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Note on A'_n-maps

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1. Introduction.

In [1] we have defined A'_n-spaces and A'_n-maps, and considered some of their properties. The purpose of the present note is to give certain complementary facts.

In § 2, we show that A'_2-notions are homotopy-categorical, and A'_3-notions are categorical.

In § 3, we consider conditions for a map of A'_3-spaces having the vanishing generalized Hopf homomorphism to be an A'_2-map. Our main theorem is as follows.

Theorem 3. 1. *Let $f: SZ \longrightarrow Y$ be a map into an A'_2-space. If $H(f)=0$, then f is an A'_2-map.*

We use the same notations and defintions as in [1], in particular, we work in the category of based spaces having the homotopy-types of CW-complexes and based maps.

2. Categories $\mathscr{A}'_2$ and $\mathscr{A}'_3$.

As easily seen, A'_2-spaces and A'_2-maps constitute a category $\mathscr{A}'_2$.

Definition 2. 1. Two A'_2-maps $f_0,\ f_1: X \longrightarrow Y$ are said to be *$\mathscr{A}'_2$-homotopic*, in notation : $f_0 \simeq f_1(\mathscr{A}'_2)$, if there exist a homotopy $F=H(f_0, f_1)$ and a homotopy $H(F): X\times I\times I \longrightarrow Y\vee Y$ satisfying the following conditions :

$$H(F)|X\times I\times\{0\}=H'_2(f_0),\quad H(F)|X\times I\times\{1\}=H'_2(f_1),$$
$$H(F)|X\times\{0\}\times I=(F\vee F)\circ\mu'_X \text{ and } H(F)|X\times\{1\}\times I=\mu'_Y\circ F.$$

An A'_2-map $f: X \longrightarrow Y$ is an *$\mathscr{A}'_2$-homotopy-equivalence* if there exists a homotopy-inverse g such that we have $g\circ f\simeq 1$ ($\mathscr{A}'_2$) and $f\circ g\simeq 1$ ($\mathscr{A}'_2$).

Proposition 2. 2. *Let $f_0: X \longrightarrow Y$ be an A'_2-map, and $f_1: X \longrightarrow Y$ is a map which is homotopic to f_0. Then, we may define $H'_2(f_1)$ so that it holds $f_0\simeq f_1$ ($\mathscr{A}'_2$).*

Proof. Put $F=H(f_0, f_1)$, then the homotopy $H'_2(f_1)$ is given by

$$H'_2(f_1)=\begin{cases}(F(\ ;1-3t)\vee F(\ ;1-3t))\circ\mu'_X(x) & \text{for } 0\leqq t\leqq \frac{1}{3}\\ H'_2(f_0)\ (x,3t-1) & \text{for } \frac{1}{3}\leqq t\leqq \frac{2}{3}.\\ \mu'_Y(F(x,3t-2)) & \text{for } \frac{2}{3}\leqq t\leqq 1\end{cases}$$

Let G_0 be the homotopy for f_0 in (3. 2. 2″) of [1], then the corresponding homotopy G_1 for f_1 is defined by the followings :

$$G_1\ (x,s,t)=\begin{cases}G_0(x,2s-1,\ 3t-1) \quad \text{for } \frac{1}{2}\leqq s\leqq 1,\ \frac{1}{3}\leqq t\leqq \frac{2}{3}\\ (F(\ ;1-t')\times F(\ ;1-t'))\circ D'_X(x,s') \ \text{ for } s=\frac{2s'+t'-s't'}{2},\ t=\frac{t'}{2}\\ F(x,1-2s)\times F(x,1-2s) \ \text{ for } s=\frac{s'}{2},\ t=\frac{s'+3t'-2s't'}{3}\\ D'_Y(F(x,t'),s') \quad \text{for } s=\frac{1+s'-t'+2s't'}{3},\ t=\frac{2+t'}{3}\end{cases}$$

where s' and t' run from 0 through 1.

Now, let X and Y be simply-connected CW-complexes and $f:X\longrightarrow Y$ be a cellular homotopy-equivalence. Since the mapping cylinder M_f is a simply-connected CW-complex and $\pi_n(M_f,X)=0$ for all $n\geqq 2$, X is a strong deformation retract of M_f. Let $r_1:M_f\longrightarrow X$ be the retraction, $i_1:X\longrightarrow M_f$ be the inclusion and $D_1=H(i_1\circ r_1,1)$. On the other hand, Y is a strong deformation retract of M_f, let $r_2:M_f\longrightarrow Y$ be the retraction, $i_2:Y\longrightarrow M_f$ be the inclusion and $D_2=H(i_2\circ r_2,1)$. Then, we have $f=r_2\circ i_1$, and $f'=r_1\circ i_2$ is a homotopy-inverse of f, moreover we have $F=H(f'\circ f,1)=r_1\circ D_2\circ(i_1\times 1)$ and $F'=H(f\circ f',1)=r_2\circ D_1\circ(i_2\times 1)$. Define $F_{(2)}:X\times I\times I\longrightarrow Y$ by $F_{(2)}(x,s,t)=r_2\circ D_2(D_1(i_1(x),s),t)$, then $F_{(2)}$ satisfies the following conditions :

$$F_{(2)}|X\times I\times\{0\}=f\circ F,\qquad F_{(2)}|X\times\{0\}\times I=F'\circ(f\times 1),$$
$$F_{(2)}|(x,0,0)=(f\circ f'\circ f)(x)\quad\text{and}$$
$$F_{(2)}|X\times I\times\{1\}=F_{(2)}|\ X\times\{1\}\times I=f.$$

By abuse of language, we say that two homotopies $f\circ F$ and $F'\circ(f\times 1)$ are homotopic. Similarly, we have that two homotopies $f'\circ F'$ and $F\circ(f'\times 1)$ are homotopic. In these situations, we say that $\{f,f',F,F'\}$ is *nice.*

Proposition 2. 3. *Let $f:X\longrightarrow Y$ be a cellular homotopy-equivalence of simply-connected CW-complexes with a homotopy-inverse f' such that $\{f,f',F=H(f'\circ f,1),F'=H(f\circ f',1)\}$ is nice. If X is an A'_2-space, then we may define an A'_2-structure of Y such that f is an $\mathscr{A}'_2$-homotopy-equivalence.*

Proof. Put $\mu'_Y=(f\vee f)\circ\mu'_X\circ f'$ and $D'_Y=\dot{-}(F'\times F')\circ\Delta_Y\dot{+}(f\times f)\circ D'_X\circ(f'\times 1)$, then $\{\mu'_Y, D'_Y\}$ defines an A'_2-*structure* of Y, *i.e.*, we have $D'_Y = H(\Delta_Y, j_Y \circ \mu'_Y)$. Moreover, we have $H'_2(f) = \dot{-}(f \vee f)\circ \mu'_X \circ F$ and $H'_2(f') = (F \vee F)(\mu'_X \circ f' \times 1)$.

Define $G(f): X \times I \times I \longrightarrow Y \times X$ by

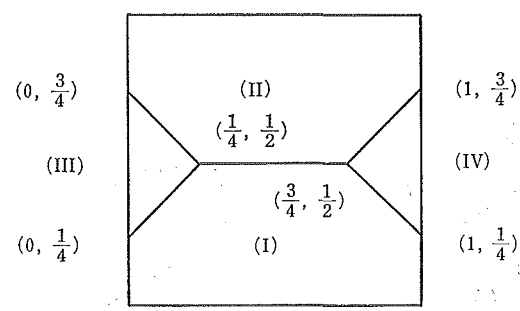
$$G(f)(x,s,t) = \begin{cases} (f \times f)\circ D'_X F(x, 1-t), \dfrac{2s-t}{2-t}) & \text{for } \dfrac{t}{2} \leqq s \leqq 1, 0 \leqq t \leqq 1 \\ \Delta_Y \circ F_{(2)}(x, 1-2s, 2-2t) & \text{for } t \geqq s + \dfrac{1}{2}, \ 0 \leqq s \leqq \dfrac{t}{2}, \\ \Delta_Y \circ F_{(2)}(x, 1-2s, 2t-4s) & \text{for } t \leqq s + \dfrac{1}{2}, \ 0 \leqq s \leqq \dfrac{t}{2} \end{cases}$$

where $F_{(2)}$ is the homotopy from $f \circ F$ to $F' \circ (f \times 1)$. Then, $G(f)$ satisfies the condition (3. 2. 2″) in [1], and f is an A'_2-map. Similarly, $G(f')$ is defined and f' is an A'_2-map. As easily seen, we have $f' \circ f \simeq 1(\mathscr{A}'_2)$ and $f \circ f' \simeq 1(\mathscr{A}'_2)$, thus f is an $\mathscr{A}'_2$-homotopy-equivalence.

Proposition 2. 4. *A'_3-spaces and A'_3-maps constitute a category $\mathscr{A}'_3$.*

Proof. It suffices to define a homotopy $H'_3(g \circ f): X \times K_3 \times I \longrightarrow W_3(Z)$ satisfying the conditions (3. 2. 1∼3) in [1] for A'_3-*maps* $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$.

Let $H'_3(f)$ and $H'_3(g)$ be the homotopies for f and g. Subdivide $I \times I$ into four domains as in the following figure.



Define $H'_3(g \circ f)|X \times (\text{I})$ using $W_3(g)\circ H'_3(f)$ and $H'_3(g \circ f)|X \times (\text{II})$ using $H'_3(g)\circ(f \times 1)$. Next, define $H'_3(g \circ f)|X \times (\text{III})$ using $(H'_2(g) \vee g)\circ H'_2(f)$ and $H'_3(g \circ f) X \times (\text{IV})$ using $(g \vee H'_2(g))\circ H'_2(f)$. Then, these maps coincide on the intersections of domains, therefore we may define $H'_3(g \circ f)$ all over $X \times K_3 \times I$. By the construction, $H'_3(g \circ f)$ satisfies the condition (3. 2. 1). Since f and g are A'_2-maps (3. 2. 2) is obvious, and (3. 2. 3) will be seen easily.

3. Generalized Hopf Homorphism.

At first, we recall the definition of the generalized Hopf homomorphism $H(f)$

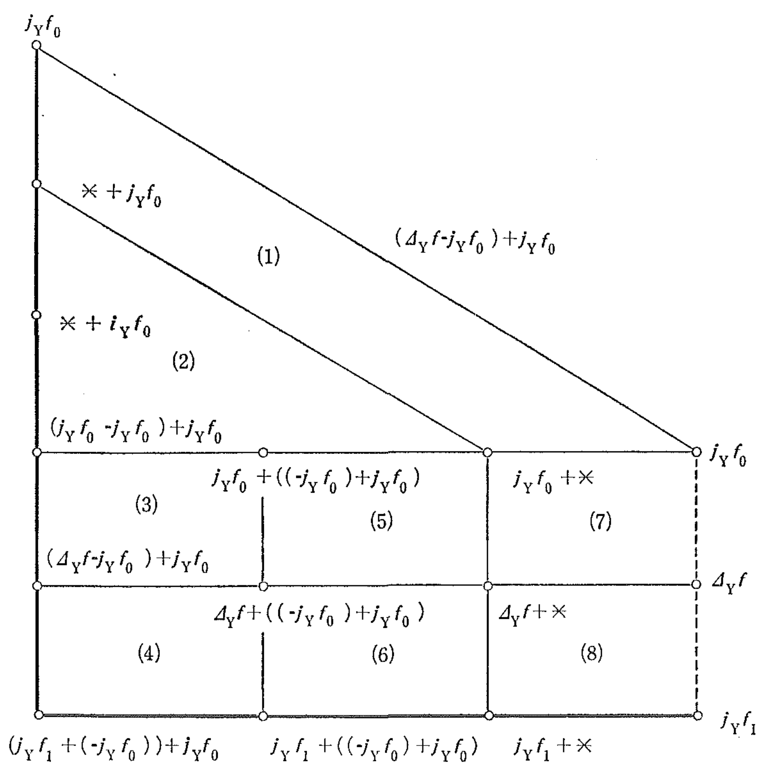
in §4 of [1]. Let X be an A'_2-space, then any map $u: X \longrightarrow Y \vee Y$ is represented as $u=(u_1 \times u_2)\circ \Delta_X$, and since there exists a homotopy $D'_X = H(\Delta_X, j_X \circ \mu'_X)$, we obtain $u=(u_1\times u_2)\circ\Delta_X \simeq (u_1\times u_2)\circ j_X \circ \mu'_X = j_Y\circ(u_1 \vee u_2)\circ \mu'_X$, *i. e.*, $j_{Y*}: [X; Y\vee Y\} \longrightarrow [X; Y\times Y]$ is surjective. Then, we have the following exact sequence ([1], Lemma 4.3):

$$0 \longrightarrow [CX, X; Y\times Y, Y\vee Y] \xrightarrow[r_*]{} [X, Y\vee Y] \xrightarrow[j_*]{} [X, Y\times Y] \longrightarrow 0.$$

And we have the isomorphism $\varphi: [CX, X; Y\times Y, Y\vee Y] \approx [X; \Omega Y * \Omega Y]$.

Let $f: X \longrightarrow Y$ be any map A'_3-cogroups. Put $f_0=(f\vee f)\circ \mu'_X$ and $f_1 = \mu'_Y \circ f$. Then, there exists a $[g'] \in [CX, X; Y\times Y, Y\vee Y]$ such that $r_*[g'] = [f_1]-[f_0]$. Define $H(f)=\varphi[g']=[g]$. If f is an A'_2-map, then $H(f)=0$, and if $H(f)=0$, we have a homotopy $H'_2(f)=H(f_0, f_1)$.

We attempt to consider these situations more precisely. Put $F' = \dot{-}(f\times f)\circ D'_X \dot{+} D'_Y \circ (f\times 1)$, then we have $F'=H(j_Y\circ f_0,\ j_Y\circ f_1)$. Define homotopies $F''=H(j_Y f_0 - j_Y\circ f_0,\ j_Y\circ f_1 - j_Y\circ f_0)$ and $F'''=H(*,\ j_Y\circ f_1 - j_Y\circ f_0)$ by $F''=F'-j_Y\circ f_0$ and $F'''=\dot{-}j_Y\circ f_0 \circ N'_R \dot{+} F''$, respectively. Then, there exists a homotopy $F: X\times I \longrightarrow Y\vee Y$ such that we have $j_Y\circ F = F'''$ and $F(x,1)=f_1(x)-f_0(x)$. The above map g is just a map defined by $g(x)=F(x,0)$. Therefore, $H(f)=0$ implies the existence of a homotopy $N_g: X\times I \longrightarrow \Omega Y * \Omega Y$ such that $N_g(x,0)=*$ and $N_g(x,1)=g(x)$. Then, we may define $H'_2(f)$



$=H(f_0, f_1)$ by the followings:

$$H'_2(f)=f_0\circ E'_L\dot{+}((N_g\dot{+}F)+f_0)\dot{+}\nabla_3\circ(f_1\vee(-f_0)\vee f_0)\circ' M_{X,3}$$
$$\dot{+}(f_1+f_0\circ N'_L)\dot{-}f_1\circ E'_R.$$

Now, consider the above diagram (the thick line segments represent the homotopy $j_Y\circ H'_2(f)$ and the broken line segments represent the homotopy F'): Squares (2)~(8) are homotopy-commutative by the similar argument as in Proposition 2. 8.

If X is a suspended space, say $X=SZ$, then the tetragon (1) is homotopy-commutative. In fact, if we define $E'_c: SZ\times K_3\times I\longrightarrow SZ$ by

$$E'_c(\langle a, z\rangle, t, s)=\begin{cases}\langle \frac{2a}{1+s}, z\rangle & \text{for } 0\leqq a\leqq \frac{(1+s)t}{2}\\ \langle t, z\rangle & \text{for } \frac{(1+s)t}{2}\leqq a\leqq \frac{1+t-s+st}{2}\\ \langle \frac{2a+s-1}{1+s}, z\rangle & \text{for } \frac{1+t-s+st}{2}\leqq a\leqq 1\end{cases},$$

then, we have $E'_c|SZ\times\{0\}\times I=H(*+1,1)$, $E'_c|SZ\times\{1\}\times I= H(1+*,1)$ and $E'_c|SZ\times K_3\times\{1\}=1_{SZ}$. Moreover, $E'_c|SZ\times K_3\times\{0\}$ defines a homotopy $E''=H(*+1,1+*)$ such that

$$E''(\langle a, z\rangle, t)=\begin{cases}\langle 2a, z\rangle & \text{for } 0\leqq a\leqq \frac{t}{2}\\ \langle t, z\rangle & \text{for } \frac{t}{2}\leqq a\leqq \frac{1+t}{2}\\ \langle 2a-1, z\rangle & \text{for } \frac{1+t}{2}\leqq a\leqq 1\end{cases}.$$

Finally, we obtain a map $\tilde{G}: SZ\times I\times I\longrightarrow SZ$ satisfying the cnnditions $\tilde{G}(\langle a, z\rangle, t, 0)=\nabla_3\circ(1\vee\nu_0\vee 1)\circ M'_{0,3}(\langle a, z\rangle, t)$ and $\tilde{G}(\langle a, z\rangle, t, 1)=E''(\langle a, z\rangle, t)$. In fact, $\tilde{G}$ is defined by the followings:

$$\tilde{G}(\langle a, z\rangle, t, s)=\begin{cases}\langle \frac{4(1-s+st)a}{1+t+st-s}, z\rangle & \text{for } 0\leqq a\leqq \frac{1+t+st-s}{4}\\ \langle 1-s+st, z\rangle & \text{for } \frac{1+t+st-s}{4}\leqq a\leqq \frac{1+t+s-st}{4}\\ \langle -4a+2+t, z\rangle & \text{for } \frac{1+t+s-st}{4}\leqq a\leqq \frac{2+t-st}{4}\end{cases}$$

$$\begin{cases} \langle st, z\rangle & \text{for } \dfrac{2+t-st}{4} \leqq a \leqq \dfrac{2+t+st}{4} \\ \left\langle \dfrac{4(1-st)a+3st-2-t}{2-t-st} \right\rangle & \text{for } \dfrac{2+t+st}{4} \leqq a \leqq 1 \end{cases} .$$

Thus, we have the following

Theorem 3.1. *Let $f: SZ \longrightarrow Y$ be a map into an A'_2-space. If $H(f)=0$, then f is an A'_2-map.*

Reference

[1] SAITO, S. On Higher Coassociativity, Hiroshima Math. J., 6(1976), 589–617