J. FAC. SCI., SHINSHU UNIVERSITY Vol. 12, No. 2 Dec., 1977

Indexes of Some Degenerate Operators[1)]

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(Received July 13, 1977)

Introduction

In I of [4], Atiyah-Patodi-Singer show the following index theorem. Let X de a Riemannian manifold with boundary Y, D an elliptic operator given near the boundary by

$$D=\sigma(\frac{\partial}{\partial u}+A), \quad on \quad Y\times[0,\ 1]\subset X,$$

where σ is a bundle isomorphism, u is the normal coordinate at Y and A is a first order selfadjoint elliptic operator on Y which does not depend on u. Then, under the boundary condition $Pf(0,\ y)=0$, P is the projection to the non-negative eigenspaces of A, D has the index and index D is given by

$$\text{index } D=\int_X \alpha(x)dz-\frac{h+\eta(0)}{2}.$$

Here, $\alpha(x)dx$ is the differential form defined from D ([3], [4], [13]), $h=$dim. ker. A and η is the η-function of A given by $\sum_{\lambda\in \mathrm{Spec.A},\ \lambda\neq 0}(\mathrm{sign}\ \lambda)|\lambda|^{-s}$.

Although the above D has no singularities at the boundary, for example, on some homogeneous symmetric domain, there exist invariant differential operators which degenerate (or to have singularity) at the boundary (cf. [7]). Therefore, it seems to have meaning to consider index of elliptic operator which degenerate (or to have singularity) at the boundary. And this study may relate rescent works on degenerate elliptic ([12], [16]) and parabolic ([11], [14], [15], [17], [19]) operators.

In this paper, we consider the following type operators

$$D_{+,k}=\sigma(\frac{\partial}{\partial u}+u^kA),\ \mathrm{D}_{-,k}=\sigma(u^k\frac{\partial}{\partial u}+A), \quad on \quad Y\times[0,\ 1]\subset X,$$

and assume X to be a real analytic Riemannian manifold. Then we have

$$\text{index } D_{+,k}=\int_X \alpha_{+,k}(x)dx-\frac{h+\eta(0)}{2},$$

1) §§ 1-4 are appear in this issue, §§ 5-11 will appear in No. 1, Vol 13.

$$\mathrm{index}_k\ \mathrm{D}_{-,k}=\int_X \alpha_{-,k}(x)dx-\frac{h+\eta(0)}{2},\quad k<1,$$

$$\mathrm{index}_0\ D_{-,k}=\int_X \alpha_{-,k}(x)dx+\frac{h-\eta(0)}{2},\quad k\geqq 1,$$

with suitable differential forms $\alpha_{\pm,k}(x)dy$ on X. Here $\mathrm{index}_k\ D_{-,k}$ is the index with the boundary condition given by $Pf(0, y)=0$, $D_{-,k}f=0$ and $lim_{u\to 0}(I-P)(u^k f(u, y))=0$, $D_{-,k}{}^*f=0$ and $\mathrm{index}_0\ D$ is the index with the 0-boundary condition. For $D_{+,k}$ and $D_{-,k}$, $k<1$, these index formulas are obtained as the limit of the index formulas of $D_{+,k,\varepsilon}=(\partial/\partial u+(u^k+\varepsilon)A)$ and $D_{-,k,\varepsilon}=((u^k+\varepsilon)\partial/\partial u+A)$ with the boundary condition $Pf(0, y)=0$. But the index formula of $D_{-,k}$, $k\geqq 1$, is not the limit of the index formula of $D_{-,k,\varepsilon}$ with the boundary condition $(I-P)f(0, y)=0$. In fact, to denote the index of $D_{-,k,\varepsilon}$ with this boundary condition by index-$D_{-,k,\varepsilon}$, we have

$$\lim_{\varepsilon\to 0}\ \mathrm{index}\text{-}D_{-,k,\varepsilon}=\mathrm{index}_0 D_{-,k}-(h_k-h_{k*}),\ k\geqq 1,$$
$$h_k=\dim H_k,\quad h_{k*}=\dim H_{k*},$$
$$H_k=\{0\}\cup\{f|D_{-,k}f=Af(0, y)=0,\ f(0, y)\neq 0\},$$
$$H_{k*}=\{0\}\cup f|\ D_{-,k}{}^*f=Af(0, y)=0,\ f(0, y)\neq 0\}.$$

It is shown that, if $D_{-,k}$ is a real analytic coefficients operator, then h_k does not depend on $D_{-,k}$ and h_{k*} depends only on k.

The method of the proof of these index formulas is same that of in I of [4]. But since our operators degenerate at the boundary, some analytic difficulty occurs. The outline of the paper is as follows; First we construct and treat the properties of the elementary solutions of $D_{\pm,k}$ and $D_{\pm,k}{}^*$ on $Y\times\mathbf{R}^+$ (§§1–5). The properties of the elementary solutions of $D_{+,k}$, $D_{-,k}$, $k<1$ and $D_{-,k}$, $k\geqq 1$, are different and the elementary solution of $D_{-,k}$ exists under some 0-boundary condition. Set $\Delta_{1,\pm,k}=D_{\pm,k}{}^*D_{\pm,k}$ and $\Delta_{2,\pm,k}=D_{\pm,k}D_{\pm,k}{}^*$, to construct the fundamental solutions of $\partial/\partial t+\Delta_{i,\pm,k}$, $i=1,2$, on $Y\times\mathbf{R}^+\times\mathbf{R}^+$, we use the following lemma which is shown in §6.

Lemma. *Let $\partial/\partial t+L$ be a parabolic operator on $\mathbf{R}^+\times D$ and has a fundamental solution with kernel $G(t, x, \xi)$ such that G satisfies* (i), *G is real analytic in t if* $t>0$, (ii), *$lim_{t\to 0}(\partial^n/\partial t^n)G(t, x, \xi)=0$, $x\neq\xi$, for all $n\geqq 0$. Then H, which give the fundamental solution of $\partial/\partial t+(L+K)$ in the form $G+G^*H$* ([5]), *is given as the solution of*

$$(1+tK_x)H(t, x, \xi)=-K_xG(t, x, \xi),$$

if H is real analytic in t, $t>0$, *and satisfies*

$$\lim_{t\to 0}(1+K_x)(\frac{\partial^n H}{\partial t^n}(t, x, \xi))=0 \text{ implies } \lim_{t\to 0}\frac{\partial^n H}{\partial t^n}(t, x, \xi)=0 \text{ for all } n\geqq 0.$$

By virtue of this lemma, we can construct the fundamental solutions of $\partial/\partial t+\Delta_{i,\pm,k}$, $i=1,2$ and show that the fundamental solutions $\partial/\partial t+\Delta_{i,\pm,k,\varepsilon}$, $i=1$, 2, converge to the fundamental solutions of $\partial/\partial t+\Delta_{i,\pm,k}$, $i=1,2$, in some function space. Here $\Delta_{1,\pm,k,\varepsilon}=D_{\pm,k,\varepsilon}{}^{*}D_{\pm,k,\varepsilon}$ and $\Delta_{2,\pm,k,\varepsilon}=D_{\pm,k,\varepsilon}D_{\pm,k,\varepsilon}{}^{*}$ (§§7–9). For $\partial/\partial t+\Delta_{i,-,k}$, $i=1,2$, analyticity is used in the definition of this function space and this is the reason to assume X to be real analytic. We note that, the fundamental solution of related operator of $\partial/\partial t+\Delta_{i,-k}$ has been constructed by Gevrey ([8], cf. [8]′, [9]). Then, since on $\hat{X}$, the double of X, $\hat{\Delta}_{i,\pm,k}$, $i=1,2$ ($\hat{\Delta}_{i,\pm,k}$ are the induced operators of $\Delta_{i,\pm,k}$ on $\hat{X}$), have parametrixes ($\hat{\Delta}_{i,-,k}$, $i=1,2$, have parametrixes only on spaces of those functions which vanish on Y with suitable degree), we obtain the index formulas (§12), together with the limit properties of index $D_{\pm,k,\varepsilon}$ which are treated in §§10–11. In §12, it is also noted for the operator $D_{(-k)}$ given by $\sigma(\partial/\partial u+u^{-k}\mathrm{A})$ on $Y\times[0, 1,]$ we have

$$\text{index } D_{(-k)}=\int_X \alpha_{(-k)}(x)dx-\frac{h+\eta(0)}{2},\quad k<1,$$

$$\text{index-}D_{(-k)}=\int_X \alpha_{(-k)}(x)dx+\frac{h-\eta(0)}{2},\quad k\geqq 1,$$

with suitable $\alpha_{(-k)}(x)dx$.

The result of this paper seems to be poor than its method and it seems there must exist other geometric quantities for the operators $D_{\pm,k}$, especially for $D_{-,k}$. But at this stage, I can not clarify them.

I would like to thank Dr. Abe who give me the occasion to consider this problem.

§1. Differential operators $D_{\pm,k,\lambda}$.

On the positive half line R^{+} given by $u\geqq 0$, we define differential operators $D_{+,k,\lambda}$ and $D_{-,k,\lambda}$ by

$$(1)_{+,k}\qquad D_{+,k,\lambda}(f_{\lambda,k})=\frac{d}{du}f_{\lambda,k}+\lambda u^{k}f_{\lambda,k}=g_{\lambda},\ \lambda\in\mathbf{R},\ f_{\lambda,k}(0)=0,\ if\lambda\geqq 0,\ k>0,$$

$$(1)_{-,k}\qquad D_{-,k,\lambda}(f_{\lambda,k})=u^{k}\frac{d}{du}f_{\lambda,k}+\lambda f_{\lambda,k}=g_{\lambda},\ \lambda\epsilon\mathbf{R},\ f_{\lambda,k}(0)=0,\ if_{\lambda}\geqq 0,\ k>0.$$

It is known that similar operator

$$(1)\qquad D_{\lambda}(f_{\lambda})=\frac{d}{du}f_{\lambda}+\lambda f_{\lambda}=g_{\lambda},\ f_{\lambda}(0)=0,\ \lambda\geqq 0,$$

has a fundamental solution

(2) $$f_\lambda(u)=Q_\lambda(g_\lambda)(u)=\int_0^u e^{\lambda(v-u)}\, g_\lambda(v)dv,\ \lambda\geqq 0,$$

$$=-\int_u^\infty e^{\lambda(v-u)} g_\lambda(v)dv,\ \lambda<0,$$

with the properties that there exist constants C_0 and C_1 such that

(3) $$||f_\lambda||\leqq C_0||g_\lambda||,\ ||f_\lambda||\leqq C_1||g_\lambda||,\ \lambda\neq 0.$$

Here, $||f||$ and $||f||_1$ are the L^2-norm and Sobolev's 1-norm of f ([4]).

To construct fundamental solutions of $(1)_{\pm,k}$ defined on some subspace of $C_C^\infty(\mathbf{R}^+)$, the space of compact support C^∞-functions on $\mathbf{R}^+$ ($f\epsilon C_C^\infty(\mathbf{R}^+)$ may not be $f(0)=0$), we use

Lemma 1. *For $a>0$, $c>0$, we have*

(4) $$\lim_{u\to 0}\int_0^u u^{-s}e^{a(u^{-c}-v^{-c})}dv=\lim_{u\to 0}\int_0^u v^{-t}u^{t-s}e^{a(u^{-c}-v^{-c})}dv=O(u^{1+c-s}),$$

(4)′ $$\lim_{u\to 0}\int_u^M u^{-s}e^{a(v^{-c}-u^{-c})}dv=\lim_{u\to 0}\int_u^M v^{-t}u^{t-s}e^{a(v^{-s}-u^{-c})}dv=O(u^{-s}).$$

Proof. Since $u^{-s}e^{a(u^{-c}-v^{-c})}\leqq v^{-s}e^{a(u^{-c}-v^{-c})}\leqq Au^{-s}e^{(a+\varepsilon)(u^{-c}-v^{-c})}$ if $u\geqq V>0$ for some $A>0$ and $\varepsilon>0$, we get the first inequality of (4). Then, since

$$\int_0^u u^{-s}e^{a(u^{-c}-v^{-c})}\, dv=\int_0^1 u^{1-s}e^{au^{-c}(1-w^{-c})}dw=$$

$$=u^{1-s}e^{au^{-c}}\frac{1}{c}\int_1^\infty e^{-au^{-c}\xi}\xi^{-1-1/c}d\xi,$$

we obtain (4) because $e^{-au^{-c}}\geqq e^{-au^{-c}\xi}\xi^{-1-1/c}\geqq e^{-(a+\varepsilon)u^{-c}}$ for some $\varepsilon>0$ on $1\leqq\xi<\infty$.

Similarly, $u^{-s}e^{a(v^{-c}-u^{-c})}\geqq v^{-s}e^{a(v^{-c}-u^{-c})}\geqq Au^{-s}e^{(a-\varepsilon)(v^{-c}-u^{-c})}$ if $v\geqq u>0$ for some $A>0$ and $a>\varepsilon>0$, we get the first inequality of (4)′. Then we have (4)′ because

$$\int_u^M u^{-s}e^{a(v^{-c}-u^{-c})}\, dv=\frac{1}{c}\int_{M^{-c}}^{u^{-c}} u^{-s}e^{a\xi-au^{-c}}\xi^{-1-1/c}d\xi,$$

$K_1e^{a\varepsilon}\geqq e^{a\varepsilon}\xi^{-1-1/c}\geqq K_2e^{(a-\varepsilon)\xi}$ *for some* $K_1,\ K_2>0$ *and* $0<\varepsilon<a$ *on* $M^{-c}\leqq\xi\leqq u^{-c}$.

Definition. *We se* $C_{c,[t]}{}^\infty(\mathbf{R}^+)=\{f|f\in C_c{}^\infty(\mathbf{R}^+)$ *and* $f=O(u^t),\ u\to 0\}$ and set $C_{c,(t)}{}^\infty(\mathbf{R}^+)=\bigcup_{\varepsilon>0} C_{c,[t+\varepsilon]}{}^\infty(\mathbf{R}^+)$.

Definition. *We define operators* $Q_{\lambda,k}$ *and* $Q_{\lambda,-k}$ *by*

$$(2)_+ \qquad Q_{\lambda,k}(g_\lambda)(u)=\int_0^u e^{\frac{\lambda}{k+1}(v^{k+1}-u^{k+1})} g_\lambda(v)dv,\ \lambda\geqq 0,$$

$$=-\int_u^\infty e^{\frac{\lambda}{k+1}(v^{k+1}-u^{k+1})} g_\lambda(v)dv,\ \lambda<0,\ g_\lambda\in C_C^\infty(\mathbf{R}^+),$$

$$(2)_- \qquad Q_{\lambda,\ -k}(g_\lambda)(u)=\int_0^u v^{-k} e^{\frac{\lambda}{1-k}(u^{1-k}-u^{1-k})} g_\lambda(v)dv,\ k\neq 1,\ \lambda\geqq 0,\ g_\lambda\in C_C^\infty(R^+),$$

$$g_0\in C_{c,\ (k-1)}{}^\infty(\mathbf{R}^+)\ if\ k>1,$$

$$=-\int_u^\infty v^{-k} e^{\frac{\lambda}{1-k}(v^{1-k}-u^{1-k})} g_\lambda(v)dv,\ k\neq 1,\ \lambda<0,\ g_\lambda\in C_C^\infty(\mathbf{R}^+),$$

$$(2)_{-,1} \qquad Q_{\lambda,-1}(g_\lambda)\ (u)=\int_0^u v^{-1}\left(\frac{v}{u}\right)^\lambda g_\lambda(v)dv,\ \lambda\geqq 0, g_0\in C_c,\ (0)^\infty(\mathbf{R}^+),$$

$$=-\int_u^\infty v^{-1}\left(\frac{v}{u}\right)^\lambda g_\lambda(v)dv,\ \lambda<0,\ g_\lambda\in C_c{}^\infty(\mathbf{R}^+).$$

By definitions and lemma 1, we have

Lemma 2. *$Q_{\lambda,k}$ and $Q_{\lambda,-k}$ are the fundamental solutions of $(1)_{\pm,k}$. They are C^∞-class on $\mathbf{R}^+ -\{0\}$ and $Q_{\lambda,k}$ and $Q_{\lambda,-h}, \lambda>0$, are continuous on $\mathbf{R}^+$. $Q_{\lambda,-k}(g_\lambda)$, $\lambda<0$, is continuous on $\mathbf{R}^+$ if $g_\lambda\epsilon\ C_{c,[k]}{}^\infty(\mathbf{R}^+)$ and $Q_{0,-k}\ (g_0)$ is continuous if $0<k\leqq 1$ or $g_0\in C_{c,[k-1]}{}^\infty(\mathbf{R}^+)$ if $k>1$. More precisely, we have*

$(n)_+$ *$Q_{\lambda,-k}(g_\lambda)\ (u)$ is C^n-class on $\mathbf{R}^+$ if and only if $g_\lambda\in C_{c,[nk-1]}{}^\infty(\mathbf{R}^+)$ for $\lambda>0$,*

$(n)_0$ *$Q_{0,-k}(g_0)\ (u)$ is C^n-class on $\mathbf{R}^+$ if and only if $g_0\in C_{c,[n+k-1]}{}^\infty(\mathbf{R}_+)$,*

$(n)_-$ *$Q_{\lambda,-k}(g_\lambda)\ (u)$ is C^n-class on $\mathbf{R}^+$ if and only if $g_\lambda\in C_{c,[(n+1)k]}{}^\infty(\mathbf{R}^+)$ for $\lambda<0$.*

Proof. We need only to show $(n)_+$, $(n)_0$ and $(n)_-$. These follows from lemma 1 because we obtain if $g_\lambda(u)=O(u^t),\ u\to 0$,

$$(5)_+ \qquad \frac{d^n}{du^n} Q_{\lambda,-k}(g_\lambda)\ (u)=O(u^{1+t-nk}),\ \lambda>0,$$

$$(5)_0 \qquad \frac{d^n}{du^n} Q_{0,-k}(g_0)\ \ (u)=O(u^{1+t-k-n}),$$

$$(5)_- \qquad \frac{d^n}{du^n} Q_{\lambda,-k}(g_\lambda)(u)=O(u^{t-(n+1)k}),\ \lambda<0.$$

Corollary. *If $Q_{\lambda,-k}(g_\lambda)$ is C^n-class on $\mathbf{R}^+$, then $(d^m/du^m(Q_{\lambda,-k}(g_\lambda))(0)=0,\ 0\leqq m \leqq n-1$ and $d^{m+1}/du^{m+1}(Q_{\lambda,-k}(g_\lambda))$ is unbounded near 0 if g_λ does not satisfy $(n+1)$.*

Note 1. Since the fundamental solutions of the adjoint operators of $(1)_{\pm,k}$ are obtained by the interchange of the variables of the kernels of $Q_{\lambda,\pm,k}$, we get same results for adjoint operators.

Note 2. We have

(6)+ $$Q_{\lambda,k}(g_\lambda)(u)=Q_\lambda(g_\lambda\{(k+1)w\}^{\frac{1}{k+1}}\{(k+1)w\}^{-\frac{1}{k+1}})(u),\ w=\frac{u^{k+1}}{k+1},\ k>0,$$

(6)- $$Q_{\lambda,-k}(g_\lambda)(u)=Q_\lambda(g_\lambda(\{(1-k)w\}^{\frac{1}{1-k}})(u),\quad w=\frac{u^{1-k}}{1-k},\ 0<k<1,$$

and to set

$$Q_{\lambda,-}(g_\lambda)(u)=\int_{-\infty}^{u}e^{\lambda(v-u)}g_\lambda(v)dv,\ \lambda\geqq 0,$$

$$=-\int_{u}^{\infty}e^{\lambda(v-u)}g_\lambda(v)dv,\ \lambda<0,$$

which is a fundamental solution of D_λ considered on $\mathbf{R}$ with the boundary condition $lim_{u\to-\infty}f_\lambda(u)=0,\ \lambda\geqq 0$, we also have

(6)-,1 $$Q_{\lambda,-1}(g_\lambda)(u)=Q_{\lambda,-}(g_\lambda(\log w))(u),\ w=\log u.$$

§2. Estimates of $Q_{\lambda,\pm,k}$.

In this §, we use the notations

$$M=M(g)=max\{u|g(u)\neq 0\},\quad m=m(g)=min\{u|g(u)\neq 0\},$$

and $||f||_{[a,b]}$, $||f||_{n,[a,b]}$, *etc.*, mean L^2—norm and Sobolev's n-th norm, *etc.*, of f on $[a,\ b]$.

Lemma 3. *There exist constants* C_k, $C_{k,L}$, $C_{-k,M,L}$, $0<k<1/2$ *and* $C_{-k,m,M,L}$, $k\geqq 1/2$, *such that*

(3)+ $$||Q_{\lambda,k}(g)||\leqq C_k||g||,\ \lambda\neq 0,\ ||Q_{0,k}(g)||_{0,L}\leqq C_{[k,L]}||g||,$$

(3)-,i $$||Q_{\lambda,-k}(g)||_{[0,L]}\leqq C_{-k,M,L}||g||,\ 0<k\leqq\frac{1}{2},$$

(3)-,ii $$||Q_{\lambda,-k}(g)||_{[0,L]}\leqq C_{-k,m,M,L}||g||,\ k\geqq\frac{1}{2},\ m(g)\neq 0.$$

Proof. Since the kernel of $Q_{\lambda,-k}$ is continuous if $u\neq 0$, $v\neq 0$, we have (3)-,ii. On the other hand, since $[g(\{(1-k)w\}^{1/(1-k)})^2dw=\{g(u)\}^2u^{-k}du,\ g(\{1-k)w\}^{1/(1-k)})\in L^2(\mathbf{R}^+)$ if $g\in C_C^\infty(\mathbf{R}^+)$ and $Q_{\lambda,-k}(g)\in L^2[0,\ L]$ if $0<k<1/2$ by (5), we have (3)-,i by (3) and (6)-. For $k>0$, we get

$$||Q_{\lambda,k}(g)||\leqq||g||+||Q_{\lambda/(k+1)}(g)||,\ \lambda\neq 0,$$

because $\exp\left[\frac{\lambda}{k+1}(v^{k+1}-u^{k+1})\right]\leqq\exp\left[\frac{\lambda}{k+1}(v-u)\right]$ if $u\geqq 1$, $u\geqq v$ and $\lambda>0$ or $v\geqq u \geqq 1$ *and* $\lambda<0$. Hence we obtain $(3)_+$ by (3).

To get boundary estimate for $Q_{\lambda,-k}$, we use

Lemma 4. *Let $g(u)$ be a C^{n+1}-class function such that*

$$g(0)=g'(0)=\cdots=g^{(n-1)}(0)=0, \tag{7$_n$}$$

and assume $g^{(n+1)}(0)\neq 0$. Then to set

$$g(u)=\frac{u^n}{n!}g^{(n)}(\theta(u)u),\ 0\leqq\theta(u)\leqq 1, \tag{8$_n$}$$

$\lim_{u\to 0}\theta(u)=1/(n+1)$, and this convergence is locally uniform in g by the C^{n+1}-topology.

Proof. By assumption, we may set

$$\begin{aligned} g(u)&=\frac{u^n}{n!}g^{(n)}(0)+\frac{u^{n+1}}{(n+1)!}g^{(n+1)}(\theta_1(u)u)\\ &=\frac{u^n}{n!}g^{(n)}(0)+\theta(u)ug^{(n+1)}(\theta_0(\theta(u)u)\theta(u)u). \end{aligned}$$

Hence we get

$$\theta(u)=\frac{1}{n+1}\frac{g^{(n+1)}(\theta_1(u)u)}{g^{(n+1)}(\theta_0(\theta(u)u)\theta(u)u)}, \tag{9}$$

which shows the lemma.

Corollary 1. *If $g(u)$ is a C^n-class function and satisfies $(7)_n$, then*

$$\max_{0\leqq u\leqq a}\left|\frac{g(u)}{u^n}\right|\leqq\frac{1}{n!}\max_{0\leqq u\leqq a}|g^{(n)}(u)|, \tag{10}$$

and if $g(u)$ is C^{n+1}-class and $g^{(n+1)}(0)\neq 0$, then there exists a constant $\alpha=\alpha(a,g)>0$ such that

$$\max_{0\leqq u\leqq a}|g^{(n)}(u)|\leqq n!\max_{0\leqq u\leqq a}\left|\frac{g(u)}{u^n}\right|, \tag{10}$$

and this α is taken to be locally uniform in g by the C^{n+1}-topology.

Proof. (10) follows from $(8)_n$. Since $\theta(u)$ is continuous near $u=0$ if $g^{(n+1)}(0)\neq 0$, $\theta(u)u$, $0\leqq u\leqq a$ covers $0\leqq u\leqq\alpha a$, $\alpha>0$, by lemma 4. This shows (10)′. The local uniformity of α also follows from lemma 4.

Corollary 2. *If* $g(u)$ *is a* C^n*−class function and satisfies* $(7)_n$, *then there exists a constant* C_1 *such that*

$$(11)\qquad ||\frac{g(u)}{u^n}||_{[0,a]}\leqq C_1||g(u)||_{n+1,[0,a]},$$

and if $g(u)$ *is a* C^{n+1}*−class function and satisfies* $(7)_n$ *and* $g^{(n+1)}(0)\neq 0$, *then there exist constants* C_2 *and* $\beta=\beta(a,g)>1$ *such that*

$$(11)'\qquad ||g(u)||_{n,[0,a]}\leqq C_2||\frac{g(u)}{u^n}||_{[0,\beta a]},$$

and this β *is taken to be locally uniform in* g *by the* C^{n+1}*−topology.*

Proof. (11)′ follows from (10)′. (11) follows from the inequalities

$$||\frac{g(u)}{u^n}||_{[0,a]}\leqq C_0 \max_{0\leqq u\leqq a}|\frac{g(u)}{u^n}|\leqq n!C_0\ (\max_{0\leqq u\leqq a}|g^{(n)}(u)|)\leqq C_1||g||_{n+1,[0,a]},$$

which are obtained by (10) and Sobolev inequality ([1]).

Lemma 5. (i). *Let* $[k]'$ *be* $-[-k]$ *and* g *an element of* $C_{c,[k]'-1}{}^{\infty}$, *then there exists a constant* $C^{[k]'}_{k,M,L}$, $M=M(g)$, *such that*

$$(12)_+\qquad ||Q_{\lambda,-k}g||_{[0,L]}\leqq C^{[k]'}_{k,M,L}\ ||g||_{[k]'+1}.$$

(ii). *If* k *is an integer,* g *satisfies* (n) *and assume* $Q_{\lambda,-k}{}^{(n)}(0)\neq 0$. *Then there exist constants* $C^n_{k,M,L,g,\lambda}$ *such that*

$$(12)'\qquad ||Q_{\lambda,-k}[g]||_{n,[0,L]}\leqq C^n_{k,M,L,g,\lambda}\ ||g||_{nk},\quad \lambda>0,$$

$$(12)'_0\qquad ||Q_{0,-k}[g]||_{n,[0,L]}\leqq C^n_{k,M,L,g,0}||g||_{n+k},$$

$$(12)'_-\qquad ||Q_{\lambda,-k}[g]||_{n,[0,L]}\leqq C^n_{k,M,L,g,\lambda}\ ||g||_{(n+1)k+1},\quad \lambda<0,$$

and these $C^n_{k,M,L,g,\lambda}$ *are locally uniform in* g *by Sobolev's* $(n+1)$*−topology and uniform in* λ *if* $|\lambda|$ *is large.*

Proof. First we note that for an integer n, $C_{c,[n]}{}^{\infty}=\{f|\ f\in C^{\infty}_C,\ f$ *satisfies* $(7)_{n+1}\}$. Hence we get (12) by (11). Because we know

$$|\left(\frac{u}{v}\right)^{\lambda}|\leqq 1,\quad \lambda>0,\ v\leqq u\ \text{or}\ \lambda<0,\ v\geqq u.$$

$$|e^{\frac{\lambda}{1-k}(v^{1-k}-u^{1-k})}|\leqq 1,\quad \lambda>0,\ v\leqq u\ \text{or}\ \lambda<0,\ v\geqq u.$$

Since we know $||u^{-s}Q_{\lambda,-k}[g]||\leqq C||Q_{\lambda,-k}[v^{-s}g]||$ for some $C>0$ by lemma 1, we have

$$||Q_{\lambda,-k}[g]||_{s,[0,a]}\leqq C_2||u^{-s}Q_{\lambda,-k}[g]||_{[0,a]}\leqq C_3||g||_{s+k+1}$$

by (12), if g satisfies (s). Therefore we get (12)′. The local uniformity of $C^n_{k,M,L,g,\lambda}$ in g follows from lemma 4. To show the uniformity in λ, we note that we have

$$Q_{\lambda,-k}(g)^{(n)}(u)=(-1)^n\lambda^n u^{-nk}Q_{\lambda,-k}(g)(u)+O(\lambda^{n-1}u^k),$$

by (5) and

$$Q_{\lambda,-k}(g)^{(s)}=\sum_{t=0}^{S-1}\frac{(s-1)!}{t!(s-t-1)!}(-1)^t\frac{(k+t)}{(k)}u^{-k-t}(g^{(s-t-1)}-\lambda Q_{\lambda,-k}(g)^{(s-t-1)}).$$

Hence by (9), $\theta\ (u)$, determined by $(8)_{n-1}$ for $Q_{\lambda,-k}(g)$, is given by

$$\theta(u)=\frac{1}{n}\frac{Q_{\lambda,-k}(g)(\theta_1(u)u)+O(\lambda^{-1}u^k)}{Q_{\lambda,-k}(g)(\theta_0(\theta(u)u)\theta(u)u)+O(\lambda^{-1}u^k)}.$$

On the other hand, the kernel of $Q_{\lambda,-k}$ tends to 0 if $|\lambda|\to\infty$. Hence we have the uniformity in λ for $|\lambda|$ is large by $(2)_-$ and $(2)_{-,1}$.

Note. For the fundamental solutions of the adjoint operators of $D_{\pm,k,}$, we have same estimates.

§3. Extensions of $D_{\pm,k,\lambda}$ and $Q_{\lambda,\pm k}$.

By lemma 3 and lemma 5, we have

Lemma 6. (i). $Q_{\lambda,k}$, $k>0$, *is extended to a continuous map* $\widetilde{Q}_{\lambda,k}: L^2\to L^2_{loc.}$.
(ii). *For any* $0<m<M$, $Q_{\lambda,-k}$ *is extended to a continuous map* $\widetilde{Q}_{\lambda,-k}: L^2m,M\to L^2$.
(iii). $Q_{\lambda,-k}$ *is extended to a continuous map* $\widetilde{Q}_{\lambda,-k}: H^{([k]'-1}{}^{[k]'+1}[0,\ L]\to L^2_{loc.}$, $k\geqq 1$.
(iv). $Q_{\lambda,-1}$ *is extended to a continuous map* $\widetilde{Q}_{\lambda,-1}: H_{(n-1)}{}^n[0,\ L]\to H^n{}_{loc.}$, *if* $\lambda>0$.

Here $H_{(s)}{}^n$, $s\leqq n-1$, *means the Sobolev space with the boundary condition* $(7)_s$.

Corollary. $D_{\pm,k,\lambda}$ *and theire formal adjoints* $D_{\pm,k,\lambda}{}^*$ *have closed extensions* $\mathscr{D}_{\pm,k,\lambda}$ *and* $\mathscr{D}_{\pm,k,\lambda}$.

Definition. *We set* $\Delta_{1,\ \pm,k,\lambda}=\mathscr{D}_{\pm,k,\lambda}{}^*\mathscr{D}_{\pm,k,\lambda}$ *and* $\Delta_{2,\pm,k,\lambda}=\mathscr{D}_{\pm,k,\lambda}\mathscr{D}_{\pm,k,\lambda}{}^*$.
Since (on $C^\infty(\mathbf{R}^+)$) $D_{\pm,k,\lambda}{}^*$are given by

$$D_{+,k,\lambda}{}^*=-\frac{d}{du}+\lambda u^k,\ D_{-,k,\lambda}{}^*=-u^k\frac{d}{du}-ku^{k-1}+\lambda,$$

we have (on $C^\infty(\mathbf{R}^+)$)

$$(13)_+ \quad \Delta_{1,+,k,\lambda}=-\frac{d^2}{du^2}-k\lambda u^{k-1}+\lambda^2u^{2k},\ \Delta_{2,+,k,\lambda}=-\frac{d^2}{du^2}+k\lambda u^{k-1}+\lambda^2u^{2k},$$

$$(13)_- \quad \Delta_{1,-,k,\lambda}=-u^{2k}\frac{d^2}{du^2}-2ku^{2k-1}\frac{d}{du}-\lambda ku^{k-1}+\lambda^2,$$

$$\Delta_{2,-,k,\lambda}=-u^{2k}\frac{d^2}{du^2}-2ku^{2k-1}\frac{d}{du}-k(k-1)u^{2k-2}-\lambda ku^{k-1}+\lambda^2.$$

By definition, $Q_{\lambda,k}Q_{\lambda,k}{}^*$ and $Q_{\lambda,k}{}^*Q_{\lambda,k}$ are defined on $C_C^\infty(\mathbf{R}^+)$ and they are the fundamental solutions of $\Delta_{1,+,k}$, and $\Delta_{2,+,k}$, with the boundary conditions

$$(14)_+ \quad f_{\lambda,k}(0)=0,\ \lambda\geqq 0,\ \frac{d}{du}f_{\lambda,k}(0)=0,\ \lambda<0,\ \textit{for}\ \Delta_{1,+,k},,$$

$$f_{\lambda,k}(0)=0,\ \lambda\leqq 0,\ \frac{d}{du}f_{\lambda,k}(0)=0,\ \lambda>0,\ \text{for}\ \Delta_{2,+,k},.$$

The boundary conditions for $\Delta_{i,-,k},$, $k<1$, $i=1,2$, are similar as $(14)_+$ (cf. § 11). But since $O_{\lambda,-k}\ Q_{\lambda,-k}$ and $Q_{\lambda,-k}\ {}^*Q_{\lambda,-k}$ are defined only on $C_{c,[2k]}{}^\infty(\mathbf{R}^+)$ if $k\geqq 1$ and therefore the boundary condition is

$$(14)_- \quad f_{\lambda,k}(0)=0,\ \textit{for all}\ \lambda.$$

For the extensions of $Q_{\lambda,\pm k}Q_{\lambda,\pm k}{}^*$ and $Q_{\lambda,\pm k}{}^*Q_{\lambda,\pm k}$, we have by lemma 3 and lemma 4

Lemma 6′. (i). *$Q_{\lambda,k}Q_{\lambda,k}{}^*$ and $Q_{\lambda,k}{}^*Q_{\lambda,k}$ are extended to a continuous maps $L^2\to L^2{}_{loc.}$.*
(ii). *For any L, $Q_{\lambda,-k}Q_{\lambda,-k}{}^*$ and $Q_{\lambda,-k}{}^*Q_{\lambda,-k}$ are extended to a continuous maps $H_{([2k]'-1)^{[2k]'+1}}[0,\ L]\to L^2{}_{loc.}$.*

§4. Continuation of $D_{\pm,k,\lambda}$ and $Q_{\lambda,\pm k}$ across 0.

If $u<0$, we define $D_{\pm,k,\lambda}$ by

$$D_{+,k,\lambda}=\frac{d}{du}+\lambda(-u)^k,\ D_{-,k,\lambda}=(-u)^k\frac{d}{du}+\lambda.$$

Hence for $u<0$, $D_{\pm,k,\lambda}$ and $\Delta_{i,\pm,k,\lambda}$, $i=1,2$, take the forms

$$D_{+,k,\lambda}{}^*=-\frac{d}{du}+\lambda(-u)^k,\quad D_{-,k,\lambda}{}^*=-(-u)^k\frac{d}{du}+k(-u)^{k-1}+\lambda,$$

$$\Delta_{1,+,k,\lambda}=-\frac{d^2}{du^2}+k\lambda(-u)^{k-1}+\lambda^2(-u)^{2k},$$

$$\Delta_{2,+,k,\lambda}=-\frac{d^2}{du^2}-k\lambda(-u)^{k-1}+\lambda^2(-u)^{2k},$$

$$\Delta_{1,-,k,\lambda}=-(-u)^{2k}\frac{d^2}{du^2}+2k(-u)^{2k-1}\frac{d}{du}+\lambda k(-u)^{k-1}+\lambda^2,$$

$$\Delta_{2,-,k,\lambda}=-(-u)^{2k}\frac{d_f^2}{du^2}+2k(-u)^{2k-1}\frac{d}{du}-k(k-1)(-u)^{2k-2}+\lambda k(-u)^{k-1}+\lambda^2.$$

The boundary conditions for $u\leqq 0$ are $f_{\lambda,k}(0)=0, \lambda\leqq 0$ and the fundamental solutions take the forms

$$Q_{\lambda,k}(g_\lambda)(u)=-\int_u^0 e^{\frac{\lambda}{k+1}((-u)^{k+1}-(-v)^{k+1})}\, g_\lambda(v)dv,\ \lambda\leqq 0,$$

$$=\int_{-\infty}^u e^{\frac{\lambda}{k+1}((u^{k+1}-(-v)^{k+1})}\, g_\lambda(v)dv,\ \lambda>0,$$

$$Q_{\lambda,-k}(g_\lambda)(u)=-\int_u^0 (-v)^{-k}e^{\frac{\lambda}{1-k}((u)^{1-k}-(-v)^{1-k})}g_\lambda(v)dv,\ \lambda\leqq 0,\ k\neq 1,$$

$$=\int_{-\infty}^u (-v)^{-k}e^{\frac{\lambda}{1-k}((-u)^{1-k}-(-v)^{-1k})}\, g_\lambda(v)dv,\ \lambda>0,$$

$$Q_{\lambda,-1}(g_\lambda)(u)=\int_u^0 v^{-1}\left(\frac{u}{v}\right)^\lambda g_\lambda(v)dv,\ \lambda\leqq 0,$$

$$=-\int_{-\infty}^u v^{-1}\left(\frac{u}{v}\right)^\lambda g_\lambda(v)dv,\ \lambda>0.$$

Therefore, to set

$$C_{c,k,\lambda}{}^\infty(\mathbf{R})=\{f\,|\,f\in C_C^\infty(\mathbf{R}),\int_{-\infty}^0 e^{-\frac{\lambda}{k+1}|v|^{k+1}} f(v)dv=0\},\ \lambda\geqq 0,$$

$$C_{c,k,\lambda}{}^\infty(\mathbf{R})=\{f\,|\,f\in C_C^\infty(\mathbf{R}),\int_0^\infty e^{\frac{\lambda}{k+1}v^{k+1}} f(v)dv=0\},\ \lambda<0,$$

$$C_{c,-k,\lambda}{}^\infty(\mathbf{R})=\{f\,|\,f\in C_C^\infty(\mathbf{R}),\int_{-\infty}^0 e^{-\lambda|v|} f(\{(1-k)v\}^{\frac{1}{1-k}})dv=0\},$$

$$C_{c,k,\lambda}{}^\infty(\mathbf{R})=\{f\,|\,f\in C_C^\infty(\mathbf{R}),\int_0^\infty e^{\lambda v} f(\{(1-k)v\}^{\frac{1}{1-k}})dv=0\},\ 0<k<1,\ \lambda<0,$$

$Q_{\lambda,k}$ and $Q_{\lambda,-k}$, $0<k<1$, are defined on $C_{c,k,\lambda}{}^\infty(\mathbf{R})$ and $C_{c,-k,\lambda}{}^\infty(\mathbf{R})$. Here $C_C^\infty(\mathbf{R})$ means the space of compact support C^∞-class functions on $\mathbf{R}$. On the other hand, $Q_{\lambda,-k}$, $k\geqq 1$, is defined on $C_{c,((n+1)k)}{}^\infty$. Here, $C_{c,((n+1)k)}{}^\infty$ means the space of compact support functions with $g(u)=O(u^{(n+1)k+\varepsilon})$, $u\to 0$, for some $\varepsilon>0$.

Since the above extended $Q_{\lambda,\pm k}$ have same properties as $Q_{\lambda,\pm k}$ on $\mathbf{R}^+$ and the L^2-completions of $C_{c,k,\lambda}{}^\infty(\mathbf{R})$, and $C_{c,-k,\lambda}{}^\infty(\mathbf{R})$ $0<k<1$, are 1-codimensional subspaces of $L^2(\mathbf{R})$, we obtain

Lemma 7. (i). $\Delta_{i,+,k,\lambda}$, $i=1,2$, *and* $\Delta_{i,-k,\lambda}$, $0<k<1$, $i=1,2$, *have fundamental* *solutions across* 0.

(ii). $\Delta_{i,-,k,\lambda}$, $k\geq 1$, $i=1,2$, *have fundamental solutions defined on* $H_{([2k]')}{}^{[2k]'+2}[-L, L]$ *for any* $L>0$.

Proof. By the definitions of $Q_{\lambda,\pm k}$ (and $Q_{\lambda,\pm k}{}^*$), if $g_\lambda\in C_C^\infty[a, b,]$ $-\infty<a<b<\infty$, then the iterations $Q_{\lambda,k}Q_{\lambda,k}{}^*$, $Q_{\lambda,k}Q_{\lambda,k}$ and $Q_{\lambda-,k}Q_{\lambda,-k},{}^*$ $Q_{\lambda,-k}{}^*Q_{\lambda,-k}$, $0<k<1$, are defined. Similarly, if $g_\lambda\in C_{C,([2k]')}{}^\infty[a, b]$, $(a<0<b)$, then $Q_{\lambda,-k}Q_{\lambda,-k}{}^*$ and $Q_{\lambda,-k}{}^*Q_{\lambda,-k}$, $k\geq 1$, are defined. Hence we have the lemma.

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