J. FAC. SCI., SHINSHU UNIVERSITY Vol. 10, No. 2 Dec., 1975

Remarks on Inverse Borel Transformation

By AKIRA ASADA

Department of Mathematics, Faculty of Science,
Shinshu University
(Received October 30, 1975)

The main purpose of this note is to show the following theorem.

Theorem. *If f is an algebraic function on* $\mathbf{C}^n$ *(may be many-valued), then the Riemann surface of* $\mathscr{B}^{-1}[f]$ *(in the sence of* [2]*) is a (branched) covering of* $\mathbf{C}^n$.

Some calculations of $\mathscr{B}^{-1}[f]$ and singular solutions of some equations which are closely related to Mittag-Leffler functions, are also given.

1. To show the above theorem, we first use

Lemma 1 (*cf.* [3], [7]). *If $f(z)$ is a finite exponential type function, then*

$$(1)_0 \qquad \mathscr{B}^{-1}[f](z) = \int_0^\infty \cdots \int_0^\infty e^{-(t_1+\cdots+t_n)} f(z_1t_1, \cdots, z_nt_n)dt_1\cdots dt_n.$$

Proof. By assumption, for suitable $c_1>0, \cdots, c_n>0$, $\mathscr{B}^{-1}[f](z)$ exists if $|z_i|<c_i$, $i=1, \cdots$, n. Then, we get for the above defined $\mathscr{B}^{-1}[f]$,

$$\mathscr{B}^{-1}[f](z) = \sum_{k_1\geqq 0, \cdots, k_n\geqq 0} k_1!\cdots k_n! a_{k_1\cdots k_n} z_1^{k_1}\cdots z_n^{k_n},$$

if $f(z)=\sum_{k_1\geqq 0, \cdots, k_n\geqq 0} a_{k_1\cdots k_n} z_1^{k_1}\cdots z_n^{k_n}$. This shows the lemma.

But, it is known that the right hand side of (1) may exist although $f(z)$ is not a finite exponential type entire function. For examples, we know ([4], [8])

$$(2) \qquad \int_0^\infty e^{-t}\log(zt)dt = \log z - \gamma,$$

$$(3) \qquad \int_0^\infty \frac{e^{-t}}{(1-\alpha zt)}dt = \frac{1}{\alpha z}e^{(-1/\alpha z)}\mathrm{Ei}\left(\frac{1}{\alpha z}\right),$$

where Ei(z) is given by

$$\mathrm{Ei}(z) = -\int_{-z}^\infty e^{-t}t^{-1}dt = -(\log z + \gamma + \sum_{m=1}^\infty \frac{z^m}{m\cdot m!}).$$

(2) has been used by Volterra as the base of his theory of logarithmic function in the convolution algebra. On the other hand, (3) is essentially known by Euler

in his caluclation of $\sum(-)^m m!$.

We note that (2) is compatible with our definition of $\mathscr{B}[\log z]$ ([1]). Moreover, under our definiton of extended Borel transformation, we get

$$\mathscr{B}\left[\frac{1}{\alpha z}e^{(-1/\alpha z)}\mathrm{Ei}\left(\frac{1}{\alpha z}\right)\right]=\mathscr{B}\left[\frac{-1}{\alpha z}e^{(-1/\alpha z)}\log z\right]$$

$$=-\sum_{m=0}^{\infty}\frac{1}{(\alpha z)^{n+1}}=\frac{1}{1-\alpha z},\quad |\alpha z|>1.$$

Definition. For $\theta=(\theta_1,\cdots,\theta_n)\varepsilon S^1\times\cdots\times S^1$, $S^1=\{z|\ |z|=1\}\subset\mathbf{C}$, we set

$$\mathbf{R}^{n,+}{}_\theta=\{t_1\theta_1,\cdots,t_n\theta_n)|\,t_1\geqq 0,\cdots,t_n\geqq 0\}\subset\mathbf{C}^n.$$

Then, if $|f(\theta t)|=O(e^{c||t||})$, $\theta t=\theta_1 t_1,\cdots,\theta_n t_n)$, for some $c>0$ on $\mathbf{R}^{n,+}{}_\theta$ for $||t||\to\infty$, we define $\mathscr{B}^{-1}[f]$ by

$$\mathscr{B}^{-1}[f](z)=\int_0^\infty\cdots\int_0^\infty e^{-(t_1+\cdots+t_n)}f(z_1t_1,\cdots,z_nt_n)dt_1\cdots dt_n. \tag{1}$$

By assumption, $\mathscr{B}^{-1}[f](z)$ exists if $z=\rho\theta$, $0\leqq\rho_i<\rho_{0,i}$, $i=1,\cdots,n$ for some $\rho_{0,i}>0$, $i=1,\cdots,n$. Moreover, if $|f(\theta t)|=O(e^{c}|^{t}||)$ on $\mathbf{R}^{n,+}{}_\theta$ if $\theta\varepsilon D$, a (non empty) open set of $S^1\times\cdots\times S^1$, then $\mathscr{B}^{-1}[f]$ is holomorphic on $U(\{0\})\cap\mathbf{R}^{n,+}{}_D$, where $U(\{0\})$ is a neighborhood of the origin of $\mathbf{C}^n$ and $\mathbf{R}^{n,+}{}_D=\cup_{\theta\varepsilon D}\mathbf{R}^{n,+}{}_\theta$.

By definition, $\mathscr{B}^{-1}$ is linear and $\mathscr{B}^{-1}[f\otimes g]=\mathscr{B}^{-1}[f]\otimes\mathscr{B}^{-1}[g]$. Moreover, since

$$\int_0^\infty e^{-t}\frac{df(w)}{dw}\Big|_{w=zt}dt=\frac{1}{z}\int_0^\infty e^{-t}\frac{df}{dt}(zt)dt=\frac{f(0)}{z}+\frac{1}{z}\int_0^\infty e^{-t}f(zt)dt,$$

we get

$$\mathscr{B}^{-1}\left[\frac{df}{dz_i}(z)\right]=\frac{1}{z_i}\mathscr{B}^{-1}[f|_{z_i=0}]+\frac{1}{z_i}\mathscr{B}^{-1}[f(z)], \tag{4}$$

where the first term in the right hand side means the inverse Borel transformation for $(z_1,\cdots,z_{i-1},\ z_{i+1},\cdots,z_n)$.

By (4), if $(\partial^{i_1+\cdots+i_n}/\partial z_1^{i_1}\cdots\partial z_n^{i_n})F(z)=f(z)$ and $\mathscr{B}^{-1}[f]$ is defined by (1), then we may define $\mathscr{B}^{-1}[f]$ by

$$\mathscr{B}^{-1}[f](z)=\frac{1}{z_1^{i_1}\cdots z_n^{i_n}}\mathscr{B}^{-1}[F](z). \tag{5}$$

Note. Since the above $F(z)$ is not unique for $f(z)$, $\mathscr{B}^{-1}[f]$, defined by (5), is not unique. But since $\mathscr{B}[\mathscr{B}^{-1}[F]]=F$, we have

$$\mathscr{B}[\mathscr{B}^{-1}[f]] = f,$$

for $\mathscr{B}^{-1}[f]$ defined by (5). Here Borel transformation is taken by extended sence.

Example. Since we know $(d^{k-1}/dz^{k-1})(1/(1-\alpha z)) = (-\alpha)^{k-1}(k-1)!/(1-\alpha z)^k$, we get

$$(6)_0 \qquad \mathscr{B}^{-1}\left[\frac{1}{(1-\alpha z)^k}\right] = \frac{(-1)^{k-1}}{(k-1)!\,(\alpha z)^k} e^{-(1/\alpha z)}\,\mathrm{Ei}\left(\frac{1}{\alpha z}\right).$$

We note that we also get

$$(6)_{0'} \qquad \mathscr{B}^{-1}\left[\frac{1}{(1-\alpha z)^k}\right] = \frac{(-1)^k}{(k-1)!(z)^k} e^{-(1/\alpha z)} \log z.$$

2. By $(6)_0$, if $f(z)$ is a rational function, then we have

$$(6) \qquad \mathscr{B}^{-1}[f(z)]$$
$$= P(z) + \sum_{i=1}^{m} \sum_{k_j=1}^{s_j} \frac{c_{j,k_j}}{(\alpha_j z)^{k_j}} e^{-(1/\alpha_j z)}\,\mathrm{Ei}\left(\frac{1}{\alpha_j z}\right) + \sum_{k_0=1}^{s_0} \frac{c_{k_0}}{z^{k_0}} \log z,$$

Where $P(z)$ is a polynomial and the poles of $f(z)$ are on $\{0, 1/\alpha_1, \cdots, 1/\alpha_m\}$ with the multiplicities $\{s_0, s_1, \cdots, s_m\}$. Therefore, the Riemann surface of $\mathscr{B}^{-1}[f(z)]$ covers $\mathbf{C}^1$ and its singularitiys (and branch point) is carried only at the origin. In general, if $f(z)$ is meromorphic on $\mathbf{C}^1$ such that

$$f(z) = g(z) + \sum \frac{c_{k_0}}{z^{k_0}} + \sum \sum \frac{a_{ij}}{(1-\alpha_j z)^{k_j}},$$

where $g(z)$ is an entire function of order 0, then, since

$$\mathscr{B}^{-1}[f(z)] = \mathscr{B}^{-1}[g(z)] + \sum \frac{(-1)^{k_0-1}}{(k_0-1)!} \frac{c_{k_0}}{z^{k_0}} \log z +$$
$$+ \sum \sum \frac{(-1)^{k_j-1}}{(k_j-1)!} \frac{a_{ij}}{(\alpha_j z_j)^{k_j}} e^{-(1/\alpha_j z)}\,\mathrm{Ei}\left(\frac{1}{\alpha_j z}\right),$$

the Riemann surface of $\mathscr{B}^{-1}[f(z)]$ covers $\mathbf{C}^1$ and has singularity (and branch point) only at the origin. Because by assumption, there exists a constant M such that $M \geq |1/\alpha_j|$ for all α_j, and hence $|e^{-(1/\alpha_j z)}| \leq e^{M/|z|}$ for all α_j.

Note. By $(6)_{0'}$, we may also set (for example, a rational function $f(z)$)

$$(6)' \qquad \mathscr{B}^{-1}[f(z)] = p(z) + \left(\sum_{j=1}^{m} \sum_{k_j=1}^{s_j} \frac{c_{j,k_j}}{(\alpha_j z)^{k_j}} e^{-(1/\alpha_j z)} + \right.$$
$$\left. + \sum_{k_0=1}^{s_0} \frac{c_{k_0}}{z^{k_0}}\right) \log z,$$

3. In this n^0, we assume $f(z)$ is a (many valued) analytic function on $\mathbf{C}^n$ such that there exists a (many valued) analytic function $F(z)$ on $\mathbf{C}^n$ which satisfies the following conditions (i), (ii), (iii).

(i) $\dfrac{\partial^{i_1+\cdots+i_n}}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}}F(z)=f(z)$, *for some* $(i_1,\cdots,i_n)$.

(ii) *For any fixed* $\theta=(\theta_1,\cdots,\theta_n)$ $S^1\times\cdots\times S^1$, $|\log F(\theta t)|=o(||t||)$, $||t||\to\infty$ *for any branch of* $F(z)$ *on* $\mathbf{R}^{n,+}{}_{\theta}$.

(iii) *Any branch of* $F(\theta t)$ *is integrable on any (relative compact) simply connected subdomain of* $\mathbf{R}^{n,+}{}_{\theta}$, *for any* θ.

Theorem. *If* $f(z)$ *satisfies the above condition, then the Riemann surface of* $\mathscr{B}^{-1}[f]$ *is a (branched) covering of* $\mathbf{C}^n$.

Proof. By (5), we may prove the theorem for $F(z)$.

First we note that the singularities (and branch points) of $F(z)$ is contained in an analytic subvariety Y of $\mathbf{C}^n$. Hence there exists a (real analytic) subvariety Z of $S^1\times\cdots\times S^1$ such that if $\theta\in Z$, then $dim(Y\cap\mathbf{R}^{n,+}{}_{\theta})\geq n-2$, Then, since $Y\cap\mathbf{R}^{n,+}{}_{\theta}$ is an analytic subvariety of $\mathbf{R}^{n,+}{}_{\theta}$, by Łojasiewicz's theorem ([5], [6]), there exists an $(n-1)$ -dimensional subset $\Gamma=\Gamma_\theta$ of $\mathbf{R}^{n,+}{}_{\theta}$ such that

$$\Gamma\supset Y\cap\mathbf{R}^{n,+}{}_{\theta},\quad \pi_1(\mathbf{R}^{n,+}{}_{\theta}-\Gamma)=\{0\}, \tag{7}$$

if $\theta\in Z$. The image of Γ_θ in $\mathbf{R}^{n,+}{}_{(1,\cdots,1)}$ by the map $t\theta\to t$, is denoted by $\Gamma_{(\theta)}$.

By (7) and (ii), (iii), if $z\in\mathbf{R}^{n,+}{}_{\theta}$, $\theta\in Z$, then we may define $\mathscr{B}^{-1}[F](z)$ by

$$\mathscr{B}^{-1}[F](z)=\int_{\mathbf{R}^{n,+}-\Gamma(\theta)}e^{-t}F(zt)dt. \tag{8}$$

Hence to show the theorem, it is sufficient to show $\mathscr{B}^{-1}(F)$, defined by (8), can be continued on whole $\mathbf{C}^n$.

Since $dim\ Z\leq n-1$, for any closed (not homotopic to 0) path γ_0' of $S^1\times\cdots\times S^1$, there exists a closed path γ_0 of $S^1\times\cdots\times S^1$ such that γ_0 is homotopic to γ_0' and $\gamma_0\cap Z$ is consisted only finite number of points. Hence for any z_0, $z_0\neq 0$, $z_0\in\mathbf{R}^{n,+}{}_{\theta}$, $\theta\in Z$ and z_1, $z_1\neq 0$, $z_1\in\mathbf{R}^{n,+}{}_{\theta'}$, $\theta'\in Z$, and a closed path γ' such that z_0, $z_1\in\gamma'$ (and by the map $\gamma'\ni z\to\theta(z)=(z_1/|z_1|,\cdots,z_n/|z_n|)\in S^1\times\cdots\times S^1$, the image of γ' is not homotopic to 0), there exists a closed path γ such that z_0, $z_1\in\gamma'$, γ is homotopic to γ', the correspondence $z\to\theta(z)$ is 1 to 1 on γ and

$$\gamma\cup(\bigcup_{\theta\in Z}(Y\cap\mathbf{R}^{n,+}{}_{\theta}))=\{z_1\}.$$

We assume $z_1\in\mathbf{R}^{n,+}{}_{\theta_1}$, and set

$$Y\cap\mathbf{R}^{n,+}{}_{\theta_1}=Y_1,\quad \bigcup_{z\in\gamma}\mathbf{R}^{n,+}{}_{\theta(z)}=\mathbf{R}^{n,+}{}_{\gamma}.$$

Since γ is a path, we assume γ is parametrized by $\gamma(s)$, $0 \leqq s \leqq 1$, such that $\gamma(0) = \gamma(1) = z_0$, $\gamma(1/2) = z_1$. We note that by the map $\tau_{a,b} : \gamma(a)t \to \gamma(b)t$, $\mathbf{R}^{n,+}{}_{\theta(\gamma(a))}$ is mapped onto $\mathbf{R}^{n,+}{}_{\theta(\gamma(b))}$ for any a, b.

Since Y_1 is an analytic set and $\mathbf{R}^{n,+}{}_{\gamma} \cong \mathbf{R}^{n,+} \times S^1/(\{0\} \times S^1)$, by Łojasiewicz's theorem, there exists a closed n -dimensional Subset Δ of $\mathbf{R}^{n,+}{}_{\gamma}$ such that

(i) $\Delta = \partial D,\ D \supset Y_1,\ in\ \mathbf{R}^{n,+}{}_{\theta},$

(ii) $dim\, \Delta \cap \mathbf{R}^{n,+}{}_{\theta(z)} \leqq n-1,\ z \in \gamma\ and\ \Delta \cap \mathbf{R}^{n,+}{}_{\theta(\gamma(a))} = \emptyset,\ a \lneqq \frac{1}{4},\ or\ a \gneqq \frac{3}{4}.$

We set $\Delta_1 = \Delta \cap \mathbf{R}^{n,+}{}_{\theta_1}$ and $D_1 = D \cap \mathbf{R}^{n,+}{}_{\theta_1}$. Then by (i), $\Delta_1 = \partial D_1$ and $Y \subset D_1$. Using these notations, we set

$$\mathbf{R}^{n,+}{}_{s}{}' = \mathbf{R}^{n,+}{}_{\theta(\gamma(s))}, \quad 0 \leqq s \leqq \frac{1}{4},$$

$$\mathbf{R}^{n,+}{}_{s}{}' = (\mathbf{R}^{n,+}{}_{\theta(\gamma(s))} - D \cap \mathbf{R}^{n,+}{}_{\theta(\gamma(s))}) \cup \Big(\bigcup_{\frac{1}{4} < a \leqq s} (\Delta \cap \mathbf{R}^{n,+}{}_{\theta(\gamma(a))}) \Big)$$

$$\frac{1}{4} < s \leqq \frac{1}{2},$$

$$\mathbf{R}^{n,+}{}_{s}{}' = (\mathbf{R}^{n,+}{}_{\theta(\gamma(s))} - \tau_{1/2,s}(D_1)) \cup \Big(\bigcup_{\frac{1}{4} < a \leqq \frac{1}{2}} (\Delta \cap \mathbf{R}^{n,+}{}_{\theta(\gamma(a))}) \cup$$

$$\cup \Big(\bigcup_{\frac{1}{2} \leqq a \leqq s} \tau_{1/2,\, a}(\Delta_1) \Big) \Big), \quad s > \frac{1}{2},$$

where, s may be larger than 1 and if $s > 1$, then $\gamma(s)$ means $\gamma(s\ mod.\ 1)$.

We set $\gamma(a) = (\alpha_1, \cdots, \alpha_n)$ and define the map $\tau_a : \mathbf{C}^n \to \mathbf{C}^n$ by

$$\tau_a(z_1, \cdots, z_n) = (\alpha_1^{-1} z_1, \cdots, \alpha_n^{-1} z_n).$$

To use this τ_a, we set

$$\mathbf{R}^{n,+}{}_{s} = \tau_s(\mathbf{R}^{n,+}{}_{s}{}').$$

Then to set

$$\mathscr{B}^{-1}[F](z)_s = \int_{\mathbf{R}^{n,+}{}_{s} - (\cap_a \tau_a(\Gamma_{\theta(\gamma)(a)}) \cap \mathbf{R}^{n,+}{}_{s})} e^{-t} F(zt)dt, \quad z = \gamma(s),$$

$\mathscr{B}^{-1}[F](z)_s$ is defined and continuous if $z \in \gamma$ and we have

$$\mathscr{B}^{-1}[F](z)_s = \mathscr{B}^{-1}[F](z), \quad 0 \leqq s < \frac{1}{2},$$

$$\mathscr{B}^{-1}[F](z)_s = \mathscr{B}^{-1}[F](z), \quad \frac{3}{2} < s < 2,$$

Where $\mathscr{B}^{-1}[F]$ in the right hand side is defined by (8). Because if $1/4 < s < 1/2$, then

$$\mathscr{B}^{-1}[F](z)_s - \mathscr{B}^{-1}[F](z) = \int_{\partial D_s} e^{-t}F(zt)dt = 0,$$

where D_s is given by $\tau_s(\cup_{1/4<a\leqq s} D \cap \mathbf{R}^{n,+}{}_{\theta(\gamma(a))})$ and if $3/2 < s < 2$, then

$$\mathscr{B}^{-1}[F](z)_s - \mathscr{B}^{-1}[F](z)_{s-1} = \int_{\tau_s(\Delta)} e^{-t}F(zt)dt,$$

if F is single valued (along γ) and the integral of $\mathscr{B}^{-1}[F](z)_s$ is taken on another branch of $F(z)$ if F is branched at Y_1. Therefore we have the theorem.

4. By the above theorem, to show the first stated theorem, it is sufficient to show the following lemma 2.

Lemma 2. *If $f(z)$ is an algebraic function on $\mathbf{C}^n$, then $f(z)$ satisfies the assumptions stated in the beginning of n^0 3.*

Proof. If $f(z)$ is integral over $\mathbf{C}[z_1, \cdots, z_n]$, then we may take $F(z) = f(z)$, If $f(z)$ is not integral over $\mathbf{C}[z_1, \cdots, z_n]$, then there exists a polynomial $P(z)$ and an algebraic function $g(z)$ which is integral over $\mathbf{C}[z_1, \cdots, z_n]$ such that

$$f(z) = \frac{g(z)}{P(z)}.$$

Moreover, by suitable change of coordinates, we my assume $P(z)$ is a Kowalevskaja type polynomial, that is $P(z) = z_1{}^m + P_1(z_2, \cdots, z_n)z_1{}^{m-1} + \cdots + P_m(z_2, \cdots, z_n)$, *deg.* $P_k \leqq m-k$. Hence we may set

$$f(z) = g_0(z) + \sum_{k_0=1}^{s_0} \frac{g_{k_0}(z)}{z_1{}^{k_0}} + \sum_{i=1}^{s} \sum_{k_j=1}^{s_j} \frac{g_{k_j}(z)}{(z_1 - \alpha_j(z_2, \cdots, z_n))^{k_j}},$$
$$k_0 \leqq m, \quad < k_j \leqq m,$$

where $g_0(z)$, $g_{k_0}(z)$ and $g_{k_j}(z)$ are integral over $\mathbf{C}[z_1, \cdots, z_n]$ and each $\alpha_j(z_2, \cdots, z_n)$ is integral over $\mathbf{C}[z_2, \cdots, z_n]$. Therefore, the m -th indefinite integral $F(z)$ of $f(z)$ with respect to z_1 is either $|F(z)|$ is bounded on any compact subset of $\mathbf{C}^n$ or to have only logarithmic singularties (with respect to z_1). Hence we have the lemma.

From the proof of theorem in n^o 3, we only conclude that to set

$$S = \{(z_1, \cdots, z_n) | z_1 \cdots z_n = 0\} \cup \left(\bigcup_{\theta \in Z} (\mathbf{R}^{n,+}{}_\theta)\right),$$

the singularities (and branch points) of $\mathscr{B}^{-1}[f]$ is contained in S. In fact, the singularities of $\mathscr{B}^{-1}[f]$ may be different from the singularities of $f(z)$. For example, we have

$$\mathscr{B}^{-1}\left[\frac{1}{z_1+\alpha_2 z_2+\cdots+\alpha_n z_n}\right]=\mathscr{B}_{(z_2,\cdots,z_n)}[e^{-(\alpha_2 z_2+\cdots+\alpha_n z_n)/z_1}]\frac{\log z_1}{z_1}$$

$$=\frac{\log z_1}{z_1{}^n}\ \frac{1}{(z_1+\alpha_2 z_2)\cdots(z_n+\alpha_n z_n)}.$$

But if $P(z)=z_1{}^m+Q(z)$ is irreducible and $deg.\ Q(z)\leq m-1$, then each root $\alpha_1(z_2, \cdots, z_n)$ of $P(z)$ (as a polynomial of z_1) satisfies

$$|\alpha_i(z)|=0(||z||),\quad ||z||\rightarrow\infty.$$

Hence $|\mathscr{B}^{-1}[1/P(z)]|$ is bounded on any compact subset of $\mathbf{C}^n-\{z|z_1=0\}$. We note that if $n=1$, then this assumption (about $P(z)$) is always satisfied.

On the other hand, if $P(z)$ is a Kowalevskaja type polynomial with respect to z_1 then to set the principal part of $P(z)$ by $P_0(z)$ and its roots (as a polynomial of z_1) by $\sigma_1(z_2, \cdots, z_n), \cdots, \sigma_k(z_2, \cdots, z_n)$, $\mathscr{B}^{-1}[1/P(z)]$ has singularities in the subvariety S_0 of $\mathbf{C}^n$ given by

$$S_0=\bigcup_{i=1}^{k}S_{0,i}, \tag{9}$$

$$S_{0,i}=\left\{(z_1, \cdots, z_n)|\limsup_{||t||\rightarrow\infty} Re.\left(\frac{\sigma_i(z_2 t_2, \cdots, z_n t_n)}{z_1}\right)=1\right\}.$$

Note. The above $\mathscr{B}^{-1}[1/P(z)]$ corresponds to $\mathscr{B}^{-1}[1/P(z)]$, where γ is the path corresponds to the Reinhaldt domain of a Laurent expansion of $1/P(z)$ which containes the plane defined by $z_1=0$ (cf. [2]).

5. Inverse Borel transformation may be used for some explicit calculations. As the simplest case, we consider

$$\frac{\partial u}{\partial x}=P\left(x, \frac{\partial}{\partial y}\right)u,\ u(0, y)=f(y). \tag{10}$$

We set $v(x, \eta)=\mathscr{B}_y[u(x, y)]$. Then we get

$$\frac{\partial v}{\partial x}=P(x,\eta^{-1})v,\ v(0, \eta)=\mathscr{B}[f]. \tag{10$'$}$$

Therefore to set

$$v(x, \eta)=C(\eta)e^{\int_0^X P(\xi,\eta^{-1})d\xi}, C(\eta)=\mathscr{B}[f],$$

$\mathscr{B}^{-1}{}_y[v(x, y)]$ is a solution of (10) if it exists. Although $\mathscr{B}^{-1}$ is not unique, if we use (1) as the definition of $\mathscr{B}^{-1}$, then to set the principal part of P by P_0, on the domain D given by

$$D=\left\{(x, y)|\ Re.\left(\int_0^X P_0(\xi, y)d\xi\right)\leqq 0\right\},$$

the inverse Borel transformation $\mathscr{B}^{-1}{}_y[v(x, y)]$ is defined and it is a solution of (10). In this case, the analytic elementary solution $K(\zeta, x, y)$ of (10) is given by

$$K(\zeta, x, y)=\frac{1}{\zeta}\int_0^\infty \exp\frac{yt}{\zeta}-t-\int_0^X P(\xi, (yt)^{-1})d\xi)dt, \tag{11}$$

$$|\frac{y}{\zeta}|<1,\quad (x, y)\in D,$$

where $(yt)^{-1}$ means $((y_1t_1^{-1}, \cdots, (y_nt_n)^{-1})$. Because we know

$$\int_{\mathbf{R}_{n,+}} e^{-t}f(zt)dt=\frac{1}{z_1\cdots z_n}\int_{\mathbf{R}_{n,+}{}_{\theta(z)}} e^{-(s/z)}f(s)ds, \tag{12}$$

$$\frac{s}{z}=\frac{s_1}{z_1}+\cdots+\frac{s_n}{z_n}.$$

For general operators, we may get analytic elementary solutions by this manner with to use many valued functions.

Example, If $P(x, \partial/\partial y)=\partial^m/\partial y^m$, then $D=\{(x, y)|Re.\ (xy^m)\leqq 0\}$ in $\mathbf{C}^2$. Especially, D containes real half space $\{(x, y)|x\leqq 0\}$ in $\mathbf{R}^2$ if and only if m is even. But in this case, the above solution may not be usual solution. For example, for the data $u(0, y)=1$, the solution obtained by this manner is $\mathscr{B}^{-1}{}_y[\exp(y^{-m}x)]\neq 1$. In fact, we have

$$\mathscr{B}^{-1}{}_y[e^{y^{-m}x}]=\mathscr{B}^{-1}{}_y[1+\sum_{k=1}^\infty\frac{y^{-mk}}{k!}x^k]$$

$$=1+\left(\sum_{k=1}^\infty\frac{(-1)^{mk-1}y^{-mk}}{(mk-1)!\,k!}x^k\right)\log y.$$

We note that $\sum_{k=1}^\infty(-1)^{mk-1}y^{-mk}x^k/(mk-1)!k!$ converges on $\{(x, y)|y\neq 0\}$ in $\mathbf{C}^2$ and therefore the Riemann surface of $\mathscr{B}^{-1}{}_y[\exp(y^{-m}x)]$ covers $\mathbf{C}^2$. Moreover, to set $u(x, y)=\mathscr{B}^{-1}{}_y[\exp(y^{-m}x)]$, we get

$$\mathscr{B}^{-1}{}_x[u(x, y)]=1+\left(\sum_{k=1}^\infty\frac{(-1)^{mk-1}y^{-mk}}{(mk-1)!}x^k\right)\log y$$

$$=1+\left(y\frac{\partial}{\partial y}\mathrm{E_m}\left(-\frac{x}{y^m}\right)\right)\log y,$$

where $E_m(x)$ is the m-th Mittag-Leffler function $\sum_{k=0}^{\infty} x^k/(mk)!$ ([8]). Hence we have

$$(13) \qquad u(x, y) = \mathscr{B}_{\xi}\left[1 + \left(y\frac{\partial}{\partial y}E_m\left(-\frac{\xi}{ym}\right)\right)\log y\right],$$

and we also have, to set

$$(13)' \qquad u_0(x, y) = \mathscr{B}_{\xi}\left[y\frac{\partial}{\partial y}E_m\left(-\frac{\xi}{ym}\right)\right]\log y,$$

$u_0(x, y)$ is a nul-solution of $\partial u/\partial x = \partial^m u/\partial^m y$.

Note. By (12), if f(t) is a real variable function with support $K(\subset \mathbf{R}^n)$ is compact, then

$$(14) \qquad \mathscr{B}^{-1}[f](z) = \sum_{k_1,\cdots,k_n} \frac{1}{k_1!\cdots k_n!\, z_1^{k_1+1}\cdots z_n^{k_n+1}} \int_K t_1^{k_1}\cdots t_n^{k_n} f(t)dt.$$

Therefore, since $|{}_k t_1^{k_1}\cdots t_n^{k_n} f(t)dt| \leq ML^{k_1+\cdots+k_n}$, for suitable M, $L>0$, $\mathscr{B}^{-1}[f]$ is holomorphic on $\mathbf{C}^*\times\cdots\times\mathbf{C}^*$, $\mathbf{C}^* = \mathbf{C} - \{0\}$.

By (14), as a function, we have

$$\mathscr{B}[\mathscr{B}^{-1}[f]] = 0,$$

in this case. On the other hand, since we know

$$\mathscr{B}[fg] = \frac{\partial^n}{\partial z_1\cdots\partial z_n}\int_0^z \mathscr{B}[f](z-t)\mathscr{B}[g](t)dt,$$

we get

$$\mathscr{B}\left[\frac{1}{z_{i_1}\cdots z_{i_k}}\right] = \delta_{i_1}\otimes\cdots\delta_{i_k},$$

where δ_i means δ in z_i-plane. Hence to set $\delta = \delta_1\otimes\cdots\otimes\delta_n$, we get

$$(14)' \qquad \mathscr{B}[\mathscr{B}^{-1}[f]] = \sum_{k_1,\cdots,k_n} \frac{1}{k_1!\cdots k_n!}\int_K t_1^{k_1}\cdots t_n^{k_n} f(t)dt \frac{\partial^{i_1+\cdots+i_n}}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}}(\delta),$$

as an operator. We note that, if we use (14)′ as the definition of $\mathscr{B}[\mathscr{B}^{-1}[f]]$, then we get

$$(15) \qquad \mathscr{B}[\mathscr{B}^{-1}[f]](g) = \int_{\mathbf{R}^n} f(t)g(t)dt,$$

if g is an entire function on $\mathbf{C}^n$. In fact, we have by (14)′

$$\begin{aligned}
&\mathscr{B}[\mathscr{B}^{-1}[f]](g)\\
&=\sum_{k_1,\cdots,k_n}\frac{1}{k_1!\cdots k_n!}\int_K t_1^{k_1}\cdots t_n^{k_n}f(t)dt\,\frac{\partial^{i_1+\cdots+i_n}g(z)}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}}\Big|_{z=0}\\
&=\int_K f(t)\Big(\sum_{i,\cdots,i_n}\frac{1}{k_1!\cdots k_n!}\frac{\partial^{i_1+\cdots+i_n}g(z)}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}}\Big|_{z=0}t_1^{k_1}\cdots t_n^{k_n}\Big)dt\\
&=\int_K f(t)g(t)dt\\
&=\int_{\mathbf{R}^n} f(t)g(t)dt.
\end{aligned}$$

Since we regard f as an operator $\int_{\mathbf{R}^n} fgdt$, (15) shows

$$\mathscr{B}[\mathscr{B}^{-1}[f]]=f, \tag{16}$$

if we regard both f and $\mathscr{B}[\mathscr{B}^{-1}[f]]$ to be operators.

References

[1] ASADA, A. : Some extensions of Borel transformation, J. Fac. Sci. Shinshn Univ., 9 (1974), 71–89.

[2] ASADA, A. : Extended Borel transformations and elementary solutions on polydisks of linear partial differential operators with holomorphic coefficients, J. Fac. Sci. Shinshu Univ., 10 (1975), 1–26.

[3] DOETSCH, G. : *Theorie und Anwendung der Laplace -Transformation*, Berlin, 1937.

[4] HARDY, G. H. : *Divergent Series*, Oxford, 1948.

[5] HIRONAKA. H. : Triangulations of algebraic sets, Proc. of Symposia in pure Math., 29 (Algebraic geometry, Arcata 1974), 165–185.

[6] ŁOJASIEWICZ, S. : Triangulation of semi-analytic sets, Ann. Scn. Norm. di Pisa, 18 (1964), 449–474.

[7] MITTAG-LEFFLER, G. : Sur la représentation analytique d'une branche uniforme d'une fonction monogène, V, Acta math. 29 (1904), 101–181.

[8] VOLTERRA, V. : Teoria delle potenze dei logaritmi e delle funzioni di composizione, Mem. Acc. Lincei, ser. 5a, 11 (1916), 167–250.

[8]′ VOLTERRA, V. : Functions of composition, The Rice Inst. Pamphlet, 7 (1920), 181–251.