JOUR. FAC. SCI., SHINSHU UNIVERSITY Vol. 12, No. 1 June, 1977

Characterizations of Division Rings

By MASAYUKI ÔHORI

Department of Mathematics, Faculty of Science, Shinshu University

(Received October 28, 1976)

Recently, R. Couture and M. Lavoie [1] proved that any ring whose left modules have free bases is a division ring. In this note, we shall prove the same more briefly by giving some reasonable equivalent conditions.

Theorem. *If* $R \neq 0$ *is a ring then the following are equivalent :*

1) *R is a division ring.*

2) *For every left ideal I of R, R/I has a free R-basis.*

3) *Every left ideal of R is a direct summand of ${}_RR$ and every non-zero left ideal of R contains a left regular element.*

4) *R is the sum of minimal left ideals and every non-zero left ideal of R contains a left regular element.*

5) *R is a left s-unital ring ($a \in Ra$ for any $a \in R$) such that every maximal left ideal is a direct summand of ${}_RR$ and every non-zero left ideal contains a left regular element.*

6) *R is a left s-unital ring such that every maximal left ideal is a left annihilator and every non-zero left ideal contains a left regular element.*

2′)-6′) *The left-right analogues of* 2)-6).

Proof. Any division ring satisfies the conditions 2)-6), and moreover the equivalence of 3) and 4) is well known.

2)⇒3) In fact, I is a direct summand of ${}_RR$, and isomorphic to R/I' for some left ideal I'.

4)⇒1) Let I be an arbitrary minimal left ideal of R. Then, I contains a left regular element a. Since $I=Ra$, there exists an element e such that $a=ea$. Evidently, e is a right identity element of R. Hence, R is left Artinian, whence it follows $Ra=R$, namely, R is the unique minimal left ideal of R. As is well known, R is then a division ring.

5)⇒4) By [3, Lemma 1], R is the sum of minimal left ideals.

6)⇒4) Since R is semiprime, R is the sum of minimal left ideals by [2, Theorem].

References

[1] R. COUTURE et M. LAVOIE : Réciproque du théorème des base sur un espace vectoriel, Canad. Math. Bull. **18** (1975), 749–751.

[2] K. KISHIMOTO and H. TOMINAGA : On decompositions into simple rings. II, Math. J. Okayama Univ. **18** (1975), 39–41.

[3] H. TOMINAGA : On decompositions into simple rings, Math. J. Okayama Univ. **17** (1975), 159–163.