n \text{ and } (\mathfrak{S}_n(F))\leqq(\mathfrak{S}_n(F'))\}.$$

Let Δ be an element of $A(C_n)$. If $\Theta(\Delta)=\pm 1_{n-1}$, then from Lemma 4. 3, we have the following formula :

$$\text{(5. 1. 1)} \qquad \lambda(F\,;\Delta)=\begin{cases}-(1/n-r)\sum_{F'\in\Phi F}\lambda(F'\,;\Delta)I(F') & \text{if } F\in\tilde{\Omega}_n^r,\ F\neq F(n-1) \text{ and } 2\leqq r\leqq n-1,\\ \pm 1-\lambda(F(n)\,;\Delta) & \text{if } F=F(n-1),\\ (-1/n)\lambda(F(2)\,;\Delta) & \text{if } F=F(1),\end{cases}$$

where $I(F')=Car.\{j \mid \phi_j(F')=F\}$. Therefore $\lambda(F\,;\Delta)$ is determied by $\{\lambda(F'\,;\Delta) \mid F' \in \widetilde{\Phi}_n(F)\}$ and $\lambda(F(n)\,;\Delta)$ if $\Theta(\Delta)=\pm 1_{n-1}$.
Let $S(\Delta)$ be a subset of $\widetilde{\Omega}_n^n$ such that

$$\lambda(F\,;\Delta)=0 \quad \text{for each} \quad F\in S(\Delta),$$

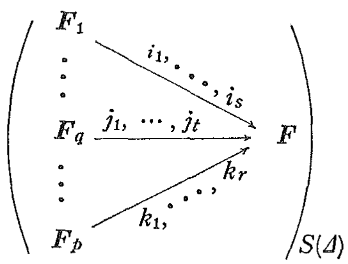
and $\widetilde{\Phi}(F)_{S(\Delta)}$ a subset

$$\widetilde{\Phi}(F)-(\{F' \mid F'\in\widetilde{\Phi}(F) \text{ and } \widetilde{\Phi}_n(F')\subset S(\Delta)\}\cup\{F(n-1),\ F(n)\})$$

of $\widetilde{\Phi}(F)$. Let

$$F_1 \xrightarrow{i_1,\ \cdots,\ i_p} F_2$$

denote the set $\{j \mid \phi_j(F_1)=F_2\}=\{i_1,\ \cdots,\ i_p\}$. Let



denote the system of the set $\Phi(F)\cap\widetilde{\Phi}(F)_{S(\Delta)}=\{F_1,\ \cdots,\ F_p\}$ and the sets $\{j \mid \phi_j(F_q)=F\}$ $(1\leqq q\leqq p)$ of indicies. From Lemma 4.6, we have the following formula:

(5.1.2) $$\lambda(F\,;\Delta^2)=\lambda(F\,;\Delta(F)^2),$$

where

$$\Delta(F)=\sum_{F'\in\Gamma(F)}\lambda(F'\,;\Delta)\langle X(F')_n\rangle.$$

Let $B(r)_p$ be the subset of $\widetilde{\Omega}_n^n$ defined by

$$B(r)_p=\begin{cases}\{F(r_1,\ \cdots,\ r_q,\ m_2,\ \cdots,\ m_p) \mid q\geqq 2,\ \sum_{i=1}^{q} r_i=r\leqq m_2 \text{ and } \sum_{i=1}^{q-1} r_i\leqq 6\} \\ \quad \cup\{F(r,\ m_2,\ \cdots,\ m_p)\} \qquad\qquad \text{if} \quad r>6, \\ F(r_1,\ \cdots,\ r_q,\ m_2,\ \cdots,\ m_p) \mid q\geqq 1 \text{ and } \sum_{i=1}^{q} r_i=r\} \quad \text{if } r\leqq 6.\end{cases}$$

For examples,

$$B(7)_2=\{F(2,5,n-7),\ F(3,4,n-7),\ F(7,n-7)\} \qquad \text{if} \quad 7\leqq n-7,$$
$$B(6)_2=\begin{cases}\{F(2,4,5),\ F(3,3,5),\ F(2,2,2,5)\} & \text{if} \quad n=11, \\ \{F(2,4,n-6),\ F(3,3,n-6),\ F(2,2,2,n-6),\ F(6,n-6)\} & \text{if} \quad n\geqq 12,\end{cases}$$

and

$$F(2,5,6)\notin B(7)_2 \quad if \quad n=13.$$

In the following Lemmas 5.2–5.10, let Δ denotes an element of $A(C_n)^*$ such that

$$\lambda(F(n)\,;\,\Delta)=1 \quad \text{and} \quad \Theta(\Delta)=1_{n-1}.$$

***Lemma* 5.2.** *We have*

$$\lambda(F(n-1)\,;\,\Delta)=0.$$

Proof. It is trivial by (5.1.1). *Q.E.D.*

***Lemma* 5.3.** *If* $4\leqq n\leqq 12$, *then we have*

$$\lambda(F\,;\,\Delta)=0 \quad \textit{for each} \quad F\in\bigcup_r B(r)_2.$$

Proof.

In the case $n=4$. From Lemma 4.8, $\lambda(F(2,2)\,;\,\Delta)=0$ or -1. Since

$$\widetilde{\Phi}(F(1))=\begin{pmatrix} F(4) & \xrightarrow{1} & F(3) & \xrightarrow{1} & F(2) & \xrightarrow{1} & F(1) \\ & & & \overset{1,2}{\nearrow} & & & \\ F(2,2) & & & & & & \end{pmatrix},$$

so we have

$$\lambda(F(1)\,;\,\Delta)=\begin{cases}-1/4 & \text{if } \lambda(F(2,2)\,;\,\Delta)=-1,\\ 0 & \text{if } \lambda(F(2,2)\,;\,\Delta)=0,\end{cases}$$

by (5.1.1) and Lemma 5.2. Since $\lambda(F(1)\,;\,\Delta)\in\mathbf{Z}$ and $\bigcup_r B(r)_2=\{F(2,2)\}$, so we have the desired result.

In the case $n=5$. From Lemma 4.9, $\lambda(F(2,3)\,;\,\Delta)=0$ or -2. Since

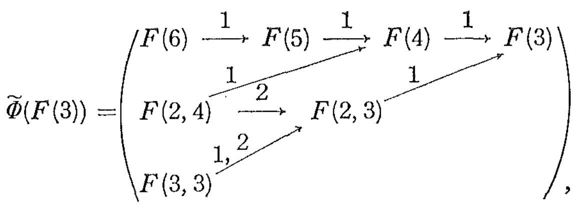
$$\widetilde{\Phi}(F(2))=\begin{pmatrix} F(5) & \xrightarrow{1} & F(4) & \xrightarrow{1} & F(3) & \xrightarrow{1} & F(2) \\ & \overset{1}{\nearrow} & & & & \overset{1,2}{\nearrow} & \\ F(2,3) & & \xrightarrow{2} & & F(2,2) & & \end{pmatrix},$$

so we have

$$\lambda(F(2)\,;\,\Delta)=\begin{cases}-5/3 & \text{if } \lambda(F(2,3)\,;\,\Delta)=-2,\\ 0 & \text{if } \lambda(F(2,3)\,;\,\Delta)=0,\end{cases}$$

by (5.1.1) and Lemma 5.2. Since $\lambda(F(2)\,;\,\Delta)\in\mathbf{Z}$ and $\bigcup_r B(r)_2=\{F(2,3)\}$, so we have the desired result.

In the case $n=6$. From Lemma 4.9, $\lambda(F(2,4)\,;\,\Delta)=0$ or -2 and $\lambda(F(3,3)\,;\,\Delta)=0$ or -1. Since



so we have

$$\lambda(F(3)\,;\,\Delta)=\begin{cases}-2/3 \ \text{ or } \ -5/3 & \text{if } \lambda(F(3,3)\,;\,\Delta)=-1,\\ 0 \ \text{ or } \ -1 & \text{if } \lambda(F(3,3)\,;\,\Delta)=0,\end{cases}$$

by (5.1.1) and Lemma 5.2. Since $\lambda(F(3)\,;\,\Delta)\in\mathbf{Z}$, so we have

$$\lambda(F(3,3)\,;\,\Delta)=0.$$

From Lemma 3.5,

$$\Gamma(F(2,2,2))=\{F(2,4),\ F(2,2,2),\ F(6)\}.$$

Therefore

$$\Delta(F(2,2,2))=x\langle X(F(2,2,2))\rangle+y\langle X(F(2,4))\rangle+1_6,$$

where $x=\lambda(F(2,2,2)\,;\,\Delta)$ and $y=\lambda(F(2,4)\,;\,\Delta)$. Since $\lambda(F(2,2,2)\,;\,\Delta^2)=0$, so we can deduce

$$\begin{aligned}&\lambda(F(2,2,2)\,;\,\Delta(F(2,2,2))^2)\\ &=x^2\lambda(F(2,2,2)\,;\,F(2,2,2),\ F(2,2,2))\\ &+y^2\lambda(F(2,2,2)\,;\,F(2,4),\ F(2,4))\\ &+2xy\lambda(F(2,2,2)\,;\,F(2,2,2),\ F(2,4))\\ &+2x\lambda(F(2,2,2)\,;\,F(2,2,2),\ F(6))\\ &+2y\lambda(F(2,2,2)\,;\,F(2,4),\ F(6))\\ &=0\end{aligned}$$

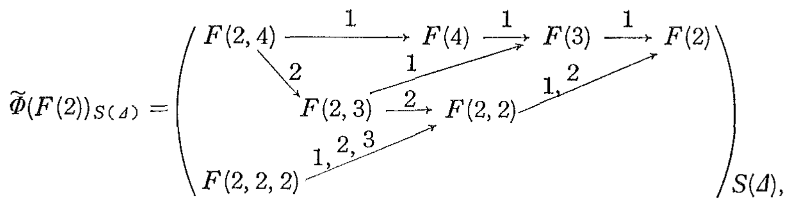
by (5.1.2). Therefore we have

$$0=\begin{cases}6x^2+4-12x+2x=2(3x-2)(x-1) & \text{if } \ y=-2,\\ 6x^2+2x & \text{if } \ y=0,\end{cases}$$

by (4.7.1). Since $x\in\mathbf{Z}$, so we have

$$\lambda(F(2,2,2)\,;\,\Delta)=\begin{cases}0 & \text{if } \lambda(F(2,4)\,;\,\Delta)=0,\\ 1 & \text{if } \lambda(F(2,4\,;\,\Delta)=-2.\end{cases}$$

we have

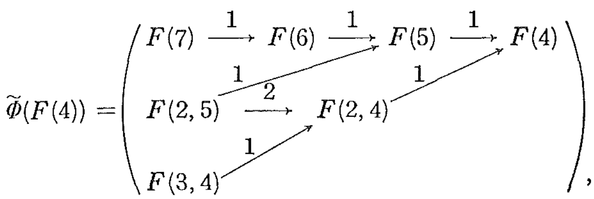


where $S(\varDelta)=\{F(3,3)\}$. Therefore we have

$$\lambda(F(2,4)\ ;\ \varDelta)=0 \quad\text{and}\quad \lambda(F(2,2,2)\ ;\ \varDelta)=0,$$

by (5. 1. 1) and Lemma 5. 2. Since $\bigcup_r B(r)_2=\{F(2,4),\ F(3,3),\ F(2,2,2)\}$, so we have the desired result.

In the case n $=7$. From Lemma 4. 9, $\lambda(F(2,5)\ ;\ \varDelta)=0$ or -2 and $\lambda(F(3,4)\ ;\ \varDelta)=0$ or -2. Since



so we have

$$\lambda(F(4)\ ;\ \varDelta)=\begin{cases}-2/3 \quad\text{or}\quad -5/3 & \text{if } \lambda(F(3,4)\ ;\ \varDelta)=-2,\\ -1 \quad\text{or}\quad 0 & \text{if } \lambda(F(3,4)\ ;\ \varDelta)=0,\end{cases}$$

by (5. 1. 1.) and Lemma 5. 2. Since $\lambda(F(4)\ ;\ \varDelta)\in\mathbf{Z}$, so we have

$$\lambda(F(3,4)\ ;\ \varDelta)=0.$$

From Lemma 3. 5,

$$\Gamma(F(2,2,3))=\{F(2,2,3),\ F(2,5),\ F(3,4),\ F(7)\}.$$

Therefore

$$\varDelta(F(2,2,3))=x\langle X(F(2,2,3))\rangle+y\langle X(F(2,5))\rangle+z\langle X(F(3,4))\rangle+1_7,$$

where $x=\lambda(F(2,2,3)\ ;\ \varDelta)$, $y=\lambda(F(2,5)\ ;\ \varDelta)$ and $z=\lambda(F(3,4)\ ;\ \varDelta)$. Since $\lambda(F(2,2,3)\ ;\ \varDelta^2)=0$, so we can deduce

$$\begin{aligned}&\lambda(F(2,2,3)\,;\,\Delta^2)\\&= x^2\lambda(F(2,2,3)\,;\,F(2,2,3),\ F(2,2,3))\\&+ y^2\lambda(F(2,2,3)\,;\,F(2,5),\ F(2,5))\\&+ 2xy\lambda(F(2,2,3)\,;\,F(2,2,3),\ F(2,5))\\&+ 2x\lambda(F(2,2,3)\,;\,F(2,2,3),\ F(7))\\&+ 2y\lambda(F(2,2,3)\,;\,F(2,5),\ F(7))\\&= 0,\end{aligned}$$

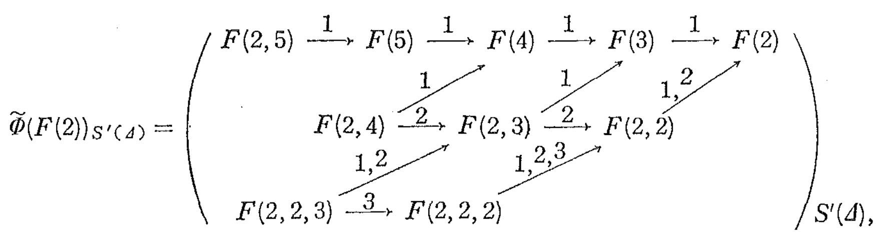
by (5. 1. 2).
Therefore we have

$$0 = \begin{cases} 2x^2+4-8x+2x = 2(x-1)(x-2) & \text{if } y=-2,\\ 2x^2+2x & \text{if } y=0,\end{cases}$$

by (4. 7. 1). From the above caluculations, we can separate four cases :

cases	(1)	(2)	(3)	(4)
$\lambda(F(2,5)\,;\,\Delta)$	0	0	−2	−2
$\lambda(F(3,4)\,;\,\Delta)$	0	0	0	0
$\lambda F(2,2,3)\,;\,\Delta)$	0	−1	1	2

We have



where $S'(\Delta) = \{F(3,4)$. Therefore we have the following table :

cases	(1)	(2)	(3)	(4)
$\lambda(F(3)\,;\,\Delta$	0	*	*	*
$\lambda(F(2,2\,;\,\Delta)$	0	−4/3	5/3	*
$\lambda(F(2)\,;\,\Delta)$	0	*	*	−7/5

(* denotes that there is no need to caluculate $\lambda(F\,;\,\Delta)$.),

by (5. 1. 1). Since $\lambda(F(2,2)\,;\,\Delta,\ \lambda(F(2)\,;\,\Delta)\in\mathbf{Z}$, so we have the case

(1) $\lambda(F(2,5)\,;\,\Delta) = \lambda(F(3,4\,;\,\Delta) = \lambda(F(2,2,3)\,;\,\Delta) = 0.$

Since $\bigcup_r B(r)_2 = \{F(2,5),\ F(3,4),\ F(2,2,3)\}$, so we have the desired result.

In the case $n=8$. From Lemma 4.9, $\lambda(F(2,6)\,;\,\Delta)=0$ or -2, $\lambda(F(3,5)\,;\,\Delta)=0$ or -2 and $\lambda(F(4,4)\,;\,\Delta)=0$ or -1. In general, we have

(5.3.1) $$\lambda(F(3,\ n-3\,;\,\Delta)=0 \quad \text{for each } n\geqq 6,$$

by the same way as in the proofs of the above cases. Since

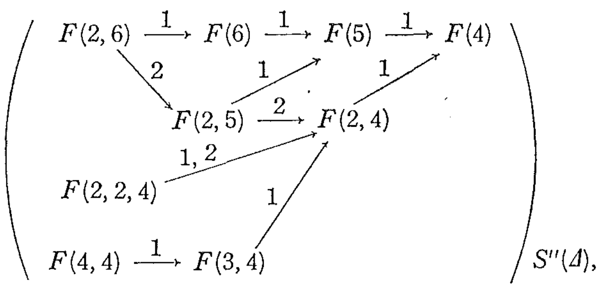
$$\Gamma(F(2,2,4))=\{F(2,2,4),\ F(2,6),\ F(4,4),\ F(8)\},$$

so we can separate eight cases :

cases	(1′)	(2′)	(3′)	(4′)	(5′)	(6′)	(7′)	(8′)
$\lambda(F(2,6)\,;\,\Delta)$	-2	-2	-2	-2	0	0	0	0
$\lambda(F(3,5)\,;\,\Delta)$	0	0	0	0	0	0	0	0
$\lambda(F(4,4)\,;\,\Delta)$	0	0	-1	-1	0	0	-1	-1
$\lambda(F(2,2,4)\,;\,\Delta)$	1	2	2	3	0	-1	0	1

by (4.7.1), (5.1.2) and the simple caluculations. We have

$\widetilde{\Phi}(F(4))_{S''(\Delta)}=$



where $S''(\Delta)=\{F(3,5)\}$. Therefore we have the following table

cases	(1′)	(2′)	(3′)	(4′)	(5′)	(6,)	(7′)	(8′)
$\lambda(F(4)\,;\,\Delta)$	3/4	1	5/4	6/4	0	$-1/4$	1/4	2/4
$\lambda(F(2,4)\,;\,\Delta)$	*	-3	*	*	0	*	*	*
$\lambda(F(3,4)\,;\,\Delta)$	*	0	*	*	0	*	*	*

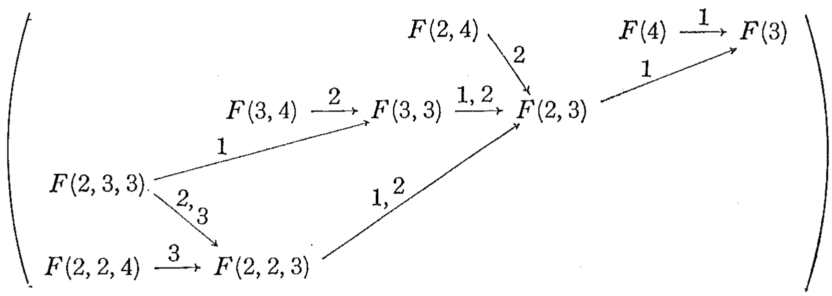
by (5.1.1). Since $\lambda(F(4)\,;\,\Delta)\in\mathbf{Z}$, so we have the two cases (2′) and (5′). Since

$$\Gamma(F(2,3,3))=\{F(2,6),\ F(3,5),\ F(8)\},$$

so we have

$$\lambda(F(2,3,3)\,;\,\Delta)=\begin{cases}0 \text{ or } -1 & \text{if } \lambda(F(2,6):\Delta)=0,\\ 0 \text{ or } 1 & \text{if } \lambda(F(2,6)\,;\,\Delta)=-2,\end{cases}$$

by (4.7.5), (5.1.2) and the simple caluculations. We have



Therefore we have the following table :

cases	(2′) and $x=0$	(2′) and $x=1$	(5′) and $x=0$	(5′) and $x=-1$
$\lambda(F(2,3)\,;\,\Delta)$	7/3	*	0	*
$\lambda(F(3)\,;\,\Delta)$	*	*	0	*
$\lambda(F(3,3)\,;\,\Delta)$	*	$-1/2$	0	1/2,

(where $x=\lambda(F(2,3,3)\,;\,\Delta)$)

by (5. 1. 1). Since $\lambda(F(2,3)\,;\,\Delta)$, $\lambda(F(3,3)\,;\,\Delta)\in\mathbb{Z}$, so we have

the case (5′) and $\lambda(F(2,3,3)\,;\,\Delta)=0$.

Moreover, we have

$$\lambda(F(2,2,2,2)\,;\,\Delta)=0,$$

by Lemma 4. 9. Since $\bigcup_r B(r)_2=\{F(2,6),\ F(3,5),\ F(4,4),\ F(2,2,4),\ F(2,3,3),\ F(2,2,2,2)\}$, so we have the desired result.

In the cases $9\leqq n\leqq 12$. We can prove by the same way as in the above cases by the use of Lemma 4. 7, Lemma 4. 9, (5. 1. 1) and (5. 1. 2). *Q. E. D.*

Lemma 5. 4. *If* $n>12$, *then we have*

$$\lambda(F\,;\,\Delta)=0 \quad \textit{for each} \quad F\in\bigcup_{r=2}^{6}B(r)_2.$$

Proof. From Lemma 4. 9 and (5. 3. 1), $\lambda(F(2,n-2)\,;\,\Delta)=0$ or -2, $\lambda(F(3,n-3)\,;\,\Delta)=0$ and $\lambda(F(4,n-4)\,;\,\Delta)=0$ or -2. Since

$$\Gamma(F(2,2,n-4))=\{F(2,2,n-4),\ F(4,n-4),\ F(2,n-2),\ F(n)\},$$

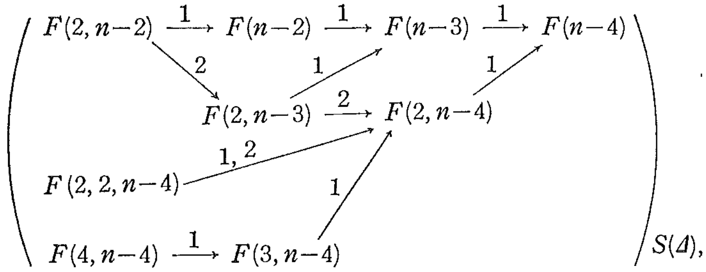
so we can separate eight cases :

cases	(I)	(I^1)	(I^2)	(I^3)	(II)	(II^1)	(II^2)	(II^3)
$\lambda(F(2,n-2)\,;\,\Delta)$	-2	-2	-2	-2	0	0	0	0
$\lambda(F(3,n-3)\,;\,\Delta)$	0	0	0	0	0	0	0	0
$\lambda(F(4,n-4)\,;\,\Delta)$	0	0	-2	-2	0	0	-2	-2
$\lambda(F(2,2,n-4)\,;\,\Delta)$	2	1	2	3	0	-1	0	1

,

by (4.7.1), Lemma 4.9, (5.1.2) and the simple caluculations. We have

$\tilde{\Phi}(F(n-4))_{S(\Delta)} =$



where $S(\Delta) = \{F(3, n-3)\}$. Therefore we have the following table:

cases	(I)	(I^1)	(I^2)	(I^3)	(II)	(II^1)	(II^2)	(II^3)
$\lambda(F(n-4)\,;\,\Delta)$	1	3/4	5/4	6/4	0	−1/4	1/4	2/4
$\lambda(F(2,n-4)\,;\,\Delta)$	−3	*	*	*	0	*	*	*
$\lambda(F(3,n-4)\,;\,\Delta)$	0	*	*	*	0	*	*	*

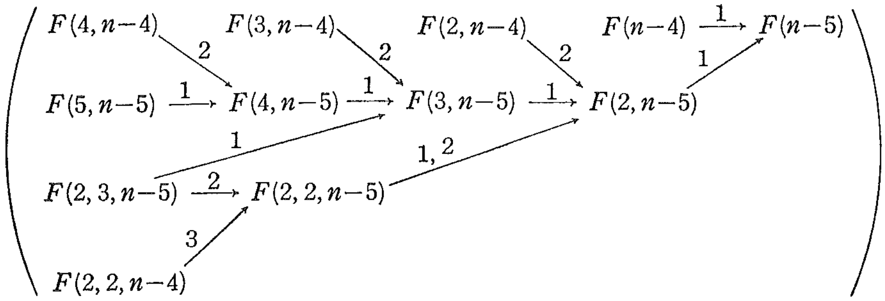
by (5.1.1). Since $\lambda(F(n-4)\,;\,\Delta) \in \mathbf{Z}$, so we have the two cases (I) and (II). Since

$$\Gamma(F(2,3,n-5)) = \{F(n),\ F(2,n-2),\ F(3,n-3),\ F(5,n-5),\ F(2,3,n-5)\}$$

and $\lambda(F(2,3,n-5)\,;\,\Delta^2) = 0$, so we can separate eight cases:

cases	(I)				(II)			
	(I_2)	(I_2^1)	(I_2^2)	(I_2^3)	(II_2)	(II_2^1)	(II_2^2)	(II_2^3)
$\lambda(F(5,n-5)\,;\,\Delta)$	0	0	−2	−2	0	0	−2	−2
$\lambda(F(2,3,n-5)\,;\,\Delta)$	2	0	2	4	0	−2	0	2

by (4.7.5), Lemma 4.9, (5.1.2) and the simple caluculations.
we have



Therfore we have the following table:

cases	(I_2)	(I_2^1)	(I_2^2)	(I_2^3)	(II_2)	(II_2^1)	(II_2^2)	(II_2^3)
$\lambda(F(n-5)\,;\,\Delta)$	-1	$*$	$*$	$-7/5$	0	$1/5$	$*$	$-2/5$
$\lambda(F(2,n-5)\,;\,\Delta)$	4	$7/3$	$13/3$	$*$	0	$*$	$1/3$	$*$
$\lambda(F(2,2,n-5)\,;\,\Delta)$	-4	$*$	$*$	$*$	0	$*$	$*$	$*$

by (5.1.1). Since $\lambda(F(n-5)\,;\,\Delta)$, $\lambda(F(2,n-5)\,;\,\Delta)\in\mathbf{Z}$, so we have the two cases (I_2) and (II_2). From Lemma 3.5,

$$\Gamma(F(3,3,n-6))=\{F(n),\ F(3,n-3),\ F(6,n-6),\ F(3,3,n-6)\},$$
$$\Gamma(F(2,4,n-6))=\{F(n),\ F(2,n-2),\ F(4,n-4),\ F(6,n-6),\ F(2,4,n-6)\},$$

and

$$\Gamma(F(2,2,2,n-6))=\{F(n),\ F(2,n-2),\ F(4,n-4),\ F(2,2,n-4),\ F(6,n-6),\ F(2,4,n-6),\ F(2,2,2,n-6)\}.$$

Since $\lambda(F(3,3,n-6)\,;\,\Delta^2)=\lambda(F(2,4,n-6)\,;\,\Delta^2)=\lambda(F(2,2,2,n-6)\,;\,\Delta^2)=0$, so we can separate sixteen cases :

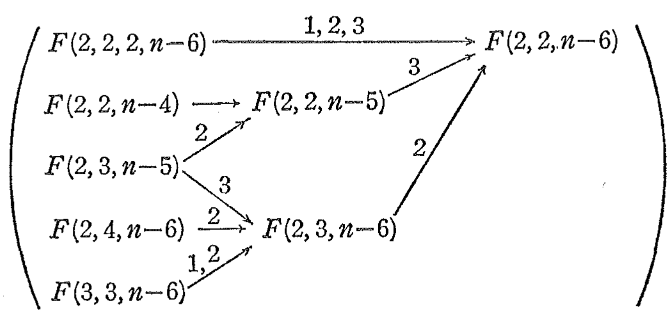
cases	(I_2)							
	(I_3)	(I_3^1)	(I_3^2)	(I_3^3)	(I_3^4)	(I_3^5)	(I_3^6)	(I_3^7)
$\lambda(F(6,n-6)\,;\,\Delta)$	-2	-2	-2	-2	0	0	0	0
$\lambda(F(2,4,n-6)\,;\,\Delta)$	2	2	4	4	0	0	2	2
$\lambda(F(3,3,n-6)\,;\,\Delta)$	0	-1	0	-1	0	-1	0	-1
$\lambda(F(2,2,2,n-6)\,;\,\Delta)$	-3	-3	$*\notin Z$	$*\notin Z$	-1	-1	-1	-1

and

cases	(II_2)							
	(II_3)	(II_3^1)	(II_3^2)	(II_3^3)	(II_3^4)	(II_3^5)	(II_3^6)	(II_3^7)
$\lambda(F(6,n-6)\,;\,\Delta)$	-2	-2	-2	-2	0	0	0	0
$\lambda(F(2,4,n-6)\,;\,\Delta)$	0	0	2	2	0	0	-2	-2
$\lambda(F(3,3,n-6)\,;\,\Delta)$	0	1	0	1	0	-1	0	-1
$\lambda(F(2,2,2,n-6)\,;\,\Delta)$	0	0	-1	-1	0	0	1	1

by Lemma 4.7, Lemma 4,9, (5.1.2) and the simple caluculation. We have

$\widetilde{\Phi}(F(2,2,n-6))=$

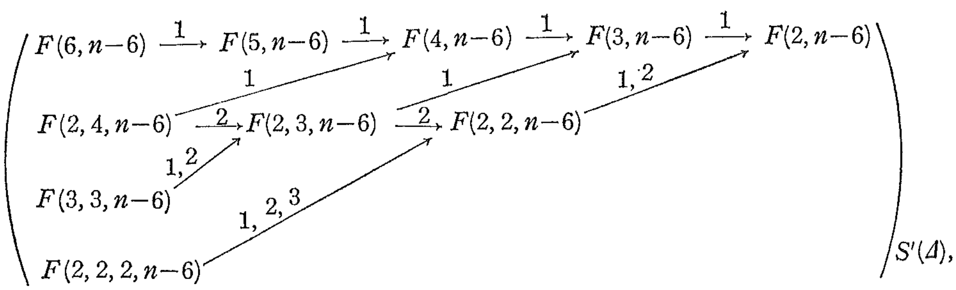


Therefore we have the following table :

cases	(I_3)	(I_3^1)	(I_3^2)	(I_3^3)	(I_3^4)	(I_3^5)	(I_3^6)	(I_3^7)
$\lambda(F(2,2,n-6)\ ;\ \Delta)$	17/2	15/2	*	*	7/2	7/2	11/2	9/2

by (5.1.1). Since $\lambda(F(2,2,n-6)\ ;\ \Delta)\in\mathbb{Z}$, so we have the cases (II_3)–(II_3^7). We have

$$\widetilde{\Phi}(F(2,n-6))_{S'(\Delta)}=$$



where $S'(\Delta)=\bigcup_{r=2}^{5}B(r)_2$. Therefore we have the following table :

cases	(II_3)	(II_3^1)	(II_3^2)	(II_3^3)	(II_3^4)	(II_3^5)	(II_3^6)	(II_3^7)
$\lambda(F(2,n-6)\ ;\ \Delta)$	*	−3/4	*	*	0	*	*	*
$\lambda(F(3,n-6)\ ;\ \Delta)$	1/3	*	4/3	*	0	−2/3	*	*
$\lambda(F(2,2,n-6)\ ;\ \Delta)$	*	*	*	7/2	0	*	−5/2	−7/2

by (5.1.1). Since $\lambda(F(2,n-6) : \Delta)$, $\lambda(F(3,n-6)\ ;\ \Delta)$, $\lambda(F(2,2,n-6)\ ;\ \Delta)\in\mathbb{Z}$, so we have the case (II_3^4). Therefore the desired result follows from (II), (II_2) and (II_3^4).

Q.E.D.

Lemma 5.5. *If $7\leqq r\leqq 12$, then we have*

$$\lambda(F\ ;\ \Delta)=0 \quad \textit{for each } F\in B(r)_2.$$

Proof. From Lemma 4.9, we have

$$\lambda(F(7,n-7)\ ;\ \Delta)=\begin{cases}0 \text{ or } -1 & \text{if } n=14,\\ 0 \text{ or } -2 & \text{if } n \quad 14.\end{cases}$$

From Lemma 3.5, we have

$$\Gamma(F(2,5,n-7))-\bigcup_{r=2}^{6}B(r)_2=\{F(n),\ F(7,n-7),\ F(2,5,n-7)\},$$

$$\Gamma(F(3,4,n-7))-\bigcup_{r=2}^{6}B(r)_2=\{F(n),\ F(7,n-7),\ F(3,4,n-7)\},$$

and

$$\Gamma(F(2,2,3,n-7))-\bigcup_{r=2}^{6} B(r)_2=\{F(n),\ F(7,n-7),\ F(3,4,n-7),\ F(2,5,n-7),\ F(2,2,3,n-7).$$

Since $\lambda(F(2,5,n-7)\ ;\ \Delta^2)=\lambda(3,4,n-7)\ ;\ \Delta^2)=\lambda(F(2,2,3,n-7)\ ;\ \Delta^2)=0$, so we can separate sixteen cases :

cases	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\lambda(F(7,n-7)\ ;\ \Delta)$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
$\lambda(F(2,5,n=7)\ ;\ \Delta)$	2	2	2	2	0	0	0	0
$\lambda(F(3,4,n-7)\ ;\ \Delta)$	2	2	0	0	2	2	0	0
$\lambda(F(2,2,3,n-7)\ ;\ \Delta)$	-2	-3	-1	-2	0	-1	0	1

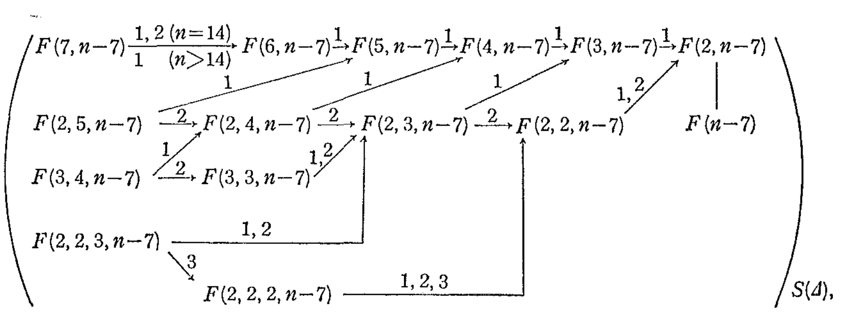
and

cases	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
$\lambda(F(7,n-7)\ ;\ \Delta)$	0	0	0	0	0	0	0	0
$\lambda(F(2,5,n-7)\ ;\ \Delta)$	-2	-2	-2	-2	0	0	0	0
$\lambda(F(3,4,n-7)\ ;\ \Delta)$	-2	-2	0	0	-2	-2	0	0
$\lambda(F(2,2,3,n-7)\ ;\ \Delta)$	2	3	1	2	0	1	0	-1

,

by Lemmas 4.7–4.9, (5.1.2), Lemma 5.4 and the simple caluculations. We have

$$\tilde{\Phi}(F(n-7))_{S(\Delta)}=$$



where $S(\Delta)=\bigcup_{r=2}^{6} B(r)_2$. Therefore we have the following table :

cases	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$\lambda(F(4,n-7)\ ;\ \Delta)$	*	*	4/3	4/3	*	*	1/3	1/3
$\lambda(F(3,n-7)\ ;\ \Delta)$	*	$-9/4$	*	*	$-5/4$	*	*	*
$\lambda(F(2,n-7)\ ;\ \Delta)$	*	*	*	*	*	3/5	*	*
$\lambda(F(n-7)\ ;\ \Delta)$	$-2/7$	*	*	*	*	*	*	*

and

cases	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
$\lambda(F(4,n-7)\,;\,\Delta)$	$-5/3$	$-5/3$	*	*	$-2/3$	$-2/3$	0	*
$\lambda(F(3,n-7)\,;\,\Delta)$	*	*	3/4	*	*	*	0	$-1/4$
$\lambda(F(2,n-7)\,;\,\Delta)$	*	*	*	$-7/5$	*	*	0	*
$\lambda(F(n-7)\,;\,\Delta)$	*	*	*	*	*	*	0	*

by (5. 1. 1). Since $\lambda(F\,;\,\Delta)\in\mathbf{Z}$, so we have case (15). Therefore we have

$$\lambda(F\,;\,\Delta)=0 \quad \text{for each} \quad F\in B(7)_2.$$

For $8\leqq r\leqq 12$, we can prove inductively, by the use of Lemmas 4. 7–4. 9, (5. 1. 1) and (5. 1. 2). *Q. E. D.*

Lemma 5.6. *If $r>12$, then we have*

$$\lambda(F\,;\,\Delta)=0 \quad \text{for each} \quad F\in B(r)_2.$$

Proof. If

$$\lambda(F\,;\,\Delta)=0 \;\; \text{for each} \;\; F\in \bigcup_{r=2}^{k-1} B(r)_2 \;\; (k\geqq 13),$$

and

$$\lambda(F\,;\,\Delta)\neq 0 \;\; \text{for some} \;\; F\in B(k)_2,$$

then by the same way as in the proofs of Lemmas 5. 3–5. 5, we have

$$\lambda(F\,;\,\Delta)\notin\mathbf{Z} \quad \text{for some} \quad F\in B,$$

where $B=\{F(k-6,\,n-k),\ F(2,\,k-6,\,n-k),\ F(3,\,k-6,\,n-k),\ F(4,\,k-6,\,n-k)\}$. Since $\lambda(F\,;\,\Delta)\in\mathbf{Z}$, so this contradiction establishes the result. *Q. E. D.*

Remark. $\lambda(F_1\,;\,F_2,F_3)$ $(F_i\in B(k)_2)$ and $\lambda(F\,;\,\Delta)$ $(F\in B(k)_2)$ are computable by Lemmas 4. 7 and 4. 8.

Combining above Lemmas 5. 2–5. 6, we have the following Lemma.

Lemma 5.7. *We have*

$$\lambda(F\,;\,\Delta)=0 \quad \textit{for each} \quad F\in \bigcup_r B(r)_2.$$

Lemma 5.8. *We have*

$$\lambda(F(2,m,m)\,;\,\Delta)=0 \quad \textit{if} \quad 2m+2=n.$$

Proof. From Lemma 4. 8 and Lemma 5. 7,

$$\lambda(F(2,m,m)\,;\,\Delta)=\begin{cases}0 \text{ or } -1 & \text{if } m>2 \ (n>6),\\ 0 & \text{if } m=2 \ (n=6).\end{cases}$$

Since

$$\widetilde{\Phi}(F(m,m)) = \left(\begin{array}{ccc} F(2,m,m) & \xrightarrow{\quad 1 \quad} & F(m,m) \\ & & \nearrow 2 \\ F(m+1,m+1) & \xrightarrow{1,2} F(m,m+1) & \\ & \nearrow 2 & \\ F(m,m+2) & & \end{array}\right)$$

so we have

$$\lambda(F(m,m\ ;\ \Delta) = 1/2 \quad \text{if} \quad \lambda(F(2,m,m\ ;\ \Delta) = -1$$

by (5.1.1). Since $\lambda(F(m,m)\ ;\ \Delta) \in \mathbf{Z}$, so we have the desired result. *Q.E.D.*

Lemma 5.9. *We have*

$$\lambda(F(2,m,m+1)\ ;\ \Delta) = 0 \quad if \quad 2m+3=n.$$

Proof. If $2+m>12$, then we have

$$\lambda(F\ ;\ \Delta) = 0 \quad \text{for each} \quad F \in B,$$

where $B = \{F(2,m,m+1),\ F(3,m-3,m+1),\ F(4,m-4,m+1),\ F(2,2,m-4,m+1),\ F(5,m-5,m+1),\ F(2,3,m-5,m+1),\ F(6,m-6,m+1),\ F(2,4,m-6,m+1),\ F(3,3,m-6,m+1),\ F(2,2,2,m-6,m+1)\}$, by the same way as in the proof of Lemma 5.4. In particular, $\lambda(F(2,m,m+1)\ ;\ \Delta) = 0$.

If $2+m \leqq 12$, then we can prove the result by the same way as in the proof of Lemma 5.3. *Q.E.D.*

Now we put

$$B_p = \{\bigcup_r B(r)_p\} \cup \{F(2,\ m_2,\ \cdots,\ m_{p+1}) \mid \Sigma\, m_i = n-2\}.$$

If $2+m_2>m_3$, $2 \leqq m_2 \leqq m_3$ and $2+m_2+m_3=n$, then $m_2=m_3$ or $m_2+1=m_3$. Therefore we have the following Lemma by Lemmas 5.7–5.9.

Lemma 5.10. *We have*

$$\lambda(F\ ;\ \Delta) = 0 \quad \textit{for each } F \in B_2.$$

Theorem 5.11 *Let Δ be an element of $A(C_n)^*$* $(n \geqq 3)$ such that

$$\lambda(F(n)\ ;\ \Delta) = 1 \quad \textit{and} \quad \Theta(\Delta) = 1_{n-1},$$

then we have

$$\Delta = 1_n.$$

Proof. If

$$\lambda(F\ ;\ \Delta) = 0 \quad \text{for each } F \in \bigcup_{p=2}^{k} B_p,$$

then we have

$$\lambda(F\,;\,\Delta)=0 \text{ for each } F\in B(r)_{k+1},$$

by the induction on r by the same way as in Lemma 5, 7. Moreover we have

$$\lambda(F(2,\ m_2,\ \cdots,\ m_{k+1})\,;\,\Delta)=0,$$

by the same way as in Lemmas 5. 8–5. 9. Since $\bigcup_p B_p=\tilde{\Omega}_n^n$, so we have

$$\lambda(F\,;\,\Delta)=0 \quad \text{for each} \quad F\in\tilde{\Omega}_n^n \tag{5. 11. 1}$$

by the induction on p and Lemma 5. 10. If

$$\lambda(F'\,;\,\Delta)=0 \quad \text{for each} \quad F'\in\tilde{\Phi}_n(F)\subset\tilde{\Omega}_n^n,$$

then we have

$$\lambda(F\,;\,\Delta)=0$$

by (5. 1. 1). Therefore the desired result follows from (5. 11. 1). *Q. E. D.*

Lemma 5. 12. *Let Δ be an element of $A(C_n)^*(n\geqq 3)$ such that*

$$\Theta(\Delta)=1_{n-1},$$

then we have

$$\lambda(F(n)\,;\,\Delta)=1.$$

Proof. Since $\lambda(F(n)\,;\,\Delta)=\pm 1$, so $\lambda(F(n-1)\,;\,\Delta)=2$ if $\lambda(F(n)\,;\,\Delta)=-1$ by (5. 1. 1). From now on we assume that

$$\lambda(F(n)\,;\,\Delta)=1 \quad (\lambda(F(n-1)\,;\,\Delta)=2).$$

In the case $n>12$. From Lemma 4. 9, $\lambda(F(2,\ n-2)\,;\,\Delta)=0$ or 2 and $\lambda(F(3,\ n-3)\,;\,\Delta)=0$ or 2. Therefore we can separate four cases :

cases	(I)	(I′)	(II)	(II′)
$\lambda(F(2,n-2)\,;\,\Delta)$	2	2	0	0
$\lambda(F(3,n-3)\,;\,\Delta)$	2	0	2	0

Since

$$\tilde{\Phi}(F(n-3))=\left(\begin{array}{ccccccc} F(n) & \xrightarrow{1} & F(n-1) & \xrightarrow{1} & F(n-2) & \xrightarrow{1} & F(n-3) \\ & & & \overset{1}{\nearrow} & & \overset{1}{\nearrow} & \\ F(2,n-2) & \xrightarrow{2} & F(2,n-3) & & & & \\ & \overset{1}{\nearrow} & & & & & \\ F(3,n-3) & & & & & & \end{array}\right)$$

so we have the following table :

cases	(I)	(I′)	(II)	(II′)
$\lambda(F(n-3)\ ;\ \Delta)$	2	4/3	2	1/3

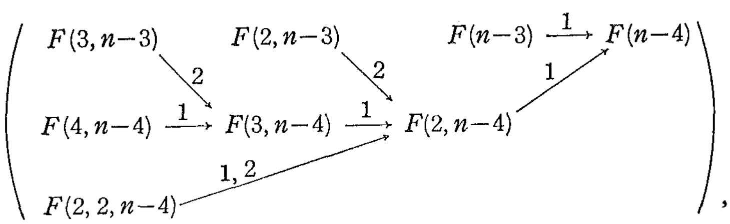
by (5.1.1). Since $\lambda(F(n-3)\ ;\ \Delta)\in\mathbf{Z}$, so we have the two cases (I) and (II). From Lemma 4.9, $\lambda(F(4,n-4)\ ;\ \Delta)=0$ or 2. Since

$$\Gamma(F(2,2,n-4))=\{F(n),\ F(2,n-2),\ F(4,n-4),\ F(2,2,n-4)\}$$

and $\lambda(F(2,2,n-4)\ ;\ \Delta^2)=0$, so we can separate eight cases :

cases	(I)				(II)			
	(I_2)	(I_2^1)	(I_2^2)	(I_2^3)	(II_2)	(II_2^1)	(II_2^2)	(II_2^3)
$\lambda(F(4,n-4)\ ;\ \Delta)$	2	2	0	0	2	2	0	0
$\lambda(F(2,2,n-4)\ ;\ \Delta)$	−2	−3	−1	−2	0	−1	0	1

by Lemma 4.7, Lemma 4.9, (5.1.2) and the simple caluculations.
Since



so we have the following table :

cases	(I_2)	(I_2^1)	(I_2^2)	(I_2^3)	(II_2)	(II_2^1)	(II_2^2)	(II_3^2)
$\lambda(F(n-4)\ ;\ \Delta)$	−2	−9/4	−6/4	−7/4	−1	−5/4	−3/4	−2/4

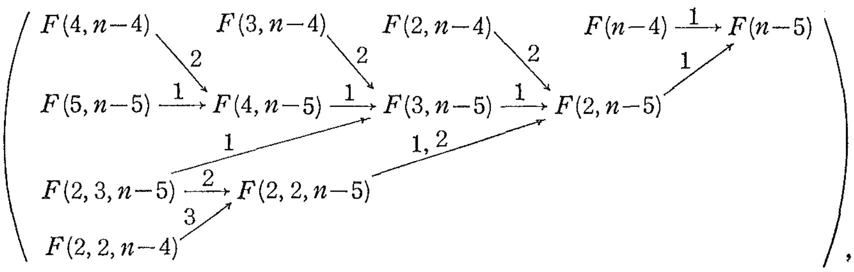
by (5.1.1). Since $\lambda(F(n-4)\ ;\ \Delta)\in\mathbf{Z}$, so we have the two cases (I_2) and (II_2). From Lemma 4.9, $\lambda(F(5,n-5)\ ;\ \Delta)=0$ or 2. Since

$$\Gamma(F(2,3,n-5))=\{F(2,n-2),\ F(3,n-3),\ F(5,n-5),\ F(2,3,n-5),\ F(n)\}$$

and $\lambda(F(2,3,n-5)\ ;\ \Delta^2)=0$, so we can separate eight cases :

cases	(I_2)				(II_2)			
	(I_3)	(I_3^1)	(I_3^2)	(I_3^3)	(II_3)	(II_3^1)	(II_3^2)	(II_3^3)
$\lambda)F(5,n-5)\ ;\ \Delta)$	2	2	0	0	2	2	0	0
$\lambda(F(2,3,n-5)\ ;\ \Delta)$	−4	−6	−2	−4	−2	−4	0	−2

by Lemma 4.7, Lemma 4.9, (5.1.2) and the simple caluculations.
Since



,

so we have the following table :

cases	(I_3)	(I_3^1)	(I_3^2)	(I_3^3)	(II_3)	(II_3^1)	(II_3^2)	(II_3^3)
$\lambda(F(2,n-5)\,;\,\Delta)$	-8	*	*	$-23/3$	-4	$-17/3$	*	$-11/3$
$\lambda(F(n-5)\,;\,\Delta)$	2	12/5	8/5	*	1	*	3/5	*

by (5.1.1). Since $\lambda(F(2,n-5)\,;\,\Delta)$, $\lambda(F(n-5)\,;\,\Delta) \in \mathbf{Z}$, so we have the two cases (I_3) and (II_3). From Lemma 4.9, $\lambda(F(6,n-6)\,;\,\Delta)=0$ or 2. Since

$$\Gamma(F(2,4,n-6)) = \{F(2,n-2),\ F(4,n-4),\ F(6,n-6),\ F(2,4,n-6),\ F(n)\},$$
$$\Gamma(F(3,3,n-6)) = \{F(3,n-3),\ F(6,n-6),\ F(3,3,n-6),\ F(n)\},$$

and

$$\lambda(F(2,4,n-6)\,;\,\Delta^2) = \lambda(F(3,3,n-6)\,;\,\Delta^2) = 0,$$

so we can separate sixteen cases :

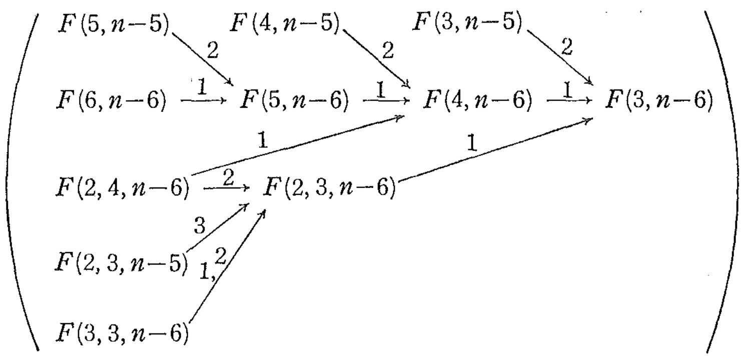
cases	(I_3)							
	(I_4)	(I_4^1)	(I_4^2)	(I_4^3)	(I_4^4)	(I_4^5)	(I_4^6)	(I_4^7)
$\lambda(F(6,n-6)\,;\,\Delta)$	2	2	2	2	0	0	0	0
$\lambda(F(2,4,n-6)\,;\,\Delta)$	-4	-4	-6	-6	-2	-2	-4	-4
$\lambda(F(3,3,n-6)\,;\,\Delta)$	-2	-3	-2	-3	-1	-2	-1	-2

and

cases	(II_3)							
	(II_4)	(II_4^1)	(II_4^2)	(II_4^3)	(II_4^4)	(II_4^5)	(II_4^6)	(II_4^7)
$\lambda(F(6,n-6)\,;\,\Delta)$	2	2	2	2	0	0	0	0
$\lambda(F(2,4,n-6)\,;\,\Delta)$	-2	-2	-4	-4	0	0	-2	-2
$\lambda(F(3,3,n-6)\,;\,\Delta)$	-2	-3	-2	-3	-1	-2	-1	-2

by Lemma 4.7, Lemma 4.9, (5.1.2) and the simple caluculations.

Since



so we have the following table :

cases	(I_4)	(I_4^1)	(I_4^2)	(I_4^3)	(I_4^4)	(I_4^5)	(I_4^6)	(I_4^7)
$\lambda(F(3,n-6)\ ;\ \Delta)$	-8	$-26/3$	-9	$-29/3$	-6	$-20/3$	-7	$-23/3$

and

cases	(II_4)	(II_4^1)	(II_4^2)	(II_4^3)	(II_4^4)	(II_4^5)	(II_4^6)	(II_4^7)
$\lambda(F(3,n-6)\ ;\ \Delta)$	-6	$-20/3$	-7	$-23/3$	-4	$-14/3$	-5	$-17/3$

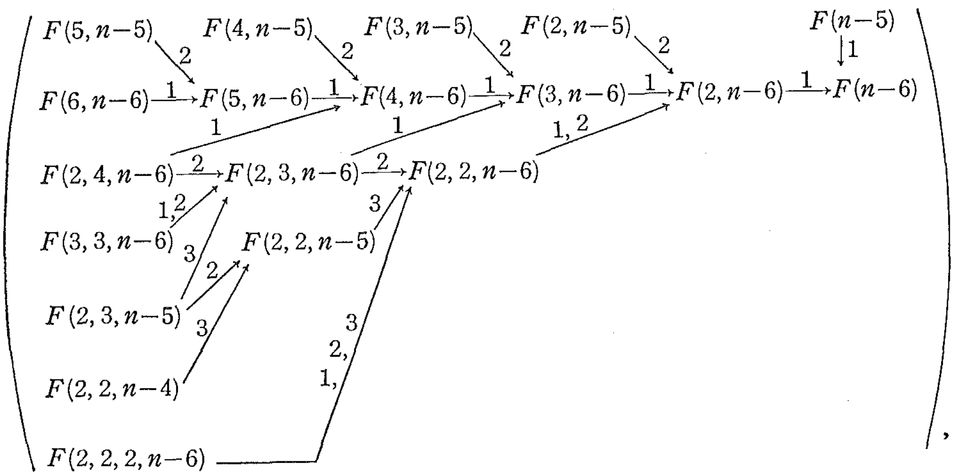
by (5.1.1). Since $\lambda(F(3,n-6)\ ;\ \Delta)\in\mathbf{Z}$, so we have the eight cases (I_4), (I_4^2), (I_4^4), (I_4^6), (II_4), (II_4^2), (II_4^4) and (II_4^6). Since

$$\Gamma(F(2,2,2,n-6)) = \{F(n),\ F(2,n-2),\ F(4,n-4),\ F(2,2,n-4),\ F(6,n-6),\ F(2,4,n-6),\ F(2,2,2,n-6)\}$$

and $\lambda(F(2,2,2,n-6)\ ;\ \Delta^2)=0$, so we have the following table :

cases	(I_4)	(I_4^2)	(I_4^4)	(I_4^6)	(II_4)	(II_4^2)	(II_4^4)	(II_4^6)
$\lambda(F(2,2,2,n-6)\ ;\ \Delta)$	0	5	$*\notin\mathbf{Z}$	4	$*\notin\mathbf{Z}$	-1	0	1

by Lemma 4.7, Lemma 4.9, (5.1.2) and the simple caluculations.
Since



so we have the following table

cases	(I_4)	(I_4^2)	(I_4^4)	(I_4^6)	(II_4)	(II_4^2)	(II_4^4)	(II_4^6)
$\lambda(F(2, n-6)\ ;\ \Delta)$	$-34/4$	*	*	$-43/4$	*	*	$14/4$	*
$\lambda(F(2, 2, n-6)\ ;\ \Delta)$	*	$-35/2$	*	*	*	$-9/2$	*	$11/2$

ay (5. 1. 1). Since $\lambda(F(2, n-6)\ ;\ \Delta)$, $\lambda(F(2, 2, n-6)\ ;\ \Delta) \in \mathbf{Z}$, so this contradiction establishes the result.

In the cases $3 \leqq n \leqq 12$. These will be proved by the same way as in the above argument. *Q. E. D.*

Theorem 5. 13. *Let Δ be an element of $A(C_n)^*$ $(n \leqq 3)$ such that*

$$\Theta(\Delta) = 1_{n-1},$$

then we have

$$\Delta = 1_n.$$

Proof. It is trivial by Theorem 5. 11. and Lemma 5. 12. *Q. E. D.*

§ 6. Proof of Theorem A.

Theorem 6. 1. *The group homomorphisms*

$$\Theta : A(C_n)^* \longrightarrow A(C_{n-1})^* \quad (n \geqq 3)$$

are the injective homomorphisms.

Proof. It is trivial by Theorem 5. 13. *Q. E. D.*

Theorem 6.2. *We have*

$$|A(C_n)^*| \geqq 4 \quad (n \geqq 2).$$

Proof. Let V_0 be the complex representation defined in 2.2. We define two $\mathfrak{S}_n$-equivariant self homotopy equivalences $\xi_i : S(V_0) \longrightarrow S(V_0)$ $(i = 1, 2)$ by

$$\xi_0 = \textit{identity map}$$

and

$$\xi_1(\mathbf{z}_1, \cdots, \mathbf{z}_n) = (\bar{\mathbf{z}}_1, \cdots, \bar{\mathbf{z}}_n),$$

where $\bar{\mathbf{z}}$ is the conjugate of a complex number $\mathbf{z}$. Since

$$S(V_0)^{\mathfrak{S}_n(F(r))} = \{(\mathbf{z}_1, \cdots, \mathbf{z}_n\} \in S(V_0) \,|\, \mathbf{z}_i = \mathbf{z}_j \text{ for any } 1 \leqq i,\ j \leqq r\}$$

and

$$Deg^{\mathfrak{S}_r(F(r))}(\xi_i) = \begin{cases} 1 & \text{if } i = 0, \\ (-1)^{n-r+1} & \text{if } i = 1, \end{cases}$$

so $\{\pm[\xi_0],\ \pm[\xi_1]\}$ is the subgroup of order 4 of $\boldsymbol{E}_n(V_0)$ by (2.1.1). Therefore the desired result follows from Theorem 2.1. *Q.E.D.*

Theorem A. *Let V be a complex representation of $\mathfrak{S}_n$ $(n \geqq 2)$ with $dim_R V^{\mathfrak{S}_n} \geqq 2$, then we have*

$$|\boldsymbol{E}_n(V)| = 4 \quad \textit{if} \quad \boldsymbol{C}_n = \boldsymbol{O}(V)_n.$$

Proof. Since $\mathfrak{S}_2$ is a group of order 2, so we have

$$|\boldsymbol{E}_2(V)| = 4$$

by Corollary 3.7 of [1]. In fact, we have

$$\boldsymbol{A}(\boldsymbol{C}_2)^* = \{\pm 1_2,\ \pm(1_2 - \langle \mathfrak{S}_2/e \rangle)\}.$$

Therefore the desired result follows from Theorems 6.1–6.2 and Theorem 2.1.

Q.E.D.

Corollary A. *Let V_0 be the complex representation defined in 2.2, then we have*

$$\boldsymbol{E}_n(V_0) = \{\pm[\xi_0],\ \pm[\xi_1]\},$$

where ξ_i $(i = 1, 2)$ are the $\mathfrak{S}_n$-equivariant self homotopy equivalences defined in the proof of Theorem 6.2.

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