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On the admissible multiplication in α-coefficient cohomology theories

Dedicated to Professor Ryoji Shizuma on his 60-th birthday

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Introduction

S. ARAKI and H. TODA [1] discussed the multiplicative structures in mod. q generalized cohomology theories. In [2], the first named author discussed the multiplicative structures in α-coefficient cohomology theories (α is stable map of spheres) and in the case $\alpha=\eta$ (a stable class of the Hopf map from S^3 to S^2) obtained a sufficient condition for existence of admissible multiplication in $\tilde{h}^*(\ ;\alpha)$ for any reduced multiplicative generalized cohomology theory $\{\tilde{h}^*, \sigma\}$ defined on the category of finite CW-complexes or of the same homotopy type with base points. In [3], a sufficient condition for existence of admissible multiplication in the case $\alpha=\eta^2$ is obtained.

In this paper we discusse the existence of the admissible multiplication in a more general case including the case $\alpha=\eta$ and η^2.

For the stable map $\alpha\in\{S^{r+k-1}, S^r\}$ satisfying the condition "k is an odd integer or $2\alpha=0$", we obtained a sufficient condition for existence of admissible multiplication in $\tilde{h}^*(\ ;\alpha)$ for any multiplicative generalized cohomology theory $\tilde{h}^*$.

In § 1, we define the notion of admissible multiplication in α-coefficient cohomology theories. In § 2, we compute some stable homotopy groups and make preparations to the existence theorem of admissible multiplication from homotopical points of view. The existence of admissible multiplication is proved in § 3 by constructing a multiplication.

§ 1. Preliminaries

First we shall fix some notations:

$X\wedge Y$: the reduced join of two spaces X and Y with base points,

$S^nX = X\wedge S^n$: the iterated reduced suspension of X,

$S^nf = f\wedge 1_{S^n}$: the iterated reduced suspension of f,

$T = T(A, B) : A\wedge B \longrightarrow B\wedge A$: the map swiching factors,

$\{X, Y\}$: the stable homotopy group of CW-complexes X and Y with base point preserving,

$G_k = \lim \pi_{r+k}(S^r)$: the k-th stable homotopy group of the sphere.

Let $\{\widetilde{h}^*, \sigma\}$ be a deduced cohomology theory defined on the category of finite CW-complexes and be equipped with an associative multiplication μ. Let α be a stable homotopy class of a map from S^{r+k-1} to S^r. Since the stable homotopy type of the reduced mapping cone of this map depends only on homotopy class α, we denote as

$$C_\alpha = S^r \cup_\alpha C(S^{r+k-1}).$$

The α-coefficient cohomology theory $\{\widetilde{h}^*(\ ;\alpha), \sigma_\alpha\}$ is defined by

$$\widetilde{h}^i(X : \alpha) = \widetilde{h}^{i+r+k}(X\wedge C_\alpha)$$

and the suspension isomorphism

$$\sigma_\alpha ; \widetilde{h}^i(X ; \alpha) \longrightarrow \widetilde{h}^{i+1}(SX ; \alpha)$$

is defined as the composition

$$\sigma_\alpha = (1_X\wedge T)^*\sigma : \widetilde{h}^i(X ; \alpha) \longrightarrow \widetilde{h}^{i+1}(SX ; \alpha),$$

where $T = T(S^1, C_\alpha)$.

Let us denote by $i : S^r \longrightarrow C_\alpha$ the canonical inclusion and let $\pi : C_\alpha \longrightarrow S^{r+k}$ be the map collapsing S^r to a point. Then we put

$$\text{(1. 1)} \qquad \begin{aligned} \rho_\alpha &= (-1)^{i(r+k)}(1_X\wedge\pi)^*\sigma^{r+k} : \widetilde{h}^i(X) \longrightarrow \widetilde{h}^i(X ; \alpha), \\ \delta_{\alpha,0} &= (-1)^{i(r+k)}\sigma^{-r}(1_X\wedge i)^* : \widetilde{h}^i(X ; \alpha) \longrightarrow \widetilde{h}^{i+k}(X) \end{aligned}$$

and

$$\delta_\alpha = \rho_\alpha\delta_{\alpha,0} : \widetilde{h}^i(X : \alpha) \longrightarrow \widetilde{h}^{i+k}(X ; \alpha)$$

which are natural and called the reduction mod. α, the Bockstein homomorphism and the mod. α Bockstein homomorphism respectively.

Moreover we put

$$\text{(1. 2)} \qquad \begin{aligned} \mu_L &= \mu : \widetilde{h}^i(X)\otimes\widetilde{h}^j(Y ; \alpha) \longrightarrow \widetilde{h}^{i+j}(X\wedge Y ; \alpha), \\ \mu_R &= (-1)^{j(r+k)}(1_X\wedge T)^*\mu : \widetilde{h}^i(X ; \alpha)\otimes\widetilde{h}^j(Y) \longrightarrow \widetilde{h}^{i+j}(X\wedge Y ; \alpha) \end{aligned}$$

where $T = T(Y, C_\alpha)$.

A multiplication

$$\mu_\alpha : \tilde{h}^i(X ; \alpha) \otimes \tilde{h}^j(Y ; \alpha) \longrightarrow \tilde{h}^{i+j}(X \wedge Y ; \alpha)$$

is said to be admissible (cf. [2] 1. 6) if it satisfy the following properties

(A_1) compatible with μ_L and μ_R through the reduction mod. α i. e.,

(1. 3) $$\mu_L = \mu_\alpha(\rho_\alpha \otimes 1) \text{ and } \mu_R = \mu_\alpha(1 \otimes \rho_\alpha) ;$$

(A_2) there exists a cohomology operation $\chi_\alpha : \tilde{h}^i(\ ) \longrightarrow \tilde{h}^{i-k}(\ ; \alpha)$ of degree $-k$ satisfying the relation

(1. 4) $$\chi_\alpha \mu(x \otimes y) = (-1)^{ik} \mu_L(x \otimes \chi_\alpha(y)) = \mu_R(\chi_\alpha(x) \otimes y)$$

for $x \in \tilde{h}^i(X)$ and it is related to μ_α by the following relation

(1. 5) $$\delta_\alpha \mu_\alpha(x \otimes y) = \mu_L(\delta_{\alpha,0}(x) \otimes y) + (-1)^{ik} \mu_R(x \otimes \delta_{\alpha,0}(y)) - (-1)^{ik} \chi_\alpha \mu(\delta_{\alpha,0}(x) \otimes \delta_{\alpha,0}(y))$$

for $x \in \tilde{h}^i(X ; \alpha)$ and $y \in \tilde{h}^j(Y ; \alpha)$;

(A_3) it is quasi-associative in the sense that

(1. 6) $$\begin{aligned} \mu_\alpha(\mu_L \otimes 1) &= \mu_L(1 \otimes \mu_\alpha), \\ \mu_\alpha(\mu_R \otimes 1) &= \mu_\alpha(1 \otimes \mu_L), \\ \mu_R(\mu_\alpha \otimes 1) &= \mu_\alpha(1 \otimes \mu_R). \end{aligned}$$

§ 2. Stable homotopy groups of some complexes

Let t be an integer. Assume that $t\alpha = 0$ for an element $\alpha \in \pi_{r+k-1}(S^r)$. Let C_α be the reduced mapping cone of α. For simplicity we denote $C = C_\alpha$. From Puppe's exact sequence and its dual associated with a cofibration

(2. 1) $$S^r \xrightarrow{i} C \xrightarrow{\pi} S^{r+k}$$

we obtain the following table

Lemma 2.1. *The groups $\{S^{r+j}, C\}$ and $\{C, S^{r+j}\}$ are isomorphic to the corresponding groups in the following table :*

		generators of free part
$\{S^r, C\}$	Z	i
$\{S^{r+k}, C\}$	$Z + i(G_k/\eta\alpha)$	$\tilde{t}$
$\{S^{r+j}, C\}$	*finite group for* $j \neq 0, k$	
$\{C, S^{r+k}\}$	Z	π
$\{C, S^r\}$	$Z + (G_k/\eta\alpha)\pi$	$\bar{t}$
$\{C, S^{r+j}\}$	*finite group for* $j \neq 0, k$	

where $\tilde{t}$ and $\bar{t}$ are defined by $\pi\tilde{t} = t\,1_{S^{r+k}}$ and $\bar{t}\,i = t\,1_{S^r}$.

Morever we may chose $\tilde{t}$ *and* $\bar{t}$ *such that the relation*

$$i\,\bar{t}+\tilde{t}\pi = t\,1_C \tag{2.2}$$

holds in $\{C,\ C\}$, *where* 1_C *is the homotopy class of the identity of* C.

From Lemma 2.1 and dual Puppe's exact sequence associated with (2.1) it follows that

Lemma 2.2. *The groups* $\{S^jC,\ C\}$ *is isomorphic to the corresponding groups in the following table* :

		generators of free part
$\{C,\ S^kC\}$	Z	$(S^ki)\pi$
$\{C,\ C\}$	$Z+Z+i(G_k/\eta\alpha)$	$\tilde{t}\pi$ (or $i\bar{t}$), 1_C
$\{S^kC,\ C\}$	Z + *finite group*	
$\{S^jC,\ C\}$	*finite group for* $j\neq k$, 0 *and* $-k$	

From dual Puppe's exact sequence associated with (2.1), we obtain the following exact sequence

$$\longrightarrow\{S^{r+2k},\ C\}\xrightarrow{(S^k\pi)^*}\{S^kC,\ C\}\xrightarrow{(S^ki)^*}\{S^{r+k},\ C\}\xrightarrow{(S^k\alpha)^*}\{S^{r+2k-1},\ C\}\longrightarrow$$

where groups $\{S^{r+2k-1},\ C\}$ and $\{S^{r+2k},\ C\}$ are finite by Lemma 2.1.

If $(S^k\alpha)^*\tilde{t}=\tilde{t}(S^k\alpha)=0$, then there exists an element δ of $\{S^kC,\ C\}$ which satisfy the following relations

$$\delta(S^ki)=\tilde{t} \text{ and } \pi\delta=S^k\tilde{t} \tag{2.3}$$

and we can take δ as generator of free part in $\{S^kC,\ C\}$.

In the following, we consider $\alpha\in\pi_{r+k-1}(S^r)$ only as the element satisfying

$$k \text{ is an odd integer or } 2\alpha=0. \tag{2.4}$$

Lemma 2.3. (Lemma 3.5 *of* [4]) *Let* α *be an element of* $\pi_{r+k-1}(S^r)$ *satsfying* (2.4). *Assume that* $k\leq 2r-1$, *then there exists an element* α' *of* $\pi_{2r+2k-1}(S^{2r})$ *such that equality*

$$1_C\wedge\alpha=(S^ri)\alpha'(S^{r+k-1}\pi) \tag{2.5}$$

holds in the homotopy set $[S^{r+k-1}C,\ S^rC]$.

Under the condition (2.4), from Lemma 2.3, we shall see that $C\wedge C$ is homotopy equivalent in stable range to the following mapping cone

$$\overline{N}_\alpha=S^rC_\alpha\underset{\bar{g}}{\cup}C(S^{r+k-1}C_\alpha) \tag{2.6}$$

where $\bar{g}=(S^ri)\alpha'(S^{r+k-1}\pi)$.

We denote also by N_α a subcomplex of $\overline{N}_\alpha$ obtained by removing the $(2r+k)$

-cell $S^rC - S^{2r}$, i.e.,

(2.7) $$N_\alpha = S^{2r} \underset{g}{\cup} C(S^{r+k-1}C_\alpha)$$

where $g = \alpha'(S^{r+k-1}\pi)$.

The cell structures of $\overline{N}_\alpha$ and N_α can be interpreted as follows:

(2.8) $$\overline{N}_\alpha = (S^rC_\alpha \vee S^{2r+k}) \cup e^{2r+2k}, \quad N_\alpha = (S^{2r} \vee S^{2r+k}) \cup e^{2r+2k},$$

where e^{2r+2k} is attached to $S^{2r} \vee S^{2r+k}$ by a map represented the sum of $\alpha' \in \{S^{2r+2k-1}, S^{2r}\}$ and $\alpha \in \{S^{r+k-1}, S^r\}$.

We use the following notations;

(2.9) $j: N_\alpha \longrightarrow \overline{N}_\alpha$, the inclusion,

$p: \overline{N}_\alpha \longrightarrow S^{2r+k}$, the map collapsing N_α,

$\bar{i}_0 : S^rC \longrightarrow \overline{N}_\alpha$, $i_0 : S^{2r} \longrightarrow N_\alpha$, the inclusions,

$\bar{\pi}_0 : \overline{N}_\alpha \longrightarrow S^{r+k}C$, $\pi_0 : N_\alpha \longrightarrow S^{r+k}C$, the map collapsing S^rC or S^{2r},

$\bar{i}_1 : S^{2r+k} \longrightarrow \overline{N}_\alpha$, $i_1 : S^{2r+k} \longrightarrow N_\alpha$, the inclusions,

$\pi_1 : N_\alpha \longrightarrow Q = S^{2r} \underset{\alpha'}{\cup} e^{2r+2k}$, the map collapsing S^{2r+k}.

Hereafter, these mapping will be fixed as to satisfy the following relations;

(2.10) $$\bar{\pi}_0 j = \pi_0, \quad j\, i_0 = \bar{i}_0(S^r i), \quad \bar{\pi}_0\, \bar{i}_1 = S^{r+k} i = \pi_0\, i_1,$$
$$\bar{i}_1 = j\, i_1, \quad p\, \bar{i}_0 = S^r \pi \quad \text{and} \quad i_0\, \alpha' = -\, i_1(S^{r+k}\alpha).$$

Lemma 2.4. There exists an element ζ *of* $\{\overline{N}_\alpha, C \wedge C\}$ *satisfying the following three conditions;*

(2.11) (i) ζ *is a homotopy equivalence*, i.e., *there is an inverse* $\xi \in \{C \wedge C, \overline{N}_\alpha\}$ *of* ζ *such that* $\xi\zeta = 1$ *and* $\zeta\xi = 1$,

(ii) $\zeta \bar{i}_0 = 1_C \wedge i$, *thus* $\xi(1_C \wedge i) = \bar{i}_0$

and

(iii) $(1_C \wedge \pi)\zeta = \bar{\pi}_0$, *thus* $\bar{\pi}_0\, \xi = 1_C \wedge \pi$.

We put

(2.12) $$\zeta_0 = \zeta \bar{i}_1 \in \{S^{2r+k}, C \wedge C\} \quad \text{and} \quad \xi_0 = p\xi \in \{C \wedge C, S^{2r+k}\}.$$

Then it follows from (ii) and (iii) of (2.11) that

(2.13) $$(1_C \wedge \pi)\zeta_0 = S^{r+k} i, \quad \xi_0(1_C \wedge i) = S^r \pi.$$

We consider the Puppe's exact sequence associated with cofibration

(2.14) $$C \wedge S^r \xrightarrow{1_C \wedge i} C \wedge C \xrightarrow{1_C \wedge \pi} C \wedge S^{r+k}.$$

Then, from Lemma 2.1, (2.12) and (2,13), we obtain the following table.

Lemma 2.5. *The groups* $\{S^j,\ C\wedge C\}$ *and* $\{C\wedge C,\ S^j\}$ *are isomorphic to the corresponding groups in the following table :*

		generators of free part
$\{S^{2r},\ C\wedge C\}$	Z	$i\wedge i$
$\{S^{2r+k},\ C\wedge C\}$	$Z+Z+(i\wedge i)(G_k/\eta\alpha)$	$\tilde{t}\wedge i,\ \zeta_0$
$\{S^{2r+2k},\ C\wedge C\}$	$Z+$ *finite group*	
$\{S^j,\ C\wedge C\}$	*finite group* $j\neq 2r,\ 2r+k$ *and* $2r+2k$	
$\{C\wedge C,\ S^{2r+2k}\}$	Z	$\pi\wedge\pi$
$\{C\wedge C,\ S^{2r+k}\}$	$Z+Z+(G_k/\eta\alpha)(\pi\wedge\pi)$	$\bar{t}\wedge\pi,\ \xi_0$
$\{C\wedge C,\ S^{2r}\}$	$Z+$ *finite group*	
$\{C\wedge C,\ S^j\}$	*finite group for* $j\neq 2r,\ 2r+k$ *and* $2r+2k$	

where ζ_0 *and* ξ_0 *are elements satisfying* $(1_C\wedge\pi)\zeta_0=S^{r+k}i$ *and* $\xi_0(1_C\wedge i)=S^r\pi$.

From the Puppe's exact sequence associated with (2.14) and Lemma 2.5 we can see easily the following lemma :

Lemma 2.6.

(i) $\{C\wedge C,\ C\wedge S^{r+k}\}=\{1_C\wedge\pi\}+\{i\,\bar{t}\wedge\pi\}+\{(S^{r+k}i)\xi_0\}+i(G_k/\eta\alpha)(\pi\wedge\pi)$
$\approx Z+Z+Z+$ *finite group,*

(ii) $\{C\wedge S^r,\ C\wedge C\}=\{1_C\wedge i\}+\{\tilde{t}\,\pi\wedge i\}+\{\xi_0(S^r\pi)\}+(i\wedge i)(G_k/\eta\alpha)\pi$
$\approx Z+Z+Z+$ *finite group.*

Lemma 2.7. *Let* $\xi\in\{C\wedge C,\ \bar{N}_\alpha\}$ *be an element satisfying* (2.11)

(i) *Any element* $\xi'\in\{C\wedge C,\ \bar{N}_\alpha\}$ *satisfies* (2.11) *if and only if*

$$\xi'=\xi+\bar{i}_0\,\omega(1_C\wedge\pi)$$

for some $\omega\in\{S^{r+k}C,\ S^rC\}$.

(ii) *For any element* $g\in G_k/\eta\alpha$, *put* $\xi'_0=\xi_0+g(\pi\wedge\pi)$ *where* $\xi_0=p\xi$. *Then there exists* $\xi'\in\{C\wedge C,\ \bar{N}_\alpha\}$ *such that satisfying* (2.11) *and* (2.12).

Proof. (i) Assume that ξ and ξ' satisfy (2.11). Since $(1_C\wedge i)^*\ (\xi'-\xi)=0$, there exists $\gamma\in\{S^{r+k}C,\ \bar{N}_\alpha\}$ such that $(1_C\wedge\pi)^*\gamma=\xi'-\xi$. From (iii) of (2.11), $(1_C\wedge\pi)^*(\bar{\pi}_0\gamma)=0$. On the other hand the homomorphism $(1_C\wedge\pi)^*:\{S^{r+k}C,\ S^{r+k}C\}\longrightarrow\{C\wedge C,\ S^{r+k}C\}$ is a monomorphism. Thus $\bar{\pi}_0\gamma=0$. Therefore γ is contained in the image of $\bar{i}_{0*}:\{S^{r+k}C,\ S^rC\}\longrightarrow\{S^{r+k}C,\ \bar{N}_\alpha\}$.

Conversely, if ξ satisfies (ii) and (iii) of (2.11), then so dose ξ'. Put $\zeta'=\zeta-(1_C\wedge i)\omega\bar{\pi}_0$, then ζ' is a homotopy inverse of ξ'.

(ii) The element $\xi' = \xi + \bar{i}_0(1_C \wedge g)(1_C \wedge \pi)$ is the required element.

We consider the ordinary homology group. Let ζ be an element of $\{\bar{N}_\alpha, C \wedge C\}$ satisfyıng (2. 11) and ξ be a homotopy inverse of ζ. Let

$$\begin{pmatrix} & e_r \wedge e_r & \\ e_r \wedge e_{r+k}, & & e_{r+k} \wedge e_r \\ & e_{r+k} \wedge e_{r+k} & \end{pmatrix}, \quad \begin{pmatrix} & e_r \wedge s_r & \\ e_r \wedge s_{r+k}, & & e_{r+k} \wedge s_r \\ & e_{r+k} \wedge s_{r+k} & \end{pmatrix}$$

be generators of the groups $H_*(C \wedge C)$ and $H_*(\bar{N}_\alpha)$ rspectively, where $e_i \wedge e_j$ and $e_i \wedge s_j$ is a generator of $(i+j)$-dim. group resp.

Using (2. 11), for the ordinary homology map ξ_* and ζ_* induced by ξ and ζ resp., we obtain that

$$\xi_* \begin{pmatrix} & e_r \wedge e_r & \\ e_r \wedge e_{r+k}, & & e_{r+k} \wedge e_r \\ & e_{r+k} \wedge e_{r+k} & \end{pmatrix} = \begin{pmatrix} & e_r \wedge s_r & \\ e_r \wedge s_{r+k} - n e_{r+k} \wedge s_r, & & e_{r+k} \wedge s_r \\ & e_{r+k} \wedge s_{r+k} & \end{pmatrix}$$

$$\zeta_* \begin{pmatrix} & e_r \wedge s_r & \\ e_r \wedge s_{r+k}, & & e_{r+k} \wedge s_r \\ & e_{r+k} \wedge s_{r+k} & \end{pmatrix} = \begin{pmatrix} & e_r \wedge e_r & \\ e_r \wedge e_{r+k} + n e_{r+k} \wedge e_r, & & e_{r+k} \wedge e_r \\ & e_{r+k} \wedge e_{r+k} & \end{pmatrix}$$

for some integer n.

Using an element $\xi_0 = p\xi$ satisfying (2. 13), we can put

(*) $(1_C \wedge \pi)T = a(1_C \wedge \pi) + b(i\bar{t} \wedge \pi) + c((S^{r+k}i)\xi_0) \qquad \text{mod. } i(G_k/\eta\alpha)(\pi \wedge \pi),$

for some integers a, b, and c by Lemma 2. 6, where $T = T(C, C)$. The homology maps induced by $(1_C \wedge \pi)T$, $1_C \wedge \pi$, $i\bar{t} \wedge \pi$ and $(S^{r+k}i)\xi_0$ can be expressed as follows:

$$(1_C \wedge \pi)_* T_* \begin{pmatrix} & e_r \wedge e_r & \\ e_r \wedge e_{r+k}, & & e_{r+k} \wedge e_r \\ & e_{r+k} \wedge e_{r+k} & \end{pmatrix} = \begin{pmatrix} & 0 & \\ 0, & & (-1)^{r(r+k)} e_r \wedge s_{r+k} \\ & (-1)^{r+k} e_{r+k} \wedge s_{r+k} & \end{pmatrix},$$

$$(1_C \wedge \pi)_* \begin{pmatrix} & e_r \wedge e_r & \\ e_r \wedge e_{r+k}, & & e_{r+k} \wedge e_r \\ & e_{r+k} \wedge e_{r+k} & \end{pmatrix} = \begin{pmatrix} & 0 & \\ e_r \wedge s_{r+k}, & & 0 \\ & e_{r+k} \wedge s_{r+k} & \end{pmatrix},$$

$$(i\bar{t} \wedge \pi)_* \begin{pmatrix} & e_r \wedge e_r & \\ e_r \wedge e_{r+k}, & & e_{r+k} \wedge e_r \\ & e_{r+k} \wedge e_{r+k} & \end{pmatrix} = \begin{pmatrix} & 0 & \\ t e_r \wedge s_{r+k}, & & 0 \\ & 0 & \end{pmatrix},$$

$$(S^{r+k}i)_* \xi_{0*} \begin{pmatrix} & e_r \wedge e_r & \\ e_r \wedge e_{r+k}, & & e_{r+k} \wedge e_r \\ & e_{r+k} \wedge e_{r+k} & \end{pmatrix} = \begin{pmatrix} & 0 & \\ -n e_r \wedge s_{r+k}, & & e_r \wedge s_{r+k} \\ & 0 & \end{pmatrix},$$

$(ig(\pi \wedge \pi))_* = 0$ for any $g \in G_k/\eta\alpha$.

From (*), we obtain the identity

$$(1_C\wedge\pi)_*\mathrm{T}_* = a(1_C\wedge\pi)_* + b(i\bar{t}\wedge\pi)_* + c(S^{r+k}i)_*\xi_{0*}$$

of homology maps. Applying this to $e_{r+k}\wedge e_{r+k}$, $e_r\wedge e_{r+k}$ and $e_{r+k}\wedge e_r$, we have

$$(-1)^{r+k}e_{r+k}\wedge s_{r+k} = ae_{r+k}\wedge s_{r+k},$$
$$0 = ae_r\wedge s_{r+k} + bte_r\wedge s_{r+k} - cne_r\wedge s_{r+k},$$
$$(-1)^{r(r+k)}e_r\wedge s_{r+k} = ce_r\wedge s_{r+k}.$$

This is, $a=(-1)^{r+k}$, $c=(-1)^{r(r+k)}$ and $b=-((-1)^{r+k}-(-1)^{r(r+k)}n)/t$ and

$$(1_C\wedge\pi)T = (-1)^{r+k}(1_C\wedge\pi) - n'(i\bar{t}\wedge\pi) + (-1)^{r(r+k)}(S^{r+k}i)\xi_0$$

mod. $i(G_k/\eta\alpha)(\pi\wedge\pi)$, where $n'=((-1)^{r+k}-(-1)^{r(r+k)}n)/t$.

Here we can put

$$(1_C\wedge\pi)T = (-1)^{r+k}(1_C\wedge\pi) - n'(i\bar{t}\wedge\pi) + (-1)^{r(r+k)}(S^{r+k}i)\xi_0 + ig(\pi\wedge\pi)$$

for some $g\in G_k/\eta\alpha$. If $g\neq 0$, put $\xi'_0=\xi_0+(-1)^{r(r+k)}g(\pi\wedge\pi)$, then ξ'_0 satisfies (2.13) and the equality

$$(1_C\wedge\pi)T = (-1)^{r+k}(1_C\wedge\pi) - n'(i\bar{t}\wedge\pi) + (-1)^{r(r+k)}(S^{r+k}i)\xi'_0$$

hold.

From (ii) of Lemma 2.7, there exists $\xi'\in\{C\wedge C,\ N_\alpha\}$ such that satisfy (2.11), (2.12) and induce the same homology maps as ξ.
Let $\zeta'\in\{N_\alpha,\ C\wedge C\}$ be the homotopy inverse of ζ' and $\zeta'_0=\zeta'\bar{i}_1$. Then ζ' and ζ'_0 induce the same homology map as ζ and ζ_0 resp. Making use of ζ', by a similar calculation we see that

$$T(1_C\wedge i) = (-1)^r(1_C\wedge i) - n''(\tilde{t}\pi\wedge i) + (-1)^{r(r+k)}\zeta'_0(S^r\pi) + (i\wedge i)g\pi$$

for some $g\in G_k/\eta\alpha$, where $n''=((-1)^r+(-1)^{r(r+k)}n)/t$.

Hence we get the Lemma :

Lemma 2.8. *There exists $\xi\in\{C\wedge C,\ \bar{N}_\alpha\}$ and its inverse $\zeta\in\{\bar{N}_\alpha,\ C\wedge C\}$ which satisfy* (2.11) *and the following relations :*

(i) $(1_C\wedge\pi)T = (-1)^{r+k}(1_C\wedge\pi) - n'(i\bar{t}\wedge\pi) + (-1)^{r(r+k)}(S^{r+k}i)\xi_0,$

(ii) $T(1_C\wedge i) = (-1)^r(1_C\wedge i) - n''(\tilde{t}\pi\wedge i) + (-1)^{r(r+k)}\zeta_0(S^r\pi) + (i\wedge i)g\pi$

for some $g\in G_k/\eta\alpha$, where $n'=((-1)^{r+k}-(-1)^{r(r+k)}n)/t$, $n''=((-1)^r+(-1)^{r(r+k)}n)/t$, $\xi_0=p\xi$ and $\zeta_0=\zeta\bar{i}_1$.

Now we consider the element $\alpha\in\pi_{r+k-1}(S^r)$ satisfying

(2.15) $$1_C\wedge\alpha=0 \text{ and } \tilde{t}(S^k\alpha)=0$$

where $t\alpha=0$ for an integer t.

Then the cell structure of $\bar{N}_\alpha$ can be interpreted as follows

$$\bar{N}_\alpha = S^rC_\alpha\vee S^{r+k}C_\alpha.$$

Thus there exists a map $\pi_0^{-1}: S^{r+k}C \longrightarrow \overline{N}_\alpha$ such that

(2. 16) $$\pi_0^{-1}(S^{r+k}i) = \bar{i}_1 \text{ and } \bar{\pi}_0\pi_0^{-1} = 1_{S^{r+k}C}$$

i. e., π_0^{-1} is the inclusion.

Making use of (2. 15) we have the following commutative diagram associated with (2. 14) and (2. 1) in which all rows and all columns are exact :

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \{S^{2r+2k}, S^rC\} & \longrightarrow & \{S^{2r+2k}, C\wedge C\} & \longrightarrow & \{S^{2r+2k}, S^{r+k}C\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle (S^{r+k}\pi)^*} & & \downarrow & & \\
0 & \longrightarrow & \{S^{r+k}C, S^rC\} & \xrightarrow{(1_C\wedge i)_*} & \{S^{r+k}C, C\wedge C\} & \xrightarrow{(1_C\wedge \pi)_*} & \{S^{r+k}C, S^{r+k}C\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle (S^{r+k}i)^*} & & \downarrow & & \\
0 & \longrightarrow & \{S^{2r+k}, S^rC\} & \longrightarrow & \{S^{2r+k}, C\wedge C\} & \longrightarrow & \{S^{2r+k}, S^{r+k}C\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

From (2. 11) (2. 12) and (2. 16) we have

(2. 17) $$(S^{r+k}i)^*(\zeta\pi_0^{-1}) = \zeta_0 \quad\text{and}\quad (1_C\wedge\pi)^*(\zeta\pi_0^{-1}) = 1_{S^{r+k}C}.$$

From (2. 2) and the commutativity of above diagram, we have

(2. 18) $$(S^{r+k}i)^*(\delta\wedge i) = \tilde{t}\wedge i.$$

Proposition 2. 9. *Let k be an odd integer. Assume that $\alpha\in\pi_{r+k-1}(S^r)$ satisfies* (2. 15). *Then there exists an element $\gamma\in\{S^{r+k}C, C\wedge C\}$ such that*

(i) $$(1_C\wedge\pi)\gamma = (-1)^{r+k}1_{S^{r+k}C},$$

(ii) $$(1_C\wedge\pi)T\gamma = 1_{S^{r+k}C}$$

and

(iii) $$T(1_C\wedge i) + (-1)^{r+1}(1_C\wedge i) = (-1)^{k(r+k)}\gamma(S^{r+k}i)(S^r\pi) + (i\wedge i)g\pi$$

where $T = T(C, C)$ and some $g\in G_k/\eta\alpha$.

Proof. From Lemma 2. 8, we have

(2. 19) $$T(1_C\wedge i) + (-1)^{r+1}(1_C\wedge i) = (-1)^{k(r+k)}\gamma_0(S^r\pi) + (i\wedge i)g\pi$$

where $\gamma_0 = (-1)^{r+k}\zeta_0 + (-1)^{(r+1)(k+1)}n_0(\tilde{t}\wedge i)$, $n_0 = (1+(-1)^{rk}n)/t$.

We consider an element

$$\gamma = (-1)^{r+k}\zeta\pi_0^{-1} + (-1)^{(r+1)(k+1)}n_0(\delta\wedge i)$$

of $\{S^{r+k}C, C\wedge C\}$. Then, from (2. 17) and (2. 18), we obtain that $(S^{r+k}i)^*\gamma = \gamma_0$.

Thus we have (iii).

Using (2.17) and $(1_C\wedge\pi)(1_C\wedge i)=0$, it follows that

$$(1_C\wedge\pi)\gamma=(-1)^{r+k}1_{S^{r+k}C}. \tag{2.20}$$

From (i) of Lemma 2.8, (2.20) and (2.3), we obtain

$$(1_C\wedge\pi)T\gamma=1_{S^{r+k}C}.$$

Proposition 2.10. *Let k be an even integer. Assume that $\alpha\in\pi_{r+k-1}(S^r)$ satisfies (2.15) and $t=2$. Then there exists an element $\gamma\in\{S^{r+k}C,\ C\wedge C\}$ such that*

(i) $(1_C\wedge\pi)\gamma=(-1)^{r+k}1_{S^{r+k}C}$,

(ii) $(1_C\wedge\pi)T\gamma=1_{S^{r+k}C}$

and

(iii) $T(1_C\wedge i)+(-1)^r(1_C\wedge i)=\gamma(S^{r+k}i)(S^r\pi)+(-1)^r(i\bar{t}\wedge i)+(i\wedge i)g\pi$

where $T=T(C,C)$ and some $g\in G_k/\eta\alpha$.

Proof. From (ii) of Lemma 2.8 and (2.2), we have

$$(2.19)' \qquad \begin{aligned} T(1_C\wedge i) &= (-1)^r(1_C\wedge i)+\gamma_0(S^r\pi)-(-1)^r(\tilde{t}\pi\wedge i)+(i\wedge i)g\pi \\ &= (-1)^{r+k}(1_C\wedge i)+\gamma_0(S^r\pi)+(-1)^r(i\bar{t}\wedge i)+(i\wedge i)g\pi \end{aligned}$$

where $\gamma_0=(-1)^r(1-n_0)\ (\tilde{t}\wedge i)+(-1)^{r(r+k)}\zeta_0$, $n_0=(1+(-1)^{rk}n)/t$.

We consider an element

$$\gamma=(-1)^{r(r+k)}\zeta\pi_0{}^{-1}+(-1)^r(1-n_0)(\delta\wedge i)$$

of $\{S^{r+k}C,\ C\wedge C\}$.

By a similar calculation as in Proposition 2.9 we have the results.

Next we consider the element $\alpha\in\pi_{r+k-1}(S^r)$ satisfying

$$1_C\wedge\alpha=(S^r i)\alpha'(S^{r+k-1}\pi) \quad\text{and}\quad \tilde{t}\alpha=0 \tag{2.21}$$

for some non trivial element $\alpha'\in\pi_{2r+2k-1}(S^{2r})$ and the integer t such that $t\alpha=0$ (c.f., Lemma 2.3).

We put

$$Q=S^{2r}\cup_{\alpha'}e^{2r+2k} \tag{2.22}$$

and denote by

$$i':S^{2r}\longrightarrow Q \quad\text{and}\quad \pi':Q\longrightarrow S^{2r+2k} \tag{2.23}$$

the canonical inclusion and the map collapsing S^{2r} to a point resp. Then from (2.9) we have following cofibrations

$$S^{2r}\xrightarrow{\ i'\ }Q\xrightarrow{\ \pi'\ }S^{2r+2k}, \tag{2.24}$$

(2.25) $$S^{2r+k} \xrightarrow{i_1} N_\alpha \xrightarrow{\pi_1} Q.$$

Making use of (2.21) we have the following commutative diagram associated with (2.14) and (2.25) in which all rows and all columns are exact :

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \{Q,\ C\wedge S^r\} & \longrightarrow & \{Q,\ C\wedge C\} & \longrightarrow & \{Q,\ C\wedge S^{r+k}\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle \pi_1{}^*} & & \downarrow & & \\
\cdots & \longrightarrow & \{N_\alpha,\ C\wedge S^r\} & \xrightarrow{(1_C\wedge i^*)} & \{N_\alpha,\ C\wedge C\} & \xrightarrow{(1_C\wedge\pi)_*} & \{N_\alpha,\ C\wedge S^{r+k}\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle i_1{}^*} & & \downarrow & & \\
0 & \longrightarrow & \{S^{2r+k},\ C\wedge S^r\} & \longrightarrow & \{S^{2r+k},\ C\wedge C\} & \longrightarrow & \{S^{2r+k},\ C\wedge S^{r+k}\} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

From the Puppe's exact sequence associated with the cofibration (2.24) and Lemma 2.1, we obtain that

(2.26) $$\{Q,\ C\wedge S^{r+k}\} = Z + i(G_k/\eta\alpha)\pi'$$

and $(S^{r+k}\bar{t})\pi'$ is a generator of free part.

On the other hand, the right column in the above diagram splits. Thus, from (2.10), Lemma 2.1 and the relation $\pi'\pi_1 = (S^{r+k}\pi)\pi_0$, we obatin that

(2.27) $$\{N_\alpha,\ C\wedge S^{r+k}\} = Z + Z + i(G_k/\eta\alpha)\pi\pi_0$$

and $(\tilde{t}\pi)\pi_0$ and π_0 are generators of free parts.

From (2.3) and (2.10), we obtain

$$i_1{}^*(S^r\delta)\pi_0 = (S^r\delta)\pi_0 i_1 = (S^r\delta)(S^{r+k}i) = S^r\tilde{t}.$$

Thus it follows from commutativity of the above diagram that

(2.8) $$i_1{}^*((1_C\wedge i)(S^r\delta)\pi_0) = \tilde{t}\wedge i.$$

Proposition 2.11. *Let k be an odd integer. Assume that an element $\alpha\in\pi_{r+k-1}(S^r)$ satisfies* (2.21). *Then there exists an element β of $\{N_\alpha,\ C\wedge C\}$ such that*

(i) $(1_C\wedge\pi)\beta = (-1)^{r+k}\pi_0$,

(ii) $(1_C\wedge\pi)T\beta = \pi_0$

and

(iii) $T(1_C\wedge i) + (-1)^{r+1}(1_C\wedge i) = (-1)^{k(r+k)}\beta i_1(S^r\pi) + (i\wedge i)g\pi$

where $T = T(C,\ C)$ and some $g\in G_k/\eta\alpha$ (c.f., Lemma 2.8).

Proof. Let ζ be a homotopy equivalence given in Lemma 2.8. Then we put

$$\beta=(-1)^{(r+1)(k+1)}n_0(1_C\wedge i)(S^r\delta)\pi_0+(-1)^{r+k}\zeta j\in\{N_\alpha,\ C\wedge C\}$$

where $n_0=(1+(-1)^{rk}n)/t$. Then using (2.19), (2.28), (2.12) and Lemma 2.8, the proof of this proposition is completely parallel to it of Proposition 2.9.

Proposition 2.12. *Let k be an even integer. Assume that an element $\alpha\in\pi_{r+k-1}(S^r)$ satisfies* (2.21) *and $t=2$. Then there exists an element β of $\{N_\alpha,\ C\wedge C\}$ such that*

(i) $(1_C\wedge\pi)\beta=(-1)^{r+k}\pi_0$,

(ii) $(1_C\wedge\pi)T\beta=\pi_0$

and

(iii) $T(1_C\wedge i)+(-1)^r(1_C\wedge i)=(-1)^{k(r+k)}\beta i_1(S^r\pi)+(-1)^r i\,\bar{t}\wedge i+(i\wedge i)g\,\pi$

where $T=T(C,\ C)$ and some $g\in G_k/\eta\alpha$ (c.f., Lemma 2.8).

Proof. For ζ in Lemma 2.8, we put

$$\beta=(-1)^r(1-n_0)(1_C\wedge i)(S^r\delta)\pi_0+(-1)^{r(r+k)}\zeta j,$$

where $n_0=(1+(-1)^{rk}n)/t$. Then using (2.19)$'$, (2.28) and (2.12), similarly as in Proposition 2.10 we have the results.

§3. Existence of the admissible multiplication in $\tilde{h}^*(\ ;\alpha)$

Let μ be an associative multiplication in a reduced generalized cohomology theory $\{\tilde{h}^*,\ \sigma\}$. In this paragraph we define a multiplication μ_α in $\tilde{h}^*(\ :\alpha)$ for some $\alpha\in\pi_{r+k-1}(S^r)$, and give a sufficient condition for μ_α to be admissible.

Let A and B be finite CW-complexes with base points, for any element φ of $\{A,\ B\}$, we define a homomorphism

$$\varphi^{**}:\tilde{h}^*(X\wedge B)\longrightarrow\tilde{h}^*(X\wedge A)$$

by the formula

$$\varphi^{**}=\sigma^{-m}(1_X\wedge f)\sigma^m$$

where $f:S^mA\longrightarrow S^mB$ is a map representing φ. The definition of φ^{**} dose not depend on the choice of f.

Making use of an element $\gamma\in\{S^{r+k}C,\ C\wedge C\}$, we define a map

$$\mu_\alpha:\tilde{h}^i(X\ ;\alpha)\otimes\tilde{h}^j(Y\ ;\alpha)\longrightarrow\tilde{h}^{i+j}(X\wedge Y\ ;\alpha) \tag{3.1}$$

as the compsition

$$\begin{aligned}\mu_\alpha=(-1)^{i(r+k)}&\sigma^{-(r+k)}(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T'\wedge 1_C)^*\mu:\\ &\tilde{h}^i(X\ ;\alpha)\otimes\tilde{h}^j(Y\ ;\alpha)=\tilde{h}^{i+r+k}(X\wedge C)\otimes\tilde{h}^{j+r+k}(Y\wedge C)\\ &\longrightarrow\tilde{h}^{i+j+2r+2k}(X\wedge C\wedge Y\wedge C)\\ &\longrightarrow\tilde{h}^{i+j+2r+2k}(X\wedge Y\wedge C\wedge C)\end{aligned}$$

$$\longrightarrow \tilde{h}^{i+j+2r+2k}(X\wedge Y\wedge C\wedge S^{r+k})$$
$$\longrightarrow \tilde{h}^{i+j+r+k}(X\wedge Y\wedge C) = \tilde{h}^{i+j}(X\wedge Y\ ;\alpha)$$

where $T' = T(Y, C)$.

Obviously μ_α is linear and natural with respect to both variable.

Proposition 3.1. *Assume that $\gamma\in\{S^{r+k}C, C\wedge C\}$ satisfies*

$$(-1)^{r+k}(1_C\wedge\pi)\gamma = 1_{S^{r+k}C} = (1_C\wedge\pi)T\gamma \tag{3.2}$$

where $T = T(C, C)$. Then the map μ_α of (3.1) *is a multiplication satisfying* (A_1) *and* (A_3).

Proof. To prove (A_1), putting $T' = T(Y, C)$, $T = T(C, C)$.
By definition of ρ_α and μ_α, we have

$$\begin{aligned}\mu_\alpha(\rho_\alpha\otimes 1) &= \sigma^{-(r+k)}(1_{X\wedge Y}\wedge\gamma)^*(1_X\wedge T'\wedge 1_C)^*\mu((1_X\wedge\pi)\sigma^{r+k}\otimes 1_{Y\wedge C})\\ &= \sigma^{-(r+k)}\gamma^{**}(1_{X\wedge Y}\wedge T)^*(1_{X\wedge Y}\wedge 1_C\wedge\pi)^*\sigma^{r+k}\mu\\ &= \sigma^{-(r+k)}((1_C\wedge\pi)T\gamma)^{**}\sigma^{r+k}\mu\\ &= \mu = \mu_L \qquad \text{by (3.2).}\end{aligned}$$

Similarly, using the relation $(1_C\wedge\pi)\gamma = (-1)^{r+k}1_{S^{r+k}C}$, we obtain that

$$\mu_\alpha(1\otimes\rho_\alpha) = \mu_R.$$

Then it follows that $\rho_\alpha(1)$ is the bilatiral unit of μ_α (see [2]).

The compatibility with suspension isomorphism σ_α and (A_3) are verified directly from the definition of μ_α, σ_α and the associativity of μ.

Proposition 3.2. *It there exists an element $\gamma\in\{S^{r+k}C, C\wedge C\}$ satisfying the relation*

$$T(1_C\wedge i) + (-1)^{r+k}(1_C\wedge i) = (-1)^{k(r+k)}\gamma(S^{r+k}i)(S^r\pi) + (-1)^{r(r+k)}(i\wedge i)(S^r\chi) \tag{3.3}$$

for some $\chi\in\{C, S^r\}$, then map μ_α of (3.1) *satisfies* (A_2) *with associated cohomology operation*

$$\chi_\alpha = (-1)^{i(r+k)}\chi^{**}\sigma^r : \tilde{h}^i(\ \ ) \longrightarrow \tilde{h}^{i-k}(\ \ ;\alpha)$$

where $T = T(C, C)$.

Proof. We put $T' = T(Y, C)$. On $\tilde{h}^i(X\ ;\alpha)\otimes\tilde{h}^j(Y\ ;\alpha)$, we have

$$\begin{aligned}&\mu_L(\delta_{\alpha,0}\otimes 1) + (-1)^{ik}\mu_R(1\otimes\delta_{\alpha,0})\\ &= (-1)^{i(r+k)}\sigma^{-r}(1_{X\wedge Y}\wedge 1_C\wedge i)^*(1_{X\wedge Y}\wedge T)^*(1_X\wedge T'\wedge 1_C)^*\mu\\ &\quad + (-1)^{i(r+k)+r+k}\sigma^{-r}(1_{X\wedge Y}\wedge 1_C\wedge i)^*(1_X\wedge T'\wedge 1_C)^*\mu\\ &= (-1)^{i(r+k)}\sigma^{-r}(T(1_C\wedge i) + (-1)^{r+k}(1_C\wedge i))^{**}(1_X\wedge T'\wedge 1_C)^*\mu\\ &= (-1)^{i(r+k)}\sigma^{-r}((-1)^{k(r+k)}\gamma(S^{r+k}i)\ (S^r\pi))^{**}(1_X\wedge T'\wedge 1_C)^*\mu\\ &\quad + (-1)^{i(r+k)}\sigma^{-r}((-1)^{r(r+k)}(i\wedge i)(S^r\chi))^{**}(1_X\wedge T'\wedge 1_C)^*\mu\end{aligned}$$

$$= (-1)^{i(r+k)}(1\wedge\pi)^*\sigma^k(1\wedge i)^*\sigma^{-(r+k)}(1\wedge\gamma)^*(1\wedge T'\wedge 1_C)^*\mu$$
$$+(-1)^{i(r+k)+r(r+k)}\chi^{**}\sigma^{-r}(i\wedge i)^{**}(1\wedge T'\wedge 1_C)^*\mu$$
$$=\delta_\alpha\mu_\alpha+(-1)^{i(r+k)+r(r+k)}\chi^{**}\sigma^r\sigma^{-2r}(i\wedge i)^{**}(1\wedge T'\wedge 1_C)^*\mu.$$

On the other hand we have

$$\mu(\delta_{\alpha,0}\otimes\delta_{\alpha,0})=(-1)^{(i+j)(r+k)+ir+r(r+k)}\sigma^{-2r}(i\wedge i)^{**}(1\wedge T'\wedge 1_C)^*\mu.$$

Here we put

$$\chi_\alpha=(-1)^{i(r+k)}\chi^{**}\sigma^r:\tilde{h}^i(\ \ )\longrightarrow\tilde{h}^{i-k}(\ \ ;\alpha),$$

then we have

$$\delta_\alpha\mu_\alpha=\mu_L(\delta_{\alpha,0}\otimes 1)+(-1)^{ik}\mu_R(1\otimes\delta_{\alpha,0})-(-1)^{ik}\chi_\alpha\mu(\delta_{\alpha,0}\otimes\delta_{\alpha,0}).$$

Clearly χ_α is a cohomology operation and the relation

$$\chi_\alpha\mu=\mu_R(\chi_\alpha\otimes 1)=(-1)^{ik}\mu_L(1\otimes\chi_\alpha)$$

holds.

As a consequence of Proposition 2.9, 2.10, 3.1 and Proposition 3.2 we obtain the following theorem:

Theorem 3.3. *Assume that an element $\alpha\in\pi_{r+k-1}(S^r)$ satisfies* (2.15) *and $t=2$ if k is even. Then there exists an admissible multiplication μ_α in $\tilde{h}^*(\ \ ;\alpha)$.*

Now we consider an element $\alpha\in\pi_{r+k-1}(S^r)$ sasisfying (2.21), i. e., $1_C\wedge\alpha=(S^r i)\alpha'(S^{r+k-1}\pi)$ and $\tilde{t}\alpha=0$ for some $\alpha'\in\pi_{2r+2k-1}(S^{2r})$ and the integer t such that $t\alpha=0$.

Using the notation (2.9), cofibration

$$S^{2r}\xrightarrow{i_0}N_\alpha\xrightarrow{\pi_0}S^{r+k}C$$

yields, for any finite CW-complex W with a base point, a cofibration

$$W\wedge S^{2r}\xrightarrow{1\wedge i_0}W\wedge N_\alpha\xrightarrow{1\wedge\pi_0}W\wedge S^{r+k}C. \tag{3.4}$$

If $(\alpha'\pi)^{**}=0$ in $\tilde{h}^*$, then the $\tilde{h}^*$-cohomology exact sequence associated to the above cofibration (3.4) breaks into the following exact sequence

$$0\longrightarrow\tilde{h}^n(W\wedge S^{r+k}C)\xrightarrow{(1\wedge\pi_0)^*}\tilde{h}^n(W\wedge N_\alpha)\xrightarrow{(1\wedge i_0)^*}\tilde{h}^n(W\wedge S^{2r})\longrightarrow 0. \tag{3.5}$$

By (3.5) for $W=S^0$ and $n=2r$, it follows that

Lemma 3.4. (i) *If $(\alpha'\pi)^{**}=0$ in $\tilde{h}^*$, then there exists $\varphi_0\in\tilde{h}^{2r}(N_\alpha)$ satisfying*

$$i_0^*\varphi_0=\sigma^{2r}1, \tag{3.6}$$

(ii) *If $\alpha'^{**}=0$ in $\tilde{h}^*$, then there exists $\varphi_0\in\tilde{h}^{2r}(N_\alpha)$ satisfying*

$$i_0^*\varphi_0=\sigma^{2r}1\quad and\quad i_1^*\varphi_0=0. \tag{3.6$'$}$$

Proof. See Lemma 4.2 of [2].

Making use of φ_0 of Lemma 3.4, hence at least under the assumption of $(\alpha'\pi)^{**}=0$, we define a homorphism

$$\varphi_W : \tilde{h}^n(W\wedge N_\alpha) \longrightarrow \tilde{h}^n(W\wedge S^{r+k}C)$$

by the formula

$$\varphi_W(x) = (1_W\wedge\pi_0)^{*-1}(x - \mu(\sigma^{-2r}(1_W\wedge i_0)^* x \otimes \varphi_0)) \tag{3.7}$$

for $x\in h^n(W\wedge N_\alpha)$.

Since $x-\mu(\sigma^{-2r}(1_W\wedge i_0)^*x\otimes\varphi_0)$ is in the kernel of $(1_W\wedge i_0)^*$ and $(1_W\wedge\pi_0)^*$ is monomorphic, the map φ_W of (3.7) is well-defined homomorphism.

Lemma 3.5. (i) *φ_W is a left inverse of $(1_W\wedge\pi_0)^*$, i.e., $\varphi_W(1_W\wedge\pi_0)^*=$ an identity map ; hence the sequence of* (3.5) *splits :*

$$\tilde{h}^n(W\wedge N_\alpha) = \tilde{h}^n(W\wedge S^{r+k}C)\oplus\tilde{h}^n(W\wedge S^{2r}).$$

(ii) *φ_W is natural in the sense that*

$$(f\wedge S^{r+k}1_C)^*\varphi_W = \varphi_{W'}(f\wedge 1_{N_\alpha})^*,$$

where $f: W' \longrightarrow W$.

(iii) *φ_W is compatible with the suspension in the sense that*

$$(1_W\wedge T')^*\sigma\varphi_W = \varphi_{SW}(1_W\wedge T'')^*\sigma,$$

where $T'=T(S^1, S^{r+k}C)$ and $T''=T(S^1, N_\alpha)$.

(iv) *The relation*

$$\mu(y\otimes\varphi_W(x)) = \varphi_{Y\wedge W}\mu(y\otimes x)$$

holds, where $x\in\tilde{h}^n(W\wedge N_\alpha)$ and $y\in\tilde{h}^m(Y)$.

(v) *If φ_0 satisfies* (3.6)$'$, *then the relation*

$$(1_W\wedge S^{r+k}i)^*\varphi_W = (1_W\wedge i_1)^*$$

holds for the inclusions $i: S^r \longrightarrow C$ and $i_1: S^{2r} \longrightarrow N_\alpha$.

(vi) *Assume that μ is commutative. Then the relations hold ;*

(a) $\mu(z\otimes\varphi_0) = T'^*\mu(\varphi_0\otimes z),$

where $z\in\tilde{h}^i(Z)$ and $T'=T(Z, N_\alpha)$.

(b) $(1_W\wedge T'')^*\mu(\varphi_W(x)\otimes z) = \varphi_{W\wedge Z}(1_W\wedge T')^*\mu(x\otimes z),$

where $x\in h^n(W\wedge N_\alpha)$, $z\in h^i(Z)$, $T'=T(Z, N_\alpha)$ and $T''=T(Z, S^{r+k}C)$.

Proof. By a similar calcuation to Lemma 4.3, 4.4, 4.5 and Lemma 4.6 of [2], we have the results.

Making use of the homomorphism φ_W defined by (3.7) and the element β of $\{N_\alpha, C\wedge C\}$, we fefine a map

$$\mu_\alpha : \tilde{h}^i(X ; \alpha) \otimes \tilde{h}^j(Y ; \alpha) \longrightarrow \tilde{h}^{i+j}(X\wedge Y ; \alpha) \tag{3.8}$$

as the composition

$$\begin{aligned}
&\mu_\alpha = (-1)^{i(r+k)}\sigma^{-(r+k)}\varphi_{X\wedge Y}(1_{X\wedge Y}\wedge\beta)^*(1_X\wedge T'\wedge 1_C)^*\mu : \\
&\tilde{h}^i(X ; \alpha)\otimes\tilde{h}^j(Y ; \alpha) = \tilde{h}^{i+r+k}(X\wedge C)\otimes\tilde{h}^{j+r+k}(Y\wedge C) \\
&\quad\longrightarrow \tilde{h}^{i+j+2r+2k}(X\wedge C\wedge Y\wedge C) \\
&\quad\longrightarrow \tilde{h}^{i+j+2r+2k}(X\wedge Y\wedge C\wedge C) \\
&\quad\longrightarrow \tilde{h}^{i+j+2r+2k}(X\wedge Y\wedge N_\alpha) \\
&\quad\longrightarrow \tilde{h}^{i+j+2r+2k}(X\wedge Y\wedge S^{r+k}C) \\
&\quad\longrightarrow \tilde{h}^{i+j+r+k}(X\wedge Y\wedge C) = \tilde{h}^{i+j}(X\wedge Y ; \alpha),
\end{aligned}$$

where $T' = T(Y, C)$.

μ_α is defined only if $(\alpha'\pi)^{**} = 0$.

The definition of μ_α depends on the choices of φ_0 and β. However we fix during the subsequent proofs of properties of an admissible multiplication.

Clearly μ_α is linear and natural with respect to both variables.

Proposition 3.6. *If an element $\beta\in\{N_\alpha, C\wedge C\}$ satisfies*

$$(1_C\wedge\pi)T\beta = \pi_0 = (-1)^{r+k}(1_C\wedge\pi)\beta, \tag{3.9}$$

where $T=T(C, C)$, $\pi_0 : N_\alpha \longrightarrow S^{r+k}C$ is the map collapsing S^{2r}, then the map μ_α of (3.8) *is a multiplication satisfying* (A_1).

Proof. (See Theorem 4.7 of [2]) From (i) of Lemma 3.5 and (3.9) we can see (A_1) directly. From (ii) and (iii) of Lemma 3.5 it follows that the map μ_α of (3.8) is compatible with the suspension isomorphism σ_α.

Proposition 3.8. *If μ is a commutative multiplication, then for any $\beta\in\{N_\alpha, C\wedge C\}$ the map μ_α of* (3.8) *satisfies* (A_3).

Proof. (See Theorem 4.10 of [2]) It follows from (iv) and (vi) of Lemma 3.5 that μ_α satisfy (A_3).

Proposition 3.8. *Let $\alpha'^{**} = 0$ in $\tilde{h}^*$. Assume that $\beta\in\{N_\alpha, C\wedge C\}$ satisfies*

$$T(1_C\wedge i) + (-1)^{r+k}(1_C\wedge i) = (-1)^{k(r+k)}\beta i_1(S^r\pi) + (-1)^{r(r+k)}(i\wedge i)(S^r\chi) \tag{3.10}$$

for some $\chi\in\{C, S^r\}$, where $T=T(C, C)$. Then the map μ_α of (3.8) *satisfies* (A_2) *with associated cohomology operation*

$$\chi_\alpha=(-1)^{i(r+k)}\chi^{**}\sigma^r : \tilde{h}^i(\ ) \longrightarrow \tilde{h}^{i-k}(\ ; \alpha).$$

Proof. (See Theorem 4.9 of [2]) It follows from (v) of Lemma 3.5 that satisfy (A_2).

As a consequence of Proposition 2.11, 2.12, 3.6, 3.7 and Proposition 3.8 we have

Theorem 3.9. *Let μ be a commutative, associative multiplication in a reduced generalized cohomology theory $\tilde{h}^*$. Assume that $\alpha \in \pi_{r+k-1}(S_r)$ satisfies* (2.21) *and the order of α is two if k is an even integer. If $\alpha'^{**}=0$ in $\tilde{h}^*$, then the admissible multiplication μ_α exist in $\tilde{h}^*(\ ;\alpha)$.*

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