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On the admissible multiplication in α-coefficient cohomology theories II

By NOBUHIRO ISHIKAWA

Institute of Mathematics, College of General Education,
Kyushu University

and HIDEYUKI KACHI

Department of Mathematics, Facaulty of Science,
Shinshu, University

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Introduction

In the previous paper [I] with the same title we discussed the admissible multiplications in α-coefficient cohomology theories and we gave a sufficient condition for existence of admissible multiplication in the case $\alpha\in\pi_{r+k-1}(S^r)$ satisfies $1_C\wedge\alpha=(S^r i)\alpha'(S^{r+k-1}\pi)$ and $\tilde{t}(S^k\alpha)=0$.

This paper is the continuations of [I] and is devoted to the discussion of commutativity and associativity of admissible multiplication μ_α which is given by [I].

In §1 and §2, we discuss the associativity and commutativity of μ_α in the case $1_C\wedge\alpha=0$. For the case $1_C\wedge\alpha\neq 0$, we discuss in §3 to §5.

We use all notations and notions defined in [I].

§1. Preparation for case $1_C\wedge\alpha=0$.

Let α be the homotopy class of a stable map $g: S^{r+k-1}\longrightarrow S^r$ $(k>1)$ of order t. Since the stable homotopy type of the mapping cone of g depends only on α we denote as

$$C_\alpha=S^r\cup_g C(S^{r+k-1}).$$

For simplicity we denote $C=C_\alpha$.

Now we consider the stable element $\alpha\in\{S^{r+k-1}, S^r\}$ satisfying

$$(1.1)\qquad 1_C\wedge\alpha=0 \quad\text{and}\quad \tilde{t}(S^k\alpha)=0.$$

Then there exists a homotopy equivalence $\xi: C\wedge C\longrightarrow\overline{N}_\alpha=S^rC\vee S^{r+k}C$ and let $\zeta:\overline{N}_\alpha\longrightarrow C\wedge C$ be a homotopy inverse of ξ.

Let

$$\pi_0^{-1}: S^{r+k}C\longrightarrow\overline{N}_\alpha$$

and

$$i_0' : \overline{N}_\alpha \longrightarrow S^r C$$

be the inclusion and the map collapsing $S^{r+k}C$, respectively.

Put

$$i'' = \zeta \pi_0^{-1} : S^{r+k}C \longrightarrow C\wedge C$$

$$\pi'' = i_0'\xi : C\wedge C \longrightarrow S^r C,$$

then we obtain the relations

(1. 2) $$\pi'' i'' = 0,\ \pi''(1_C\wedge i) = 1_{S^r C} \text{ and } (1_C\wedge \pi) i'' = 1_{S^{r+k}C}.$$

Then we see immediately

Lemma 1. 1. $(1_C\wedge\pi)^*$, $(1_C\wedge i)_*$, π''^* *and* i''_* *are monomorphic and we have the following direct sum decompositions* ;

(i) $$\{W,\ C\wedge C\} = (1_C\wedge i)_*\{W,\ S^r C\} \oplus i''_*\{W,\ S^{r+k}C\},$$

(ii) $$\{C\wedge C,\ W\} = (1_C\wedge\pi)^*\{S^{r+k}C,\ W\} \oplus \pi''^*\{S^r C,\ W\}$$

for any W, *and in particular*

(iii) $$\{C\wedge C,\ C\wedge C\} = (1_C\wedge i)^*(1_C\wedge\pi)^*\{S^{r+k}C,\ S^r C\} \oplus (1_C\wedge i)_*\pi''^*\{S^r C,\ S^r C\} \oplus i''_*(1_C\wedge\pi)^*\{S^{r+k}C,\ S^{r+k}C\} \oplus i''^*\pi''^*\{S^r C,\ S^{r+k}C\}.$$

From Lemma 1. 1 and Lemma 2. 2 of [I],

Corollary 1. 2. *We have the following direct sum decomposition*

$$\begin{aligned}\{S^{r+k}C,\ C\wedge C\} &= (1_C\wedge i)_*\{S^{r+k}C,\ S^r C\} \oplus i''_*\{S^{r+k}C,\ S^{r+k}C\}\\ &= \{\delta\wedge i\} + \{i''\} + \{i''(\widetilde{t\pi}\wedge 1_{S^{r+k}})\} + \text{finite group}\\ &\cong Z+Z+Z+\text{finite group}\end{aligned}$$

and the relations

$$i''(S^{r+k}i) = \zeta_0,\ i''(S^{r+k}(\widetilde{t\pi}))(S^{r+k}i) = 0 \text{ and } (\delta\wedge i)\ (S^{r+k}i) = \widetilde{t}\wedge i.$$

In this section, we consider only the element $\gamma\in\{S^{r+k}C,\ C\wedge C\}$ satisfying the following relations

(1. 3) $$(1_C\wedge\pi)\gamma = (-1)^{r+k}1_{S^{r+k}C},$$

(1. 4) $$(1_C\wedge\pi)T\gamma = 1_{S^{r+k}C}$$

and

(1. 5) $$T(1_C\wedge i) + (-1)^{r+k}(1_C\wedge i) = (-1)^{k(r+k)}\gamma(S^{r+k}i)(S^r\pi) + 1/2\{(-1)^r + (-1)^{r+k}\}(i\overline{t}\wedge i) + (i\wedge i)g(S^r\pi)$$

for some $g\in G_k/(\eta\alpha)$ (see Proposition 2. 9 of [I]) and $T = T(C, C)$.

By Corollary 1. 2, we can put

(*) $$\gamma=(1_C\wedge i)a_1+i''a_2$$

with $a_1\in\{S^{r+k}C,\ S^rC\}$ and $a_2\in\{S^{r+k}C,\ S^{r+k}C\}$. Compose $(1_C\wedge\pi)$ on both sides of (*) from the left, then we get

$$\begin{aligned}(-1)^{r+k}1_{S^{r+k}C}&=(1_C\wedge\pi)\gamma && \text{by (1.3)}\\ &=(1_C\wedge\pi)i''a_2=a_2 && \text{by (1.2)}.\end{aligned}$$

Making use of Lemma 2.8 of [I] and the fact that $\xi_0 i''=0$, we have

$$\begin{aligned}(1_C\wedge\pi)Ti''&=(-1)^{r+k}(1_C\wedge\pi)i''-n'(i\bar{t}\wedge\pi)i''+(-1)^{r(r+k)}(S^{r+k}i)\xi_0 i''\\ &=(-1)^{r+k}1_{S^{r+k}C}-n'(i\bar{t}\wedge 1_{S^{r+k}}) && \text{by (1.2)},\end{aligned}$$

and

$$(1_C\wedge\pi)T(1_C\wedge i)=(-1)^{r(r+k)}(S^{r+k}i)(S^r\pi).$$

Now compose $(1_C\wedge\pi)T$ on (*) from the left. Using the above remarks, we have

$$\begin{aligned}1_{S^{r+k}C}&=(1_C\wedge\pi)T\gamma\\ &=(1_C\wedge\pi)T(1_C\wedge i)a_1+(1_C\wedge\pi)Ti''a_2\\ &=(-1)^{r(r+k)}(S^{r+k}i)(S^r\pi)a_1+1_{S^{r+k}C}-(-1)^{r+k}n'(i\bar{t}\wedge 1_{S^{r+k}}).\end{aligned}$$

Thus,

$$(-1)^{r(r+k)}(S^{r+k}i)(S^r\pi)a_1=(-1)^{r+k}n'(i\bar{t}\wedge 1_{S^{r+k}}).$$

Since $(S^{r+k}i)_*:\{S^{r+k}C,\ S^r\wedge S^{r+k}\}\longrightarrow\{S^{r+k}C,\ C\wedge S^{r+k}\}$ is an isomorphism into, we have

$$(S^r\pi)a_1=(-1)^{(r+1)(r+k)}n'(\bar{t}\wedge 1_{S^{r+k}}).$$

If $\tilde{t}(S^k\alpha)=0$, then, from (2.3) of [I], there exists an element δ of $\{S^kC,\ C\}$ satisfying the relation

$$(S^r\pi)(S^r\delta)=S^{r+k}\bar{t}.$$

Thus we have

$$(S^r\pi)a_1=(-1)^{(r+1)(r+k)}n'(S^r\pi)(S^r\delta).$$

Therefore we can put

$$a_1=(-1)^{(r+1)(r+k)}n'(S^r\delta)\qquad \text{mod } (S^ri)_*\{S^{r+k}C,\ S^{2r}\}$$

$$(\text{or } a_1=(-1)^{(r+1)(r+k)}n'(S^r\delta)+(S^ri)g_1\qquad \text{for some } g_1\in\{S^{r+k}C,\ S^{2r}\})$$

where $\{S^{r+k}C,\ S^{2r}\}$ is torsion group..

Proposition 1.3. *Assume that* $\tilde{t}(S^k\alpha)=0$. *Let* $\gamma\in\{S^{r+k}C,\ C\wedge C\}$ *be an element satisfying* (1.3) *and* (1.4). *Then there holds the relation*

(1.6) $$\gamma=(-1)^{r+k}i''+(-1)^{(r+1)(r+k)}n'(\delta\wedge i)+(i\wedge i)g_1$$

for some $g_1\in\{S^{r+k}C, S^{2r}\}$.

Let γ be an element satisfying (1.3) and (1.4). By Lemma 1.1, we can put

$$(**)\qquad T\gamma=(1_C\wedge i)b_1+i''b_2$$

with $b_1\in\{S^{r+k}C,\ S^rC\}$ and $b_2\in\{S^{r+k}C,\ S^{r+k}C\}$. Compose $1_C\wedge\pi$ on both sides of (**) from the left, then, by (1.4) and (1.2), we get

$$b_2=1_{S^{r+k}C}.$$

Compose $(1_C\wedge\pi)T$ from the left, then by the similary caluculation as in Proposition 1.3 we have

$$b_1=(-1)^{r+k}(-1)^{(r+1)(r+k)}n'(S^r\delta)\qquad \mathrm{mod}\ (S^ri)_*\{S^{r+k}C, S^{2r}\}.$$

Thus, from Proposition 1.3, we obtain

Proposition 1.4. *Assume that* $\tilde{t}(S^k\alpha)=0$. *Let* γ *be an element satisfying* (1.3) *and* (1.4). *Then*

$$(-1)^{r+k}T\gamma=\gamma\qquad mod\ (i\wedge i)_*\{S^{r+k}C, S^{2r}\}$$

where $T=T(C,\ C)$.

Lemma 1.5. *Let* γ *be an element of* $\{S^{r+k}C,\ C\wedge C\}$ *satisfying* (1.3). *Then we have*

$$(-1)^{r+k}(1_C\wedge\gamma)\ (\gamma\wedge 1_{S^{r+k}})=(\gamma\wedge 1_C)\ (1_C\wedge T')\ (\gamma\wedge 1_{S^{r+k}})\qquad mod\ G$$

where $G=(1_{C\wedge C}\wedge i)_*\{S^{2r+2k}C,\ C\wedge C\wedge S^r\}$ *and* $T'=T(C,\ S^{r+k})$.

Proof. By (1.3) and $1_{C\wedge C}=1_C\wedge 1_C$,

$$(-1)^{r+k}(1_{C\wedge C}\wedge\pi)\ (1_C\wedge\gamma)\ (\gamma\wedge 1_{S^{r+k}})=\gamma\wedge 1_{S^{r+k}}.$$

Compose $(1_{C\wedge C}\wedge\pi)$ on the right hand side from the left, we have

$$\begin{aligned}&(1_{C\wedge C}\wedge\pi)(\gamma\wedge 1_C)\ (1_C\wedge T')\ (\gamma\wedge 1_{S^{r+k}})\\ &=(\gamma\wedge 1_{S^{r+k}})\ (1_{C\wedge S^{r+k}}\wedge\pi)(1_C\wedge T')\ (\gamma\wedge 1_{S^{r+k}})\\ &=(\gamma\wedge 1_{S^{r+k}})\ (1_C\wedge T(S^{r+k},\ S^{r+k}))\ (1_C\wedge\pi\wedge 1_{S^{r+k}})(\gamma\wedge 1_{S^{r+k}})\\ &=\gamma\wedge 1_{S^{r+k}}\qquad\text{by (1.3).}\end{aligned}$$

From the exact sequence associated with cofibration $C\wedge C\wedge S^r\xrightarrow{1_{C\wedge C}\wedge i}C\wedge C\wedge C\xrightarrow{1_{C\wedge C}\wedge\pi}C\wedge C\wedge S^{r+k}$;

$$\longrightarrow\{S^{2r+2k}C,\ C\wedge C\wedge S^r\}\xrightarrow{(1_{C\wedge C}\wedge i)^*}\{S^{2r+2k}C,\ C\wedge C\wedge C\}\xrightarrow{(1_{C\wedge C}\wedge\pi)^*}\{S^{2r+2k}C,\ C\wedge C\wedge S^{r+k}\}\longrightarrow$$

we obtain the result. q.e.d.

Lemma 1.6. *Let* γ *be an element of* $\{S^{r+k}C,\ C\wedge C\}$ *satisfying* (1.3) *and* (1.4). *Then we have*

$$(-1)^{r+k}(1_C\wedge\gamma)(\gamma\wedge 1_{S^{r+k}})=(\gamma\wedge 1_C)(1_C\wedge T')(\gamma\wedge 1_{S^{r+k}}) \qquad mod\ G'$$

where $G'=(1_C\wedge i\wedge 1_C)_*\{S^{2r+2k}C,\ C\wedge S^r\wedge C\}$ *and* $T'=T(C,\ S^{r+k})$.

Proof. We put $T=T(C,\ C)$. We have

$$\begin{aligned}
&(-1)^{r+k}(1_C\wedge\pi\wedge 1_C)(1_C\wedge\gamma)(\gamma\wedge 1_{S^{r+k}})\\
&\quad=(-1)^{r+k}(1_C\wedge T')(1_{C\wedge C}\wedge\pi)(1_C\wedge T)(1_C\wedge\gamma)(\gamma\wedge 1_{S^{r+k}})\\
&\quad=(-1)^{r+k}(1_C\wedge T')(1_C\wedge(1_C\wedge\pi)T\gamma)(\gamma\wedge 1_{S^{r+k}})\\
&\quad=(-1)^{r+k}(1_C\wedge T')(\gamma\wedge 1_{S^{r+k}}). \qquad \text{by (1. 4).}
\end{aligned}$$

On the other hand, by (1. 3),

$$\begin{aligned}
&(1_C\wedge\pi\wedge 1_C)(\gamma\wedge 1_C)(1_C\wedge T')(\gamma\wedge 1_{S^{r+k}})\\
&\quad=((1_C\wedge\pi)(\gamma\wedge 1_C)(1_C\wedge T')(\gamma\wedge 1_{S^{r+k}})\\
&\quad=(-1)^{r+k}(1_C\wedge T')(\gamma\wedge 1_{S^{r+k}}).
\end{aligned}$$

From the exact se quence associated with cofibration $C\wedge S^r\wedge C\xrightarrow{1_C\wedge i\wedge 1_C}C\wedge C\wedge C$ $\xrightarrow{1_C\wedge\pi\wedge 1_C} C\wedge S^{r+k}\wedge C$;

$$\longrightarrow\{S^{2r+2k}C,\ C\wedge S^r\wedge C\}\xrightarrow{(1_C\wedge i\wedge 1_C)^*}\{S^{2r+2k}C,\ C\wedge C\wedge C\}\xrightarrow{(1_C\wedge\pi\wedge 1_C)^*}\{S^{2r+2k}C,\ C\wedge S^{r+k}\wedge C\}\longrightarrow$$

we obtain Lemma. q. e. d.

Proposition 1. 7. *Let γ be an element of $\{S^{r+k}C,\ C\wedge C\}$ satisfying* (1. 3) *and* (1. 4). *Then we have*

$$(-1)^{r+k}(1_C\wedge\gamma)(\gamma\wedge 1_{S^{r+k}})=(\gamma\wedge 1_C)(1_C\wedge T')(\gamma\wedge 1_{S^{r+k}}) \qquad mod\ G''$$

where $G''=(1_C\wedge i\wedge i)_*\{S^{2r+k}C,\ C\wedge S^r\wedge S^r\}$ *and* $T'=T(C,\ S^{r+k})$.

Proof. For simplicity we put $a=(-1)^{r+k}(1_C\wedge\gamma)(\gamma\wedge 1_{S^{r+k}})$ and $b=(\gamma\wedge 1_C)(1_C\wedge T')(\gamma\wedge 1_{S^{r+k}})$.

From Lemmas 1. 5 and 1. 6,

$$\begin{aligned}
a-b&=(1_C\wedge 1_C\wedge i)c\\
&=(1_C\wedge i\wedge 1_C)c'
\end{aligned}$$

for some $c\in\{S^{2r+2k}C,\ C\wedge C\wedge S^r\}$ and $c'\in\{S^{2r+2k}C,\ C\wedge S^r\wedge C\}$.

Thus we have

$$\begin{aligned}
0&=(1_C\wedge\pi\wedge 1_C)(1_C\wedge i\wedge 1_C)c'=(1_C\wedge\pi\wedge 1_C)(1_C\wedge 1_C\wedge i)c\\
&=(1_C\wedge 1_{S^{r+k}}\wedge i)(1_C\wedge\pi\wedge 1_{S^r})c.
\end{aligned}$$

Since $(1_C\wedge 1_{S^{r+k}}\wedge i)_*:\{S^{2r+2k}C,\ C\wedge S^{r+k}\wedge S^r\}\longrightarrow\{S^{2r+2k}C,\ C\wedge S^{r+k}\wedge C\}$ is an isomorphism into, we have

$$(1_C\wedge\pi\wedge 1_{S^r})c=0.$$

Thus, from the homotopy exact sequence associated with cofibration $C\wedge S^r\wedge S^r \xrightarrow{1_C\wedge i\wedge 1_{S^r}} C\wedge C\wedge S^r \xrightarrow{1_C\wedge\pi\wedge 1_{S^r}} C\wedge S^{r+k}\wedge S^r$, there exists an element $c''\in\{S^{2r+2k}C, C\wedge S^r\wedge S^r\}$ such that $(1_C\wedge i\wedge 1_{S^r})c''=c$.

Hence we obtain that

$$\begin{aligned} a-b &= (1_C\wedge 1_C\wedge i)c \\ &= (1_C\wedge 1_C\wedge i)\,(1_C\wedge i\wedge 1_{S^r})c'' \\ &= (1_C\wedge i\wedge i)c''. \end{aligned}$$

q. e. d.

§ 2. Commutativity and associativity for case $1_C\wedge\alpha=0$.

Hereafter, we use the following convention : for each $x\in\tilde{h}^i(X;\alpha)$ which is the same as $\tilde{h}^{i+r+k}(X\wedge C)$ by definintion, we denote x as $\bar{x}$ when we consider it as an element of $\tilde{h}^{i+r+k}(X\wedge C)$.

Let μ be the associative and commutative multiplication in the reduced generalized cohomology theory $\{\tilde{h}^*,\sigma\}$ defined on the category of finite CW-complexes and $\{\tilde{h}^*(\ ;\alpha),\ \sigma_\alpha\}$ be the α-coefficient cohomology theory associated $\{\tilde{h}^*,\sigma\}$ defined in [I].

Making use of an element $\gamma\in\{S^{r+k}C, C\wedge C,\}$ we define a map

$$\mu_\alpha : \tilde{h}^i(X;\alpha)\otimes\tilde{h}^j(Y;\alpha) \longrightarrow \tilde{h}^{i+j}(X\wedge Y;\alpha) \tag{2. 1}$$

as the composition

$$\mu_\alpha=(-1)^{i(r+k)}\sigma^{-(r+k)}(1_{X\wedge Y}\wedge\gamma)^*\ (1_X\wedge T''\wedge 1_C)^*\mu$$

where $T''=T(Y,\ C)$.

For any element $\beta=\{g\}$ of $\{A,\ B\}$, $\beta^{**} : \tilde{h}^*(X\wedge B) \longrightarrow \tilde{h}^*(X\wedge A)$ is denoted by $\beta^{**}=\sigma^{-n}(1_X\wedge g)^*\sigma^n$.

Proposition 2. 1. *If $\gamma\in\{S^{r+k}C,\ C\wedge C\}$ satisfyies*

$$(T\gamma)^{**}=(-1)^{r+k}\gamma^{**} \tag{2. 2}$$

in $\tilde{h}^*$, then the relation

$$T_1{}^*\mu_\alpha(x\otimes y)=(-1)^{ij}\mu_\alpha(y\otimes x)$$

holds for $x\in\tilde{h}^i(X;\alpha)$ and $y\in\tilde{h}^j(Y;\alpha)$, where $T=T(C,\ C)$ and $T_1=T(Y,\ X)$.

Proof. On $\tilde{h}^i(X;\alpha)\otimes\tilde{h}^j(Y;\alpha)$, we have

$$\begin{aligned} T_1{}^*\mu_\alpha(x\otimes y) &= (-1)^{i(r+k)}(T_1\wedge 1_C)^*\sigma^{-(r+k)}(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T''\wedge 1_C)^*\mu(\overline{x}\otimes\overline{y}) \\ &= (-1)^{i(r+k)}\sigma^{-(r+k)}\ (T_1\wedge 1_C\wedge 1_{S^{r+k}})^*(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T''\wedge 1_C)^*\mu(\overline{x}\otimes\overline{y}) \\ &= (-1)^{i(r+k)}\sigma^{-(r+k)}\ (1_{Y\wedge X}\wedge\gamma)^*(T_1\wedge 1_{C\wedge C})^*\ (1_X\wedge T''\wedge 1_C)\ ^*\mu(\overline{x}\otimes\overline{y}) \\ &= (-1)^{i(r+k)+r+k}\sigma^{-(r+k)}\ (1_{Y\wedge X}\wedge\gamma)^*(1_{Y\wedge X}\ \wedge T)^*(T_1\wedge 1_{C\wedge C})\ ^*(1_X\wedge T''\wedge 1_C)\ ^*\mu(\overline{x}\otimes\overline{y}) \\ &\qquad\text{by (2. 2)} \\ &= (-1)^{i(r+k)+r+k}\sigma^{-(r+k)}\ (1_{Y\wedge X}\wedge\gamma)^*(1_Y\wedge T(X,C)\wedge 1_C)^*T(Y\wedge C, X\wedge C)^*\mu(\overline{x}\otimes\overline{y}) \\ &= (-1)^{j(r+k)+ij}\sigma^{-(r+k)}\ (1_{Y\wedge X}\wedge\gamma)^*(1_Y\wedge T(X,C)\wedge 1_C)^*\mu(\overline{y}\otimes\overline{x}) \end{aligned}$$

by commutativity of μ

$=(-1)^{ij}\mu_\alpha(y\otimes x)$. q. e. d.

As a consequence of Propositions 1. 4, 2. 1 and Theorem 3. 3 of [I], we obtain the following Theorem.

Theorem 2. 2 *Assume that an element* $\alpha\in\pi_{r+k-1}(S^r)$ *satisfies* (1. 1) *and* $t=2$ *if* k *is even. If* $\varepsilon^{**}=0$ *in* $\tilde{h}^*$ *for any* $\varepsilon\in\{S^{r+k}C, S^{2r}\}$, *then there exists a commutative admissible multiplication in* $\tilde{h}^*(\ ;\alpha)$.

Proposition 2. 3. *If* $\gamma\in\{S^{r+k}C, C\wedge C\}$ *satisfies the relation*

$$(-1)^{r+k}(1_C\wedge\gamma)\,(\gamma\wedge 1_{S^{r+k}}))^{**}=((\gamma\wedge 1_C)\,(1_C\wedge T')\,(\gamma\wedge 1_{S^{r+k}}))^{**} \tag{2. 3}$$

in $\tilde{h}^*$ *where* $T'=T(C,S^{r+k})$, *then the map* μ_α *of* (2. 1) *satisfies*

$$\mu_\alpha(\mu_\alpha\otimes 1)\ =\ \mu_\alpha(1\otimes\mu_\alpha).$$

Proof. Put $W=X\wedge Y\wedge Z$, the map $U: W\wedge C\wedge C\wedge C\longrightarrow X\wedge C\wedge Y\wedge C\wedge Z\wedge C$ is given by permutation of factors as $U(x, y, z, p, p', p'')=(x, p, y, p', z, p'')$.

On $\tilde{h}^i(X;\alpha)\otimes\tilde{h}^j(Y;\alpha)\otimes\tilde{h}^l(Z;\alpha)$, by the definition of μ_α and a simple calculations, we obtain that

$$\begin{aligned}
&\mu_\alpha(\mu_\alpha\otimes 1)\\
&\quad=(-1)^{j(r+k)}\sigma^{-(r+k)}\,(1_W\wedge\gamma)^*(1_{X\wedge Y}\wedge T(Z,C)\wedge 1_C)^*\\
&\qquad\qquad\qquad\mu(\sigma^{-(r+k)}\,(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T(Y,C)\wedge 1_C)^*\mu\otimes 1)\\
&\quad=(-1)^{j(r+k)}\sigma^{-2(r+k)}\,(\gamma\wedge 1_{S^{r+k}})^{**}T(C,S^{r+k})^{**}(\gamma\wedge 1_C)^{**}U^*\mu(\mu\otimes 1)
\end{aligned}$$

and

$$\begin{aligned}
&\mu_\alpha(1\otimes\mu_\alpha)\\
&\quad=(-1)^{(i+j)(r+k)}\sigma^{-(r+k)}(1_W\wedge\gamma)^*(1_X\wedge T(Y\wedge Z,C)\wedge 1_C)^*\\
&\qquad\qquad\qquad\mu(1\otimes\sigma^{-(r+k)}(1_{Y\wedge Z}\wedge\gamma)^*\ (1_Y\wedge T(Z,C)\wedge 1_C)^*\mu)\\
&\quad=(-1)^{(j+1)(r+k)}\sigma^{-2(r+k)}(\gamma\wedge 1_{S^{r+k}})^{**}(1_C\wedge\gamma)^{**}U^*\mu(1\otimes\mu).
\end{aligned}$$

Thus, from the associativity of μ and (2. 3), it follows that

$$\mu_\alpha(\mu_\alpha\otimes 1)\ =\mu_\alpha(1\otimes\mu_\alpha).$$ q. e. d.

From Propositions 1. 7, 2. 3 and Theorem 3. 3 of [I], we obtain

Theorem 2. 4. *Let* α *be an element of* $\pi_{r+k-1}(S^r)$ *satisfying* (1. 1) *and* $t=2$ *if* k *is even. If* $\varepsilon^{**}=0$ *in* $\tilde{h}^*$ *for any* $\varepsilon\in\{S^{2r+2k}C, S^{2r}C\}$, *then there exists an associative admissible multiplication in* $\tilde{h}^*(\ ;\alpha)$.

Let η be the stable homotopy class of Hopf map $S^3\longrightarrow S^2$ and $\eta^2=\eta(S\eta)$ be a generator of stable homotopy group $\{S^{r+2}, S^r\}\cong Z_2$. Then η^2 satisfy (1. 1). From Theorem 3. 3 of [I], there exists the admissible multiplication μ_{η^2} in $\tilde{h}^*(\ ;\eta^2)$.

From Puppe's exact sequence associated with a cofibration

$$S^r \xrightarrow{i} C_{\eta^2} \xrightarrow{\pi} S^{r+3},$$

we obtain that

(2.4) $$\{S^3C_{\eta^2}, S^r\} = \{\nu^2\pi\} + \{\bar{\nu}\} \cong Z_2 + Z_{24}$$

where ν is the generator of stable homotopy group $\{S^{r+3}, S^r\} \cong Z_{24}$ and $\bar{\nu}$ is defined by $\bar{\nu}(S^3i) = \nu$.

Since $\eta^2\nu^2\pi = 0$ and $\eta^2\bar{\nu} = 0$, it follows that $\pi_* : \{S^6C_{\eta^2}, C_{\eta^2}\} \longrightarrow \{S^6C_{\eta^2}, S^{r+3}\}$ is an epimorphism. Thus we have

Corollary 2.5. *If $\varepsilon^{**} = 0$ in $\tilde{h}^*$ for any $\varepsilon \in \{S^6C_{\eta^2}, C_{\eta^2}\}$, then there exists a commutative and associative admissible multiplication in $\tilde{h}^*(\ ;\eta^2)$.*

§ 3. Stable homotopy of some complexes

In this section, let α be the element of $\pi_{r+k-1}(S^r)$ satisfying

(3.1) $$1_C \wedge \alpha = (S^r i)\alpha'(S^{r+k}\pi) \neq 0 \quad \text{and} \quad \tilde{t}(S^k\alpha) = 0$$

for some non-trivial element α' of $\pi_{2r+2k-1}(S^{2r})$ and the integer t such that $t\alpha = 0$ and $t = 2$ if k is even (cf. Lemma 2.3 of [I]).

We put $N\alpha = (S^{2r} \vee S^{2r+k}) \cup e^{2r+2k}$, where e^{2r+2k} is attached to $S^{2r} \vee S^{2r+k}$ by a map represented by sum of α' and $S^{r+k}\alpha$. Let Q be the mapping cone of α', i.e., $Q = S^{2r} \cup e^{2r+2k}$. Let $i' : S^{2r} \longrightarrow Q$ and $\pi' : Q \longrightarrow S^{2r+2k}$ be the inclusion and the map collapsing S^{2r} respectively. Then we have a cofibration

(3.2) $$C \wedge S^{2r} \xrightarrow{1_C \wedge i'} C \wedge Q \xrightarrow{1_C \wedge \pi'} C \wedge S^{2r+2k}.$$

From Puppe's exact sequence associated with (3.2) and Lemma 2.2 of [I], we obtain

Lemma 3.1. *The groups $\{C \wedge Q, C \wedge S^{2r+j}\}$ are isomorphic to the corresponding groups in the following table;*

		generators of free part
$\{C\wedge Q, C\wedge S^{2r-k}\}$	Z+*finite group*	
$\{C\wedge Q, C\wedge S^{2r}\}$	$Z+Z$+*finite group*	u, v
$\{C\wedge Q, C\wedge S^{2r+k}\}$	$Z+Z$+*finite group*	$\delta\wedge\pi', w$
$\{C\wedge Q, C\wedge S^{2r+2k}\}$	$Z+Z$+*finite group*	$\tilde{i}\pi\wedge\pi', 1_C\wedge\pi'$
$\{C\wedge Q, C\wedge S^{2r+3k}\}$	Z	$(S^{2r+3k}i)(\pi\wedge\pi')$
$\{C\wedge Q, C\wedge S^{2r+j}\}$	*finite group for* $j \neq -k, 0, k, 2k, 3k$.	

The elements u, v *and* w *are defined by* $u(1_C\wedge i')=t_1 1_C$, $v(1_C\wedge i')=t_2\tilde{t}\pi$ *and* $w(1_C\wedge i')= t_3(S^{2r+k}i)(S^{2r}\pi)$ *where* t_1, t_2 *and* t_3 *are order of elements* $1_C\wedge\alpha'$, $\tilde{t}\pi\wedge\alpha'$ *and* $(1_C\wedge\alpha')(S^k i)\pi$ *respectively.*

Lemma 3.2.

$$\{C\wedge Q, C\wedge Q\}=\{\tilde{t}\pi\wedge 1_Q\}+\{1_{C\wedge Q}\}+\{(1_C\wedge i')u\}+\{(1_C\wedge i')v\}+ \text{finite group}$$
$$\cong Z+Z+Z+Z+\text{finite group}.$$

Since the complex $C\wedge C$ is homotopically equivalent with the complex $N_\alpha\cup_{i_0\alpha} e^{2r+k}$ in stable range, we shall see that

$$C\wedge C\wedge C \simeq C\wedge(N_\alpha\cup_{i_0\alpha}e^{2r+k}) \simeq (C\wedge N_\alpha)\cup_f C(S^{2r+k-1}C),$$

where $f=1_C\wedge i_0\alpha$. From the complex structure of N_α, $i_0\alpha'$ is homotopic to $i_1\alpha$. Thus we have

$$\begin{aligned} f=1_C\wedge i_0\alpha &=(1_C\wedge i_0)\,(S^{2r}i)\,(S^r\alpha')\,(S^{2r+k-1}\pi)\\ &=(i\wedge i_0\alpha')\,(S^{2r+k-1}\pi)\\ &=(i\wedge i_1\alpha)\,(S^{2r+k-1}\pi)\\ &=(1_C\wedge i_1)\,(S^{2r+k}i)\,(S^{2r+k}\alpha)\,(S^{2r+k-1}\pi)\\ &=0. \end{aligned}$$

Consider the cofibration

$$S^{2r+k}\xrightarrow{i_1}N_\alpha\xrightarrow{\pi_1}Q,$$

we have

$$C\wedge N_\alpha \simeq C\wedge(S^{2r+k}\cup_{\alpha\pi'}C(S^{-1}Q)) \simeq (C\wedge S^{2r+k})\cup_g C(C\wedge S^{-1}Q),$$

where

$$\begin{aligned} g=1_C\wedge\alpha\pi' &=(S^{2r+k}i)\,(S^{r+k}\alpha')\,(S^{2r+2k-1}\pi)\,(1_C\wedge S^{-1}\pi')\\ &=(S^{2r+k}i)\,(\pi\wedge\alpha'\pi')\\ &=0. \end{aligned}$$

Thus, we obtain

Lemma 3.3. *The complex* $C\wedge C\wedge C$ *is homotopically equivalent with* $(S^{2r+k}C)\vee(C\wedge Q)\vee(S^{2r+k}C)$ *in stable range.*

Then, from above Lemma, it follows that

Proposition 3.4.

$$\{C\wedge Q, C\wedge C\wedge C\}=\{C\wedge Q, S^{2r+k}C\}\oplus\{C\wedge Q, C\wedge Q\}\oplus\{C\wedge Q, S^{2r+k}C\}$$
$$\cong Z+Z+Z+Z+Z+Z+Z+Z+\text{finite group}.$$

The following two lemmas will be used in the later sections.

Lemma 3.5. *We can choose an element* $p_0 \in \{C\wedge Q,\ N\alpha \wedge S^{r+k}\}$ *satisfying the relations;*

(3.3) $$p_0(1_C\wedge i')=(i_0\wedge 1_{S^{r+k}})(\pi\wedge 1_{S^{2r}}),$$

(3.4) $$(\pi_0\wedge 1_{S^{r+k}})p_0=1_C\wedge\pi'$$

and

(3.5) $$(\pi_1\wedge 1_{S^{r+k}})p_0=(1_Q\wedge\pi)T \qquad \textit{for } T=T(C,Q).$$

Proof. Since $N\alpha$ is the reduced mapping cone of the map $\alpha'(S^{r+k-1}\pi) : S^{r+k-1}C \longrightarrow S^{2r}$, we have

$$\begin{aligned}(i_0\wedge 1_{S^{r+k}})(\pi\wedge 1_{S^{2r}})(1_C\wedge\alpha') &= (i_0\wedge 1_{S^{r+k}})(\alpha'\wedge 1_{S^{r+k}})(\pi\wedge 1_{S^{2r+2k-1}})\\ &= S^{r+k}(i_0\alpha'(S^{r+k-1}\pi))\\ &=0.\end{aligned}$$

From the Puppe's exact sequence

$$\longrightarrow \{C\wedge S^{2r+2k}, N\alpha\wedge S^{r+k}\} \xrightarrow{(1_C\wedge\pi')^*} \{C\wedge Q,\ N\alpha\wedge S^{r+k}\} \xrightarrow{(1_C\wedge i')^*} \{C\wedge S^{2r},\ N\alpha\wedge S^{r+k}\} \xrightarrow{(1_C\wedge\alpha')^*}$$

associated to (3.2), it follows that there exists an element $p_0'' \in \{C\wedge Q,\ N\alpha\wedge S^{r+k}\}$ such that

$$\begin{aligned}p_0''(1_C\wedge i') &= (i_0\wedge 1_{S^{r+k}})(\pi\wedge 1_{S^{2r}})\\ &= (i_0\wedge 1_{S^{r+k}})T(S^{r+k}, S^{2r})(\pi\wedge 1_{S^{2r}}).\end{aligned}$$

Consider the Puppe's exact sequence

$$\longrightarrow \{C\wedge S^{2r+2k}, Q\wedge S^{r+k}\} \xrightarrow{(1_C\wedge\pi')^*} \{C\wedge Q, Q\wedge S^{r+k}\} \xrightarrow{(1_C\wedge i')^*} \{C\wedge S^{2r},\ Q\wedge S^{r+k}\}.$$

Since

$$\begin{aligned}(1_C\wedge i')^*(1_Q\wedge\pi)T(C,\ Q) &= (1_Q\wedge\pi)T(C,\ Q)\,(1_C\wedge i')\\ &= (\pi_1\wedge 1_{S^{r+k}})(i_0\wedge 1_{S^{r+k}})T(S^{r+k},\ S^{2r})(\pi\wedge 1_{S^{2r}})\\ &= (\pi_1\wedge 1_{S^{r+k}})p_0''(1_C\wedge i')\\ &= (1_C\wedge i)^*((\pi_1\wedge 1_{S^{r+k}})p_0''),\end{aligned}$$

it follows from the exactness of the above sequence that

$$(\pi_1\wedge 1_{S^{r+k}})p_0''-(1_Q\wedge\pi)T(C,\ Q)\in(1_C\wedge\pi')^*\{C\wedge S^{2r+2k},\ Q\wedge S^{r+k}\}.$$

Since $(\pi_1\wedge 1_{S^{r+k}})_* : \{C\wedge S^{2r+2k},\ N\alpha\wedge S^{r+k}\} \longrightarrow \{C\wedge S^{2r+2k},\ Q\wedge S^{r+k}\}$ *is* an epimorphism, we have

$$(\pi_1\wedge 1_{S^{r+k}})p_0''-(1_Q\wedge\pi)T(C,Q)=(\pi_1\wedge 1_{S^{r+k}})d(1_C\wedge\pi')$$

for some $d\in\{C\wedge S^{2r+2k},\ N\alpha\wedge S^{r+k}\}$.

Now we put

$$p_0' = p_0'' - d(1_C \wedge \pi').$$

Then p_0' satisfy (3.3) and (3.5).

Discussing the Puppe's exact sequence associated with the cofibration (3.2), we have

$$(1_C \wedge \pi')^* : \{C \wedge S^{2r+2k}, C \wedge S^{2r+2k}\} \cong \{C \wedge Q, C \wedge S^{2r+2k}\}.$$

Thus from Lemma 2.2 of [I] we can put

$$(*) \qquad (\pi_0 \wedge 1_{S^{r+k}}) p_0' = a(1_C \wedge \pi') + (i \wedge 1_{S^{2r+2k}}) e (1_C \wedge \pi')$$

for some $e \in \{C \wedge S^{2r+2k}, S^r \wedge S^{2r+2k}\}$ and some integer a. Composing $\pi \wedge 1_{S^{2r+2k}}$ on both sides of (*) from left, we heve

$$\begin{aligned}(\pi \wedge 1_{S^{2r+2k}})(\pi_0 \wedge 1_{S^{r+k}}) p_0' &= (\pi' \wedge 1_{S^{r+k}})(\pi' \wedge 1_{S^{r+k}}) p_0' \\ &= (\pi' \wedge 1_{S^{r+k}})(1_Q \wedge \pi) T(C, Q) \\ &= \pi \wedge \pi'\end{aligned}$$

and

$$(\pi \wedge 1_{S^{2r+2k}})(a(1_C \wedge \pi') + (i \wedge 1_{S^{2r+2k}})\, e\, (1_C \wedge \pi')) = a(\pi \wedge \pi').$$

Thus we see that $a=1$.

For $i \wedge 1_{S^{r+k}} = \pi_0 i_1'$

$$(\pi_0 \wedge 1_{S^{r+k}}) p_0' = 1_C \wedge \pi' + (\pi_0 \wedge 1_{S^{r+k}})(i_1 \wedge 1_{S^{r+k}})\, e\, (1_C \wedge \pi').$$

Now we put

$$p_0 = p_0' - (i_1 \wedge 1_{S^{r+k}}) e (1_C \wedge \pi').$$

Then we can see that p_0 satisfy the relations (3.3)–(3.5).

Lemma 3.6. *For any $\beta' \in \{N\alpha, C \wedge C\}$ satisfying $(1_C \wedge \pi)\beta' = \pi_0$, there exists an element $\kappa = \kappa_{\beta'} \in \{C \wedge Q, C \wedge N\alpha\}$ such that*

$$(3.6) \qquad (1_C \wedge \pi_0)\kappa = (\beta' \wedge 1_{S^{r+k}}) p_0 \text{ and } (1_C \wedge \pi_1)\kappa = 1_{C \wedge Q}.$$

Proof. From Lemma 3.5, we have

$$\begin{aligned}(1_C \wedge (S^1\alpha')\pi)(\beta' \wedge 1_{S^{r+k}}) p_0 &= (1_C \wedge S^1\alpha')(1_C \wedge \pi \wedge 1_{S^{r+k}})(\beta' \wedge 1_{S^{r+k}}) p_0 \\ &= (1_C \wedge S^1\alpha')(\pi_0 \wedge 1_{S^{r+k}}) p_0 \\ &= (1_C \wedge S^1\alpha')(1_C \wedge \pi') \\ &= 0.\end{aligned}$$

Thus from the exact sequence

$$\{C \wedge Q, C \wedge N\alpha\} \xrightarrow{(1_C \wedge \pi_0)_*} \{C \wedge Q, C \wedge C \wedge S^{r+k}\} \xrightarrow{(1_C \wedge (S^1\alpha')\pi)_*} \{C \wedge Q, C \wedge S^{2r+1}\}$$

it follows that there exists an element $\kappa' \in \{C \wedge Q, C \wedge N\alpha\}$ such that

$$(1_C\wedge\pi_0)\kappa'=(\beta'\wedge 1_{S^{r+k}})p_0.$$

Now

$$\begin{aligned}(1_C\wedge\pi')((1_C\wedge\pi_1)\kappa'-1_{C\wedge Q}) &= (1_C\wedge\pi\wedge 1_{S^{r+k}})(1_C\wedge\pi_0)\kappa'-1_C\wedge\pi' \\ &= (1_C\wedge\pi\wedge 1_{S^{r+k}})(\beta'\wedge 1_{S^{r+k}})p_0-1_C\wedge\pi' \\ &= (\pi_0\wedge 1_{S^{r+k}})p_0-1_C\wedge\pi' \qquad \text{by assumption} \\ &= 0 \qquad \text{by (3. 4).}\end{aligned}$$

Thus, for some element $x\in\{C\wedge Q,\ C\wedge S^{2r}\}$,

$$\begin{aligned}(1_C\wedge\pi_1)\kappa'-1_{C\wedge Q} &= (1_C\wedge i')x \\ &= (1_C\wedge\pi_1)(1_C\wedge i_0)x,\end{aligned}$$

because $i'=\pi_1 i_0$. Put

$$\kappa=\kappa'-(1_C\wedge i_0)x,$$

then

$$(1_C\wedge\pi_1)\kappa=(1_C\wedge\pi_1)\kappa'-(1_C\wedge\pi_1)(1_C\wedge i_0)x=1_{C\wedge Q}$$

and

$$\begin{aligned}(1_C\wedge\pi_0) &= (1_C\wedge\pi_0)\kappa'-(1_C\wedge\pi_0)(1_C\wedge i_0)x \\ &= (\beta'\wedge 1_{S^{r+k}})p_0. \qquad \text{q. e. d.}\end{aligned}$$

§ 4. Commutativity of μ_α for the case $1_C\wedge\alpha\neq 0$.

From Propositions 2. 11 and 2. 12 of [I], there exists an element β of $\{N\alpha,\ C\wedge C\}$ which satisfies

(4. 1) $(1_C\wedge\pi)\beta=(-1)^{r+k}\pi_0$,

(4. 2) $(1_C\wedge\pi)T\beta=\pi_0$

and

(4. 3) $$\begin{aligned}&T(1_C\wedge i)+(-1)^{r+k}(1_C\wedge i) \\ &=(-1)^{k(r+k)}\beta i_1(S^r\pi)+1/2\{(-1)^r+(-1)^{r+k}\}(\overline{it}\wedge i)+(i\wedge i)g(S^r\pi)\end{aligned}$$

for some $g\in G_k$, where $T=T(C,\ C)$.

Composing $T=T(C,\ C)$ on both sides (4. 3) from the left, it follows that

(4. 4) $$T\beta i_1-(-1)^{r+k}\beta i_1=(-1)^{k(r+k)}\{(-1)^{r+k}+(-1)^{r+1}\}(i\wedge i)g,$$

since $(S^r\pi)^*:\{S^{2r+k},\ C\wedge C\}\longrightarrow\{C\wedge S^r,\ C\wedge C\}$ is an isomorphism into.

From (4. 1) and (4. 2), there exists an element of β_1 of $\{N\alpha,\ S^rC\}$ such that

(4. 5) $$T\beta-(-1)^{r+k}\beta=(1_C\wedge i)\beta_1.$$

Thus, it follows from (4. 4) and (4. 5) that

$$\begin{aligned}(1_C\wedge i)\beta_1 i_1 &= (T\beta-(-1)^{r+k}\beta)i_1\\ &=\{(-1)^{(r+k)(k+1)}+(-1)^{(k+1)(r+1)}\}(i\wedge i)g\\ &=\{(-1)^{k+1}+1\}(i\wedge i)\}g.\end{aligned}$$

Since $(1_C\wedge i)_* : \{S^{2r+k}, C\wedge S^r\}\longrightarrow\{S^{2r+k}, C\wedge C\}$ is an isomorphism into, we obtain that

$$\beta_1 i_1=\{(-1)^{k+1}+1\}(i\wedge 1_{S^r})g. \tag{4.6}$$

Paticularly, if k is even, then $\beta_1 i_1=0$. Thus, from the exactness of Puppe's sequence

$$\{Q,\ S^rC\}\xrightarrow{\pi_1{}^*}\{N_\alpha,\ S^rC\}\xrightarrow{i_1{}^*}\{S^{2r+k},\ S^rC\}$$

we obtain

Proposition 4.1. *If k is even, then*

$$T\beta-(-1)^{r+k}\beta\in(1_C\wedge i)\{Q, S^rC\}\pi_1. \tag{4.7}$$

If k is odd, we put

$$\beta=(-1)^{(r+1)(k+1)}n_0(\delta\wedge i)\pi_0+(-1)^{r(r+k)}\zeta j$$

and if k is even, we put

$$\beta=(-1)^r(1-n_0)\,(\delta\wedge i)\pi_0+(-1)^{r(r+k)}\zeta j$$

(c. f., Propositions 2.11 and 2.12 of [I]).
Then this element satisfies (4.1), (4.2) and (4.3).

Proposition 4.2. *For ordinary homology maps induced by $(-1)^{r+k}\beta$ and $T\beta$, we have*

$$(T\beta)_*=(-1)^{r+k}\beta_* \qquad \textit{if } k \textit{ is even}$$

and

$$(T\beta)_*\neq(-1)^{r+k}\beta_* \qquad \textit{if } k \textit{ is odd.}$$

Proof. Let

$$\begin{pmatrix} e_r\wedge s_r\\ e_r\wedge s_{r+k}\\ e_{r+k}\wedge s_{r+k}\end{pmatrix}\qquad\begin{pmatrix} e_r\wedge e_r\\ e_r\wedge e_{r+k},\ e_{r+k}\wedge e_r\\ e_{r+k}\wedge e_{r+k}\end{pmatrix}$$

be generators systems of groups $H_*(N\alpha)$ and $H_*(C\wedge C)$ respectively, where $e_i\wedge s_j$ and $e_i\wedge e_j$ are generators of $H_{i+j}(N_\alpha)$ and $H_{i+j}(C\wedge C)$ respectively.

For ordinary homology maps $(\zeta j)_*$ and $(\delta\wedge i)_*\pi_{0*}$ induced by ζj and $(\delta\wedge i)\pi_0$, we have

$$(\zeta j)_*\begin{pmatrix} e_r\wedge s_r \\ e_r\wedge s_{r+k} \\ e_{r+k}\wedge s_{r+k}\end{pmatrix} = \begin{pmatrix} e_r\wedge e_r \\ e_r\wedge e_{r+k}+ne_{r+k}\wedge e_r \\ e_{r+k}\wedge e_{r+k}\end{pmatrix}$$

and

$$(\delta\wedge i)_*\pi_{0*}\begin{pmatrix} e_r\wedge s_r \\ e_r\wedge s_{r+k} \\ e_{r+k}\wedge s_{r+k}\end{pmatrix} = \begin{pmatrix} 0 \\ te_{r+k}\wedge e_r \\ 0\end{pmatrix}.$$

For the ordinary homology map $T_*: H_*(C\wedge C)\longrightarrow H_*(C\wedge C)$ induced by a switching map $T: C\wedge C\longrightarrow C\wedge C$, we have

$$T_*\begin{pmatrix} e_r\wedge e_r \\ e_r\wedge e_{r+k},\ e_{r+k}\wedge e_r \\ e_{r+k}\wedge e_{r+k}\end{pmatrix} = \begin{pmatrix} (-1)^r e_r\wedge e_r \\ (-1)^{r(r+k)}e_{r+k}\wedge e_r,\ (-1)^{r(r+k)}e_r\wedge e_{r+k} \\ (-1)^{r+k}e_{r+k}\wedge e_{r+k}\end{pmatrix}$$

Computing $(T\beta)_*$ and $(-1)^{r+k}\beta_*$, we obtain the results. q. e. d.

Consider the Puppe's exact sequence

$$\longrightarrow \{S^{2r+2k}, S^rC\} \xrightarrow{\pi'^*} \{Q, S^rC\} \xrightarrow{i'^*} \{S^{2r}, S^rC\} \xrightarrow{\alpha'^*} \{S^{2r+2k-1}, S^rC\}$$

associated to the fibration $S^{2r} \xrightarrow{i'} Q \xrightarrow{\pi'} S^{2r+2k}$. From this exact sequence and Lemma 2. 1 of [I], it follows that

$$\text{(4. 8)} \qquad \{Q, S^rC\}\cong\{\alpha_1\}+\{S^{2r+2k}, S^rC\}\pi' \\ \cong Z+\textit{finite group},$$

where let t' be the order of $(S^ri)\alpha'$ and α_1 is defined by $\alpha_1 i'=t'(S^ri)$.

For the ordinary homology map $\alpha_{1*}: H_{2r}(Q)\longrightarrow H_{2r}(S^rC)$ induced by α_1 which is the generator of free part of $\{Q, S^rC\}$ we have the following relation

$$\text{(4. 9)} \qquad \alpha_{1*}(e'_{2r})=t'e_r\wedge s_r \qquad (t'\neq 0)$$

where e'_{2r} and $e_r\wedge s_r$ are generators of $H_{2r}(Q)$ and $H_{2r}(S^rC)$ respectively.

Theorem 4. 3. *If k is even, then*

$$T\beta=(-1)^{r+k}\beta+(1_C\wedge i)\varepsilon\pi'\pi_1$$

for some $\varepsilon\in\{S^{2r+2k}, S^rC\}$, *where* $\{S^{2r+2k}, S^rC\}$ *is torsion group.*

Proof. From Proposition 4. 1 and (4. 8), we can put

$$(*) \qquad T\beta-(-1)^{r+k}\beta=a(1_C\wedge i)\alpha_1\pi_1+b(1_C\wedge i)\varepsilon\pi'\pi_1$$

for some integers a and b.

From $(*)$, we obtain the identity

$$(T\beta)_*-(-1)^{r+k}\beta_*=a(1_C\wedge i)_*\alpha_{1*}\pi_{1*}$$

of ordinary homology map. Thus, from Proposition 4.2 and (4.9), we see that $a=0$. q. e. d.

Let μ be the associative and commutative multiplication in $\tilde{h}^*$ and μ_α an admissible multiplication in $\tilde{h}^*(\ ;\alpha)$ constructed in § 3 of [I] (assuming that $\alpha'^{**}=0$ and fixing an element β such that satisfies (4.1)–(4.3)).

For $x\in\tilde{h}^i(X;\alpha)$ and $y\in\tilde{h}^j(Y;\alpha)$, we put

$$\mu_\alpha(x\otimes y)=(-1)^{i(r+k)}\sigma^{-(r+k)}\varphi_W(1_W\wedge\beta)^*(1_X\wedge T'\wedge 1_C)^*\mu(\bar{x}\otimes\bar{y}) \tag{4.10}$$

then μ_α is an admissible multiplication in $\tilde{h}^*(\ ;\alpha)$, where $W=X\wedge Y$, $T'=T(Y, C)$ and φ_W is defined on (3.7) of [I].

Put

$$\mu'_\alpha(x\otimes y)=(-1)^{ij}T''^*\mu_\alpha(y\otimes x)$$

for $T''=T(X, Y)$. μ'_α is also an admissible multiplication in $\tilde{h}^*(\ ;\alpha)$. In fact by routine calculations making use of the commutativity of μ and the naturality of φ_W (Lemma 3.5 of [I]) etc., we see that

$$\mu'_\alpha(x\otimes y)=(-1)^{i(r+k)}\sigma^{-(r+k)}\varphi_W(1_W\wedge(-1)^{r+k}T\beta)^*\,(1_X\wedge T'\wedge 1_C)^*\mu(\bar{x}\otimes\bar{y})$$

where $T=T(C,C)$. Thus we have

Theorem 4.4. *Let μ be the associative and commutative multiplication in $\tilde{h}^*$ and assume that $\alpha'^{**}=0$ in $\tilde{h}^*$. If there exists an element β of $\{N_\alpha, C\wedge C\}$ satisfying* (4.1)–(4.3) *and the relation*

$$(-1)^{r+k}(T\beta)^{**}=\beta^{**}$$

in $\tilde{h}^*$, then the admissible multiplication μ_α which is given by (4.10) *is commutative.*

Corollary 4.5. *Suppose that k is even. Assume that $\alpha\in\pi_{r+k-1}(S^r)$ satisfies $1_C\wedge\alpha=(S^r i)\alpha'(S^{r+k-1}\pi)\neq 0$, $\tilde{t}\alpha=0$ and $t=2$. Let μ be the commutative and associative multiplication in $\tilde{h}^*$. If $\alpha'^{**}=0$ and $\varepsilon^{**}=0$ in $\tilde{h}^*$ for any $\varepsilon\in\{S^{2r+2k}, S^rC\}$, then there exists a commutative admissible multiplication μ_α in $\tilde{h}^*(\ ;\alpha)$.*

Let η be the stable homotopy class of Hopf map $S^3\longrightarrow S^2$ and $C_\eta=S^r\cup e^{r+2}$ be the mapping cone of η. Then we have

$$1_{C_\eta}\wedge\eta=(S^r i)\,(3\upsilon)\,(S^{r+1}\pi)\ \text{and}\ \tilde{2}\eta=0 \tag{4.11}$$

where υ is the generator of stable homotopy group $\{S^{r+3}, S^r\}\cong Z_{24}$

Since $\{S^{r+4}, C_\eta\}=0$, we obtain

Corollary 4.6. (*See Theorem* 1.5 *of* [5]). *Let μ be the commutative and associative multiplication in $\tilde{h}^*$. If $(3\upsilon)^{**}=0$, then there exists a commutative admissible multiplication in $\tilde{h}^*(\ ;\eta)$.*

§ 5. Associativity of μ_α for the case $1_C\wedge\alpha\neq 0$.

Under the assumption of $\alpha'^{**}=0$ in $\widetilde{h}^*$, the exact sequence of $\widetilde{h}^*$ associated to the cofibration

$$W\wedge S^{2r}\xrightarrow{1_W\wedge i'}W\wedge Q\xrightarrow{1_W\wedge\pi'}W\wedge S^{2r+2k}$$

breaks into the following short exact sequence

$$0\longrightarrow\widetilde{h}^m(W\wedge S^{2r+2k})\xrightarrow{(1_W\wedge\pi')^*}\widetilde{h}^m(W\wedge Q)\xrightarrow{(1_W\wedge i')^*}\widetilde{h}^m(W\wedge S^{2r})\longrightarrow 0 \tag{5.1}$$

for any W and integer m. In paticular, for $W=S^0$ and $m=2r$, we can shoose an element $\varphi_1\in\widetilde{h}^{2r}(Q)$ such that

$$i'^*\varphi_1=\sigma^{2r}(1).$$

Put

$$\varphi_0=\pi_1{}^*\varphi_1. \tag{5.2}$$

Then φ_0 satisfies the relations

$$i_0{}^*\varphi_0=\sigma^{2r}(1)\quad\text{and}\quad i_1{}^*\varphi_0=0.$$

Then the multiplication μ_α constructed by making use above $\varphi_0=\pi_1{}^*\varphi_1$ and $\beta\in\{N_\alpha, C\wedge C\}$ satisfying (4.1)–(4.3) is admissible from Theorem 3.9 of [I]. Now we discuss the associativity of such a multiplication μ_α.

For $x\in\widetilde{h}^m(W\wedge Q)$, we have

$$\begin{aligned}&(1_W\wedge i')^*(x-\mu(\sigma^{-2r}(1_W\wedge i')^*x\otimes\varphi_1)\\&=(1_W\wedge i')^*x-\mu(\sigma^{-2r}(1_W\wedge i')^*x\otimes i'^*\varphi_1)\\&=0.\end{aligned}$$

By (5.1), $(1_W\wedge\pi')^*$ is isomorphism into. Thus we can define a homomorphism

$$\widetilde{\varphi}_W:\widetilde{h}^m(W\wedge Q)\longrightarrow\widetilde{h}^m(W\wedge S^{2r+2k})$$

by the formula

$$\widetilde{\varphi}_W(x)=(1_W\wedge\pi')^{*-1}(x-\mu(\sigma^{-2r}(1_W\wedge i')^*x\otimes\varphi_1)) \tag{5.3}$$

for any W and $x\in\widetilde{h}^m(W\wedge Q)$.

Similarly as in Lemma 3.5 in [I], we see

Lemma 5.1. (i) *$\widetilde{\varphi}_W$ is a left inverse of $(1_W\wedge\pi')^*$, i.e., $\widetilde{\varphi}_W(1_W\wedge\pi')^*=$ an identity map; hence the sequence of (5.1) splits :*

$$\widetilde{h}^m(W\wedge Q)=\widetilde{h}^m(W\wedge S^{2r})\oplus\widetilde{h}^m(W\wedge S^{2r+rk}).$$

(ii) *$\widetilde{\varphi}_W$ is natural in the sense that*

$$(f\wedge 1_{S^{2r+2k}})^*\widetilde{\varphi}_W=\widetilde{\varphi}_W(f\wedge 1_Q)^*$$

where $f: W'\longrightarrow W$.

(iii) $\widetilde{\varphi}_W$ *is compatible with the suspension in the sense that*

$$(1_W\wedge T_1)^*\sigma\widetilde{\varphi}_W=\widetilde{\varphi}_{W\wedge S^1}(1_W\wedge T_2)^*\sigma$$

where $T_1=T(S^1, S^{2r+2k})$ *and* $T_2=T(S^1, Q)$.

Lemma 5.2. *For the element* $p_0\in\{C\wedge Q,\ N_\alpha\wedge S^{r+k}\}$ *of Lemma* 3.5 *there holds the relation*

$$\widetilde{\varphi}_{W\wedge C}(1_W\wedge p_0)^*=\sigma^{r+k}\varphi_W\sigma^{-(r+k)}$$

where φ_W *is defined on* (3.7) *of* [I].

Proof. For any $x\in\widetilde{h}^m(W\wedge N_\alpha\wedge S^{r+k})$ we have

$$\begin{aligned}
&(1_{W\wedge C}\wedge\pi')^*\sigma^{r+k}\varphi_W\sigma^{-(r+k)}(x)\\
&=(1_{W\wedge C}\wedge\pi')^*\sigma^{r+k}(1_W\wedge\pi_0)^{*-1}(\sigma^{-(r+k)}x-\mu(\sigma^{-2r}(1_W\wedge i_0)^*\sigma^{-(r+k)}x\otimes\varphi_0))\\
&=(1_{W\wedge C}\wedge\pi')^*(1_W\wedge S^{r+k}\pi_0)^{*-1}(x-\sigma^{r+k}\mu(\sigma^{-2r}(1_W\wedge i_0)^*\sigma^{-(r+k)}x\otimes\varphi_0))\\
&=(1_W\wedge p_0)^*x-(1_W\wedge p_0)^*\sigma^{r+k}\mu(\sigma^{-2r}(1_W\wedge i_0)^*\sigma^{-(r+k)}x\otimes\pi_1{}^*\varphi_1) \qquad \text{by (3.4).}
\end{aligned}$$

Now

$$\begin{aligned}
&(1_W\wedge p_0)^*\sigma^{r+k}\mu(\sigma^{-2r}(1_W\wedge i_0)^*\sigma^{-(r+k)}x\otimes\pi_1{}^*\varphi_1)\\
&=(1_W\wedge p_0)^*(1_W\wedge\pi_1\wedge 1_{S^{r+k}})^*\sigma^{r+k}\mu(\sigma^{-2r}(1_W\wedge i_0)^*\sigma^{-(r+k)}x\otimes\varphi_1)\\
&=(1_W\wedge p_0)^*(1_W\wedge\pi_1\wedge 1_{S^{r+k}})^*\sigma^{r+k}\mu(\sigma^{-(r+k)}\sigma^{-2r}(1_W\wedge S^{r+k}i_0)^*x\otimes\varphi_1)\\
&=(1_W\wedge p_0)^*(1_W\wedge\pi_1\wedge 1_{S^{r+k}})^*(1_W\wedge T_3)^*\mu(\sigma^{-2r}(1_W\wedge S^{r+k}i_0)^*x\otimes\varphi_1)\\
&\qquad\qquad \text{where } T_3=T(Q, S^{r+k})\\
&=(1_W\wedge\pi\wedge 1_Q)^*\mu(\sigma^{-2r}(1_W\wedge S^{r+k}i_0)^*x\otimes\varphi_1) \qquad \text{by (3.5)}\\
&=\mu(\sigma^{-2r}(1_W\wedge S^{2r}\pi)^*(1_W\wedge S^{r+k}i_0)^*x\otimes\varphi_1)\\
&=\mu(\sigma^{-2r}(1_{W\wedge C}\wedge i')^*(1_W\wedge p_0)^*x\otimes\varphi_1) \qquad \text{by (3.3).}
\end{aligned}$$

Thus we have

$$\begin{aligned}
(1_{W\wedge C}\wedge\pi')^*\sigma^{r+k}\varphi_W\sigma^{-(r+k)}x&=(1_W\wedge p_0)^*x-\mu(\sigma^{-2r}(1_{W\wedge C}\wedge i')^*(1_W\wedge p_0)^*x\otimes\varphi_1)\\
&=(1_{W\wedge C}\wedge\pi')^*\widetilde{\varphi}_{W\wedge C}((1_W\wedge p_0)^*x).
\end{aligned}$$

Since $(1_{W\wedge C}\wedge\pi')^*$ is monomorphic, we have the result

Lemma 5.3. *For* $\beta'\in\{N_\alpha,\ C\wedge C\}$ *and* $\kappa=\kappa_{\beta'}\in\{C\wedge Q,\ C\wedge N_\alpha\}$ *of Lemma* 3.6, *there holds the relation*

$$\varphi_W(1_W\wedge\beta')^*\sigma^{-(r+k)}\varphi_{W\wedge C}=\sigma^{-(r+k)}\widetilde{\varphi}_{W\wedge C}(1_W\wedge\kappa)^*.$$

Proof. For $x\in\widetilde{h}^m(W\wedge C\wedge N_\alpha)$, we put

$$x'=\mu(\sigma^{-2r}(1_{W\wedge C}\wedge i_0)^*x\otimes\varphi_1)\in\widetilde{h}^*(W\wedge C\wedge Q).$$

Then we have

$$(5.4)\qquad (1_{W\wedge C}\wedge\pi_0)^*\varphi_{W\wedge C}x=x-\mu(\sigma^{-2r}(1_{W\wedge C}\wedge i_0)^*x\otimes\pi_1{}^*\varphi_1)$$
$$=x-(1_{W\wedge C}\wedge\pi_1)^*x'$$

and

$$(5.5)\qquad (1_{W\wedge C}\wedge i')^*x'=\mu(\sigma^{-2r}(1_{W\wedge C}\wedge i_0)^*x\otimes i'^*\varphi_1)$$
$$=\sigma^{-2r}\mu((1_{W\wedge C}\wedge i_0)^*x\otimes i'^*\varphi_1)$$
$$=(1_{W\wedge C}\wedge i_0)^*x \qquad \text{by } i'^*\varphi_1=\sigma^{2r}(1).$$

Thus we obtain

$$(5.6)\qquad \tilde{\varphi}_{W\wedge C}x'=(1_{W\wedge C}\wedge\pi')^{*-1}(x'-\mu(\sigma^{-2r}(1_{W\wedge C}\wedge i')^*x'\otimes\varphi_1))$$
$$=(1_{W\wedge C}\wedge\pi')^{*-1}(x'-\mu(\sigma^{-2r}(1_{W\wedge C}\wedge i_0)^*x\otimes\varphi_1)) \qquad \text{by (5.5)}$$
$$=0.$$

Now

$$\varphi_W(1_{W\wedge C}\wedge\beta')^*\sigma^{-(r+k)}\varphi_{W\wedge C}x=\varphi_W\sigma^{-(r+k)}(1_W\wedge\beta'\wedge 1_{S^{r+k}})^*\varphi_{W\wedge C}x$$
$$=\sigma^{-(r+k)}\tilde{\varphi}_{W\wedge C}(1_W\wedge p_0)^*(1_W\wedge\beta'\wedge 1_{S^{r+k}})^*\phi_{W\wedge C}x \qquad \text{by Lemma 5.2}$$
$$=\sigma^{-(r+k)}\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa)^*(1_{W\wedge C}\wedge\pi_0)^*\varphi_{W\wedge C}x \qquad \text{by Lemma 3.6}$$
$$=\sigma^{-(r+k)}\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa)^*(x-(1_{W\wedge C}\wedge\pi_1)^*x') \qquad \text{by (5.4)}$$
$$=\sigma^{-(r+k)}\tilde{\varphi}_{W\wedge C}((1_W\wedge\kappa)^*x-x') \qquad \text{by Lemma 3.6}$$
$$=\sigma^{-(r+k)}\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa)^*x \qquad \text{by (5.6).}$$

q. e. d.

For any element $\omega\in\{C\wedge Q,\ C\wedge C\wedge C\}$ we define a triple *product*

$$(5.7)\qquad \tau_\omega:\tilde{h}^i(X;\alpha)\otimes\tilde{h}^j(Y;\alpha)\otimes\tilde{h}^l(Z;\alpha)\longrightarrow\tilde{h}^{i+j+l}(W;\alpha)$$

as composition

$$\tau_\omega=(-1)^{j(r+k)}\sigma^{-2(r+k)}\tilde{\varphi}_{W\wedge C}(1_W\wedge\omega)^*U^*\mu(1\otimes\mu):$$
$$\tilde{h}^i(X;\alpha)\otimes\tilde{h}^j(Y;\alpha)\otimes\tilde{h}^l(Z;\alpha)=\tilde{h}^{i+r+k}(X\wedge C)\otimes\tilde{h}^{j+r+k}(Y\wedge C)\otimes\tilde{h}^{l+r+k}(Z\wedge C)$$
$$\longrightarrow\tilde{h}^{i+j+l+3(r+k)}(X\wedge C\wedge Y\wedge C\wedge Z\wedge C)$$
$$\longrightarrow\tilde{h}^{i+j+l+3(r+k)}(W\wedge C\wedge C\wedge C)$$
$$\longrightarrow\tilde{h}^{i+j+l+3(r+k)}(W\wedge C\wedge Q)$$
$$\longrightarrow\tilde{h}^{i+j+l+3(r+k)}(W\wedge C\wedge S^{2r+2k})$$
$$\longrightarrow\tilde{h}^{i+j+l+r+k}(W\wedge C)\ =\tilde{h}^{i+j+l}(W:\alpha)$$

where $W=X\wedge Y\wedge Z$ and $U:W\wedge C\wedge C\wedge C\longrightarrow X\wedge C\wedge Y\wedge C\wedge Z\wedge C$ is the map given by a permutation of factors as $U(x, y, z, c_1, c_2, c_3)=(x, c_1, y, c_2, z, c_3)$. τ_ω is defined for all (i, j, l) and natural with respect to three variable X, Y and Z. $\tau_\omega=\tau_{\omega'}$ if and only if they are equal as natural transformations for all (i, j, l). Clerly

$$\tau_{\omega+\omega'} = \tau_{\omega}+\tau_{\omega'}$$

Let $\beta\in\{N_{\alpha}, C\wedge C\}$ be an element satisfying (4.1)–(4.3) and μ_{α} be the multiplication in $\tilde{h}^{*}(\ ;\alpha)$ defined by (3.8) of [I] by making use of this β and φ_0 of (5.2). Let κ_0 be the element which satisfy (3.6) for $\beta_0=(-1)^{r+k}\beta$.

Lemma 5.4. $\mu_{\alpha}(1\otimes\mu_{\alpha})=\tau_{\omega_0}$ *for* $\omega_0=(1_C\wedge\beta)\kappa_0$.

Proof. By definition,

$$\begin{aligned}
&\mu_{\alpha}(1\otimes\mu_{\alpha})\,(x\otimes y\otimes z)\\
&=(-1)^{i(r+k)}\sigma^{-(r+k)}\varphi_W(1_W\wedge\beta)^{*}(1_X\wedge T_2\wedge 1_C)^{*}\\
&\qquad \mu(\bar{x}\otimes(-1)^{j(r+k)}\sigma^{-(r+k)}\varphi_{X\wedge Y}(1_{X\wedge Y}\wedge\beta)^{*}(1_X\wedge T_1\wedge 1_C(^{*}\mu(\bar{y}\otimes\bar{z}))\\
&\qquad\qquad \text{for } T_1=T(Z,\ C) \text{ and } T_2=T(Y\wedge Z, C)\\
&=(-1)^{(j+1)(r+k)}\sigma^{-(r+k)}\varphi_W(1_W\wedge\beta)^{*}\sigma^{-(r+k)}\varphi_{W\wedge C}(1_{W\wedge C}\wedge\beta)^{*}U^{*}\mu(\bar{x}\otimes\mu(\bar{y}\otimes\bar{z}))\\
&\qquad\qquad \text{by Lemma 3.5 of [I]}\\
&=(-1)^{j(r+k)}\sigma^{-2(r+k)}\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa_0)^{*}(1_{W\wedge C}\wedge\beta)^{*}U^{*}\mu(1\otimes\mu)\,(\bar{x}\otimes\bar{y}\otimes\bar{z})\\
&\qquad\qquad \text{by Lemma 5.3}\\
&=\tau_{\omega_0}(x\otimes y\otimes z),
\end{aligned}$$

where $x\in\tilde{h}^{i}(X;\alpha)$, $y\in\tilde{h}^{j}(Y;\alpha)$ and $z\in\tilde{h}^{l}(Z;\alpha)$. q. e. d.

Let κ_1 be the element which satisfy (3.6) for $T\beta$ where $T=T(C, C)$.

Lemma 5.5. $\mu_{\alpha}(\mu_{\alpha}\otimes 1) = \tau_{\omega_1}$ *for* $\omega_1=T'(1_C\wedge\beta)\kappa_1$, *where* $T'=T(C,\ C\wedge C)$.

Proof. By definition, on $\tilde{h}^{i}(X;\alpha)\otimes\tilde{h}^{j}(Y;\alpha)\otimes\tilde{h}^{l}(Z;\alpha)$ we have

$$\begin{aligned}
&\mu_{\alpha}(\mu_{\alpha}\otimes 1)\\
&=(-1)^{j(r+k)}\sigma^{-(r+k)}\varphi_W(1_W\wedge\beta)^{*}(1_{X\wedge Y}\wedge T_1\wedge 1_C)^{*}\mu(\sigma^{-(r+k)}\varphi_{X\wedge Y}(1_X\wedge T_2\wedge 1_C)\mu\otimes 1)\\
&\qquad\qquad \text{for } T_1=T(Z,\ C) \text{ and } T_2=T(Y,\ C)\\
&=(-1)^{j(r+k)}\sigma^{-(r+k)}\varphi_W(1_W\wedge\beta)^{*}(1_W\wedge T)^{*}\sigma^{-(r+k)}\varphi_{W\wedge C}(1_{W\wedge C}\wedge\beta)^{*}(1_W\wedge T')^{*}U^{*}\mu(1\otimes\mu)\\
&\qquad\qquad \text{by Lemma 3.5 of [I] and associativity of } \mu\\
&=(-1)^{j(r+k)}\sigma^{-2(r+k)}\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa_1)^{*}(1_{W\wedge C}\wedge\beta)^{*}(1_W\wedge T')^{*}U^{*}\mu(1\otimes\mu)\\
&\qquad\qquad \text{by Lemma 5.3}\\
&=\tau_{\omega_1}
\end{aligned}$$

q. e. d.

From above Lemmas, $\tau_{\omega_0}-\tau_{\omega_1}=(-1)^{j(r+k)}\sigma^{-2(r+k)}\varphi_{W\wedge C}\{(1_W\wedge\kappa_0)\}^{*}(1_{W\wedge C}\wedge\beta)^{*}-(1_W\wedge\kappa_1)^{*}(1_{W\wedge C}\wedge\beta)^{*}(1_W\wedge T')^{*}\}U^{*}\mu(1\otimes\mu)$, then we have

Theorem 5.6. *Let μ be a commutative and associative multiplication in $\tilde{h}^{*}$, and assume that $\alpha'^{**}=0$ in $\tilde{h}^{*}$. Suppose that there exists an element β satisfying* (4.1)–(4.3). *Let κ_0 and κ_1 be the elements of Lemma* 3.5 *for $(-1)^{r+k}\beta$ and $T\beta$ respectively. If β, κ_0 and κ_1 satisfy the relation*

(5. 8) $$(T'(1_C\wedge\beta)\kappa_1)^{**}=((1_C\wedge\beta)\kappa_0)^{**}$$

in $\tilde{h}^*$, *then the admissible multiplication* μ_α *(defined by* (3. 8) *in* [I] *by making use of* β *and* φ_0 (5. 2)) *is associative.*

By Lemmas 3. 1, 3. 2 and Proposition 3. 4, we obtain that if $\omega^*=\omega'^*$ as ordinary homology maps for ω, $\omega'\in\{C\wedge Q,\ C\wedge C\wedge C\}$ then $\omega-\omega'$ is an element of finite order. Put $\omega'=T'(1_C\wedge\beta)\kappa_0$ and $\omega_0=(1_C\wedge\beta)\kappa_0$. Since $\omega_0{}^*=\omega'^*$ in ordinary homology, we obtain

$$\tau\omega_0-\tau\omega'=\tau\varepsilon$$

for some $\varepsilon'\in\{C\wedge Q,\ C\wedge C\wedge C\}$ which is an element of finite order.

If k is even, making use of Theorem 4. 3, Lemma 3. 5 and Lemma 3. 6, we have

(5. 9) $$\begin{aligned}(1_C\wedge\pi_0)\kappa_1&=(T\beta\wedge 1_{S^{r+k}})p_0\\&=((-1)^{r+k}\beta\wedge 1_{S^{r+k}})p_0+(1_C\wedge i)\varepsilon\pi'\pi_1\wedge 1_{S^{r+k}})p_0\\&=((-1)^{r+k}\beta\wedge 1_{S^{r+k}})p_0+((1_C\wedge i)\varepsilon(S^{r+k}\pi)\pi_0\wedge 1_{S^{r+k}})p_0\\&=(1_C\wedge\pi_0)\kappa_0+(1_C\wedge S^{r+k}i)\ (S^{r+k}\varepsilon)\ (\pi\wedge\pi').\end{aligned}$$

And, we obtain

$$\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa_1)^*=\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa_1)^*(1_{W\wedge C}\wedge\pi_0)^*\varphi_{W\wedge C}$$

by (5. 4), (5. 6) and Lemma 3. 6

$$=\tilde{\varphi}_{W\wedge C}\{(1_W\wedge\kappa_0)^*(1_{W\wedge C}\wedge\pi_0)^*+(1_W\wedge\pi\wedge\pi')^*(1_W\wedge S^{r+k}\varepsilon)^*(1_{W\wedge C}\wedge S^{r+k}i)^*\}\varphi_{W\wedge C}$$

by (5. 9)

$$=\tilde{\varphi}_{W\wedge C}(1_W\wedge\kappa_0)^*+\varphi_{W\wedge C}(1_W\wedge\pi\wedge\pi')^*(1_W\wedge S^{r+k}\varepsilon)^*(1_W\wedge S^{r+k}i_1)^*$$

by (v) of Lemma 3. 5 of [I].

Then

$$\tau\omega_1=\tau\omega'+(-1)^{j(r+k)}\sigma^{-2(r+k)}\tilde{\varphi}_{W\wedge C}(\pi\wedge\pi')^{**}\varepsilon^{**}i_1{}^{**}(1_W\wedge\beta)^*(1_W\wedge T')^*U^*\mu(1\otimes\mu).$$

Thus we have the following theorem as a corollary of Theorem 5. 6.

Theorem 5. 7. *Suppose that k is even. Assume that* $\alpha\in\pi_{r+k-1}(S^r)$ *satisfies* $1_C\wedge\alpha=(S^ri)\alpha'(S^{r+k-1}\pi)\neq 0$, $\tilde{t}\alpha=0$ *and* $t=2$. *Let* μ *be the commutative and associative multiplication in* $\tilde{h}^*$. *If* $\alpha'^{**}=0$, $\varepsilon^{**}=0$ *and* $\varepsilon'^{**}=0$ *in* $\tilde{h}^*$ *for any* $\varepsilon\in\{S^{2r+2k},\ S^rC\}$ *and* $\varepsilon'\in$ *finite group of* $\{C\wedge Q,\ C\wedge C\wedge C\}$, *then there exists a commutative and associative multiplication* μ_α *in* $\tilde{h}^*(\ ;\alpha)$.

Put $\alpha=\eta$. Then we have $\{S^{r+4},\ C\eta\}=0$ and Tor $\{C\eta\wedge Q,\ C\eta\wedge C\eta\wedge C\eta\}=0$. Thus we have

Corollary 5. 8 (*See Theorem* 1. 6 *of* [5]). *Let* μ *be the commutative and associative multiplication in* $\tilde{h}^*$. *If* $(3\nu)^{**}=0$, *then there exists an associative admissible multiplication in* $\tilde{h}^*(\ ;\alpha)$.

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