J. FAC. SCI., SHINSHU UNIVERSITY Vol. 10, No. 2 Dec., 1975

On a non compact simple Lie group $F_{4,1}$ of type F_4

By ICHIRO YOKOTA

Department of Mathematics, Faculty of Science,
Shinshu University
(Received October 8, 1975)

In this paper, we investigate some properties of a non compact simple Lie group $F_{4,1}$ which is the invariance group of

$$\langle X, Y\rangle = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + 2(x_1, y_1) - 2(x_2, y_2) - 2(x_3, y_3)$$

To do this we consider a Freudenthal's perspective mapping $\phi : \{A \in \Pi \mid \langle A, A\rangle \neq 0\} \to F_{4,1}$ [2] $\left(\phi(A) = \dfrac{2}{\langle A, A\rangle} \Pi^{-1}_{A,\hat{A}}\right)$ such that $A, X ; \hat{A} \times (A \times X), \phi(A)X$ are harmonic in the octavian projective plane Π. Throughout this paper, we refer to the definitions and notations in Freudenthal [1]. This group $F_{4,1} = F_{4(-20)}$ is also considered in [4] using a hyperbolic polarity in the plane Π.

1. Preliminaries [1], [5], [6]

Let $\mathfrak{C}$ be the alternative field of octaves over real numbers $\boldsymbol{R}$ and $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$ be the Jordan algebra consisting of all 3×3 Hermitian matrices X with components in $\mathfrak{C}$

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \boldsymbol{R}, \ x_i \in \mathfrak{C}$$

with respect to the composition

$$X \circ Y = \frac{1}{2}(XY + YX)$$

In $\mathfrak{J}$ we define another multiplication

$$X \times Y = \frac{1}{2}(2X\circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(X\circ Y))E)$$

(where E is the 3×3 unit matrix) and a positive definite symmetric inner product $(\ , \ )$ and a symmetric trilinear product $(\ , \ , \ )$ by

$$(X, Y) = \mathrm{tr}(X\circ Y)$$
$$(X, Y, Z) = (X \times Y, Z) = (X, Y \times Z)$$

Especially

$$(X,\ X,\ X)=3\det X$$
$$=3(\xi_1\xi_2\xi_3+2\mathrm{Re}(x_1x_2x_3)-\xi_1|x_1|^2-\xi_2|x_2|^2-\xi_3|x_3|^2)$$

The group $E_6=E_{6,0,*}$ consisting of all linear isomorphisms of $\mathfrak{J}$ which preserve $(X,\ Y,\ Z)$:

$$E_6=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})|(\alpha X,\ \alpha Y,\ \alpha Z)=(X,\ Y,\ Z)\}$$
$$=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})|\det(\alpha X)=\det X\}$$

is a simply connected simple Lie group and a non compact real form of type E_6. And the automorphism group F_4 of $\mathfrak{J}$:

$$F_4=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J},\ \mathfrak{J})|\alpha X\circ\alpha Y=\alpha(X\circ Y)\}$$
$$=\{\alpha\in E_6\mid \alpha E=E\}$$
$$=\{\alpha\in E_6\mid (\alpha X,\ \alpha Y)=(X,\ Y)\}$$

is a simply connected compact simple Lie group of type F_4 and has a subgroup $Spin(9)$ which a universal covering group of rotation group $SO(9)$:

$$Spin\,(9)=\{\alpha\in F_4\mid \alpha E_1=E_1\},\qquad E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Any element α of E_6 is uniquely represented by the form

$$\alpha=\beta\exp\widetilde{A},\qquad \beta\in F_4,\quad A\in\mathfrak{J},\qquad \mathrm{tr}(A)=0$$

where $\widetilde{A}$ is the endomorphism of $\mathfrak{J}$ which is defined by $\widetilde{A}X=A\circ X$. From this we see that E_6 is homeomorphic to the product space of F_4 and euclidean space $\boldsymbol{R}^{26}$:

$$E_6\simeq F_4\times \boldsymbol{R}^{26}$$

The octavian projective plane Π is defined by

$$\Pi=\{A\in\mathfrak{J}\,|\,A^2=A,\quad \mathrm{tr}(A)=1\}$$
$$=\{A\in\mathfrak{J}\,|\,A\times A=0,\quad \mathrm{tr}(A)=1\}$$
$$=\left\{A\in\mathfrak{J}\ \middle|\ \begin{array}{l}\alpha_2\alpha_3=|a_1|^2,\ \ \alpha_3\alpha_1=|a_2|^2,\ \ \alpha_1\alpha_2=|a_3|^2\\ a_2a_3=\alpha_1\bar{a}_1,\ \ a_3a_1=\alpha_2\bar{a}_2,\ \ a_1a_2=\alpha_3\bar{a}_3\\ \alpha_1+\alpha_2+\alpha_3=1\end{array}\right\}$$

This space Π can be embedded into the group F_4 by a mapping $\varphi:\Pi\to F_4$ [5]

$$\varphi(A)X=16A\times(A\times X)+4A\circ X-3X$$
$$=X-4A\circ X+4(A,\ X)A \qquad \text{(cf. §2 formula (I))}$$

and φ has the following properties.

Proposition 1. *For* $A \in \Pi$ *we have*

(1) $\varphi(A)A = A$

(2) $\varphi(A)^2 = 1$ (1 *is the identity mapping of* $\mathfrak{J}$)

(3) $\alpha\varphi(A)\alpha^{-1} = \varphi(\alpha A), \quad \alpha \in F_4$

(4) $\varphi : \Pi \to F_4$ *is injective.*

The feact that the group F_4 is simple is well known. However using this mapping φ, we shall give another proof.

Theorem 2. *The group* F_4 *is simple and is generated by* $\{\varphi(A) | A \in \Pi\}$.

Proof As is well known the Lie algebra of F_4 is simple, hence the group F_4 is simple as Lie group, that is, F_4 has only discrete normal subgroups. Since F_4 is connected, any discrete normal subgroup of F_4 is central. Now, if α is an element of center of F_4 then $\alpha\varphi(A)\alpha^{-1} = \varphi(A)$, hence $\varphi(\alpha A) = \varphi(A)$. Since φ is injective we have

$$\alpha A = A \qquad \text{for all} \quad A \in \Pi$$

Especially $\alpha E_1 = E_1$. Hence $\alpha \in Spin(9)$ and α is an element of center of $Spin(9)$. The center of $Spin(9)$ consists of two elements $\{1, \gamma\}$ and

$$\gamma A = \begin{pmatrix} \alpha_1 & -a_3 & -\bar{a}_2 \\ -\bar{a}_3 & \alpha_2 & a_1 \\ -a_2 & \bar{a}_1 & \alpha_3 \end{pmatrix} \neq \begin{pmatrix} \alpha_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \bar{a}_1 & \alpha_3 \end{pmatrix} = A \text{ for some } A \in \Pi$$

Hence we have $\alpha = 1$ and prove the simplicity of F_4. The remainder of the theorem is obvious because the subgroup generated by $\{\varphi(A) | A \in \Pi\}$ is a normal subgroup of F_4 by proposition 1 (3).

2. Construction of elements of E_6

For elements $A, B \in \Pi$ such that $(A, B) \neq 0$, we shall construct an element $\psi(A, B)$ of E_6. Define a mapping $\psi : \{(A, B) \in \Pi \times \Pi | (A, B) \neq 0\} \to E_6$ by

$$\psi(A, B)X = \frac{1}{(A, B)}(8B \times (A \times X) + 2(B, X)A - (A, B)X)$$

and also define

$$\psi(rA, sB) = \psi(A, B) \qquad r, s \in \mathbf{R} - \{0\}$$

In order to show that $\psi(A, B)$ belongs to E_6 we need the following formulae (I), (II).

$$\text{(I)} \quad A \times (Y \times (A \times X)) = \frac{1}{4}(A, Y)A \times X, \qquad A \in \Pi,\ X, Y \in \mathfrak{J}$$

$$\text{(II)} \quad X \times (Y \times (X \times X)) = \frac{1}{12}(X, X, X)Y + \frac{1}{4}(X, Y)X \times X, \qquad X, Y \in \mathfrak{J}$$

(III) $A\times(A\times X)=\frac{1}{4}(X-2A\circ X+(A,\ X)A),\qquad A\in\Pi,\ X\in\mathfrak{J}$

To prove these formulae, we may assume that $A=E_1$ in (I), (III) and X is diagonal in (II) respectively by means of the transformation of some element of F_4 [1]. Then these formulae are easily obtained by the straightforward calculations.

Propostion 3. $\phi(A,\ B)^2=1$. *Especially* $\phi(A,\ B)$ *is a linear isomorphism of* $\mathfrak{J}$: $\phi(A,B)\in\mathrm{Iso}_{R}(\mathfrak{J},\ \mathfrak{J})$.

Proof $\phi(A,\ B)^2X=\dfrac{1}{(A,\ B)^2}(8B\times(8B\times(A\times X)+2(B,\ X)A-(A,B)X))$

$$+2(B,\ 8B\times(A\times X)+2(B,\ X)A-(A,\ B)X)A$$
$$-(A,\ B)(8B\times(A\times X)+2(B,\ X)A-(A,\ B)X))$$

using $A\times A=B\times B=0$ and the formula (I)

$$=\frac{1}{(A,\ B)^2}(16(B,A)B\times(A\times X)-8(A,B)B\times(A\times X)+4(B,X)(B,A)A$$
$$-2(A,B)(B,X)A-8(A,B)B\times(A\times X)$$
$$-2(A,B)(B,X)A+(A,B)^2X)$$
$$=X$$

Theorem 4. *For* $A,\ B\in\Pi$ *such that* $(A,\ B)\neq 0$ *we have* $\phi(A,\ B)\in E_6$, *that is,* $\phi(A,\ B)$ *satisfies*

$$\det(\phi(A,\ B)X)=\det X$$

Proof Put $\phi'(A,\ B)X=8B\times(A\times X)+2(B,\ X)A-(A,B)X=P+Q+R$, then we must show

$$(\phi'(A,B)X,\ \phi'(A,B)X,\ \phi'(A,B)X)=(A,B)^3(X,X,X)$$

Now, $(\phi'(A,B)X,\ \phi'(A,B)X,\ \phi'(A,B)X)=(P,P,P)+(Q,Q,Q)+(R,R,R)+3(P,P,Q)$

$$+3(P,P,R)+3(Q,Q,P)+3(Q,Q,R)+3(R,R,P)+3(R,R,Q)+6(P,Q,R)$$

$$(P,P,P)=8^3(B\times(A\times X),\ B\times(A\times X),\ B\times(A\times X))$$
$$=8^3(A\times X,\ B\times((B\times(A\times X))\times(B\times(A\times X)))$$
$$=2.\ 8^2(B,B\times(A\times X))(A\times X,\ B\times(A\times X))\qquad\text{(by (I))}$$
$$=0\qquad\text{(since }B\times B=0)$$

$$(Q,Q,P)=(Q,Q,Q)=(Q,Q,R)=0\qquad\text{(since }A\times A=0)$$

$$(R,R,R)=-(A,B)^3(X,X,X)$$

$$3(P,P,Q)=3.\ 8^2.\ 2(B,X)(B\times(A\times X),B\times(A\times X),A)$$
$$=3.\ 8^2.\ 2(B,X)(A\times X,B\times(A\times(B\times(A\times X)))$$
$$=3.\ 8.\ 4(B,X)(B,A)(A\times X,B\times(A\times X))\qquad\text{(by (I))}$$
$$=3,\ 8.\ 4(B,X)(B,A)(X,A\times(B\times(A\times X)))$$

$$=24(B, X)(A, B)^2(X, A\times X) \qquad \text{(by (I))}$$

$$3(P, P, R)=-12(B, X)(A, B)^2(X, A\times X) \quad \text{as similar to the above}$$

$$\begin{aligned}3(R, R, P)&=3.\,8(A, B)^2(X, X, B\times(A\times X))\\ &=24(A, B)^2(A, X\times(B\times(X\times X)))\\ &=2(A, B)^3(X, X, X)+6(A, B)^2(X, B)(A, X\times X) \qquad \text{(by (II))}\end{aligned}$$

$$3(R, R, Q)=6(B, X)(A, B)^2(X, X, A)$$

$$\begin{aligned}6(P, Q, R)&=-6.\,8.\,2(B, X)(A, B)(B\times(A\times X), A, X)\\ &=-6.\,8.\,2(B, X)(A, B)(A\times(B\times(A\times X)), X)\\ &=-24(B, X)(A, B)^2(A\times X, X) \qquad \text{(by (I))}\end{aligned}$$

Adding these formulae, we have the required result.

For $\alpha\in E_6$, we denote by ${}^t\alpha\in E_6$ the transpose of α relative to (X, Y) : $(\alpha X, Y)=(X, {}^t\alpha Y)$. Then we have

$$\alpha(X\times Y)={}^t\alpha^{-1}X\times{}^t\alpha^{-1}Y, \qquad X, Y\in\mathfrak{J}$$

Proposition 5. (1) $\psi(A, B)A=A$

(2) ${}^t\psi(A, B)=\psi(B, A)$

(3) $\alpha\psi(A, B)\alpha^{-1}=\psi(\alpha A, {}^t\alpha^{-1}B), \qquad \alpha\in E_6$

Proof (1) is easy.

(2) $$(\psi(A, B)X, Y)=\frac{1}{(A, B)^2}(8(A\times X, B\times Y)+2(B, X)(A, Y)-(A, B)(X, Y))$$
$$=(X, \psi(B, A)Y)$$

(3) First, note $\alpha A/\mathrm{tr}(\alpha A)\in\Pi$ because of $\alpha A\times\alpha A={}^t\alpha^{-1}(A\times A)=0$, $\mathrm{tr}(\alpha A/\mathrm{tr}(\alpha A))=1$ and also ${}^t\alpha^{-1}B/\mathrm{tr}({}^t\alpha^{-1}B)\in\Pi$. Hence $\psi(\alpha A/\mathrm{tr}(\alpha A), {}^t\alpha^{-1}B/\mathrm{tr}({}^t\alpha^{-1}B))$ is defined and is equal to $\psi(\alpha A, {}^t\alpha^{-1}B)$. Now

$$\begin{aligned}\alpha\psi(\mathrm{A}, \mathrm{B})\alpha^{-1}X&=\frac{1}{(A, B)}\alpha(8B\times(A\times\alpha^{-1}X)+2(B, \alpha^{-1}X)A-(A, B)\alpha^{-1}X))\\ &=\frac{1}{(\alpha A, {}^t\alpha^{-1}B)}(8{}^t\alpha^{-1}B\times(\alpha A\times X)+({}^t\alpha^{-1}B, X)A-(\alpha A, {}^t\alpha^{-1}B)X)\\ &=\psi(\alpha A, {}^t\alpha^{-1}B)X\end{aligned}$$

Proposition 6. *If $\psi(A, B)$ belongs to F_4, then $A=B$ and $\psi(A, A)$ coincides with $\varphi(A)$.*

Proof The condition $\psi(A, B)\in F_4$ imples

$$\psi(A, B)=\psi(A, B)^{-1}={}^t\psi(A, B)=\psi(B, A)$$

That is, $\psi(A, B)X=\psi(B, A)X$ holds for all $X\in\mathfrak{J}$. Put $X=A$, then we hawe

$$A=\frac{1}{(A, B)}(8A\times(B\times A)+2(A, A)B-(B, A)A)$$

$$= \frac{1}{(A, B)}(2B - 4A \circ B + 2(A, B)A + 2B - (B, A)A) \qquad \text{(by (III))}$$

From this we have $B = A \circ B$. Similarly put $X = B$, then $A = B \circ A$. Therefore $A = B$. Then

$$\begin{aligned} \psi(A, A)X &= \frac{1}{(A, A)}(8A \times (A \times X) + 2(A, X)A - (A, A)X) \\ &= 2X - 4A \circ X + 2(A, X)A + 2(A, X)A - X \qquad \text{(by (III))} \\ &= X - 4A \circ X + 4(A, X)A = \varphi(A)X \end{aligned}$$

N. Jacobson [3] proved that the simplicity of group E_6 using the properties of elations in the projective plane Π. From Proposition 5 (3), the subgroup generated by $\{\psi(A, B) | A, B \in \Pi, (A, B) \neq 0\}$ is a normal subgroup of E_6. Therefore we have the following

Theorem 7. *E_6 is a simple group generated by* $\{\psi(A, B) | A, B \in \Pi, (A, B) \neq 0\}$.

3. Group $F_{4,1}$ and its polar decomposition

In $\mathfrak{J}$ we define a symmetric inner product $\langle \ , \ \rangle$ by

$$\langle X, Y \rangle = (X, \gamma Y)$$

where γY (denoted also by $\hat{Y}$ later) is

$$\gamma \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_1 & -y_3 & -\bar{y}_2 \\ -\bar{y}_3 & \eta_2 & y_1 \\ -y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}$$

($\gamma = \varphi(E_1)$ is an element of center of $Spin(9)$ (cf. Theorem 2)). Let $F_{4,1}$ be the invariance subgroup of E_6 of this inner product :

$$F_{4,1} = \{\alpha \in E_6 | \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$$

Using the notation of the transpose ${}^t\alpha$ of $\alpha \in E_6$, then the necessary and sufficient condition that an element $\alpha \in E_6$ belongs to $F_{4,1}$ is

$${}^t\alpha\gamma\alpha = \gamma$$

We shall give a polar decmposition of $F_{4,1}$ into a compact subgroup and a euclidean space.

Theorem 8. *The group $F_{4,1}$ contains the group $Spin(9)$ as the isotropy subgroup of E_1 :*

$$Spin(9) = \{\alpha \in F_{4,1} | \alpha E_1 = E_1\}$$

and any element α of $F_{4,1}$ is uniquely represented by the form

$$\alpha = \beta \exp \widetilde{A}, \qquad \beta \in Spin(9), \qquad A = \begin{pmatrix} 0 & a_3 & \bar{a}_2 \\ \bar{a}_3 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}$$

Therefore $Spin(9)$ is a maximal compact subgroup of $F_{4,1}$ and the space $F_{4,1}$ is homeomorphic to the product space of $Spin(9)$ and euclidean space R^{16} :

$$F_{4,1} \simeq Spin(9) \times R^{16}$$

Especially $F_{4,1}$ is a simply connected Lie group.

Proof Represent an element α of $F_{4,1}$ as an element of E_6 in the form

$$\alpha = \beta \exp \widetilde{A}, \qquad \beta \in F_4, \ A \in \mathfrak{J}, \ \mathrm{tr}(A) = 0$$

Since $\beta \in F_4$ and $A \in \mathfrak{J}$ we have ${}^t\beta = \beta^{-1}$ and ${}^t(\exp \widetilde{A}) = \exp \widetilde{A}$ respectively. Now, the condition of $\alpha \in F_{4,1}$ is ${}^t\alpha\gamma\alpha = \gamma$, from which ${}^t(\beta \exp \widetilde{A})\gamma(\exp \widetilde{A}) = \gamma$, $(\exp \widetilde{A})$ $(\beta^{-1}\gamma\beta)(\exp \widetilde{A}) = \gamma$ and hence

$$(\beta^{-1}\gamma\beta) \exp \widetilde{A} = (\exp(-\widetilde{A}))\gamma = \gamma \exp(-\widetilde{\gamma A})$$

(using the formula $\delta(\exp \widetilde{A})\delta^{-1} = \exp(\widetilde{\delta A})$ for $\delta \in F_4$). By the uniqueness of the representation of an element of E_6 by the above form we have

$$\beta^{-1}\gamma\beta = \gamma, \qquad \mathrm{A} = -\gamma \mathrm{A}$$

The first condition is $\beta\varphi(E_1)\beta^{-1} = \varphi(E_1)$, hence $\varphi(\beta E_1) = \varphi(E_1)$. Since φ is injective we have $\beta E_1 = E_1$, therefore $\beta \in Spin(9)$. The second condition $A = -\gamma A$ is equivalent to $\alpha_1 = \alpha_2 = \alpha_3 = a_1 = 0$. Next we shall prove the remainder of the theorem. If an element α of $F_{4,1}$, $\alpha = \beta \exp A$, $\beta \in Spin(9)$, A is the form in the theorem, satisfies the condition $\alpha E_1 = E_1$, then

$$(\exp \widetilde{A})E_1 = \beta^{-1}E_1 = E_1$$

Note $\widetilde{A}E_1 = \dfrac{1}{2}A$ and hence $\widetilde{A}^n E_1 = \dfrac{1}{2}A^n$, then the above condition gives

$$E_1 + \frac{1}{2}(A + A^2/2! + A^3/3! + A^4/4! + \cdots) = E_1$$

form which $\exp A = E$, hence $A = 0$, therefore we have $\alpha = \beta \in Spin(9)$.

4. Element $\psi(A)$ and simplicity of $F_{4,1}$

To investigate some algebraic properties of the group $F_{4,1}$, we shall construct an element $\psi(\mathrm{A})$ of $F_{4,1}$ for $A \in \Pi$ such that $\langle A, A \rangle \neq 0$ as the specification of $\psi(A, B)$ of E_6. Define a mapping $\psi : \{A \in \Pi | \langle A, A \rangle \neq 0\} \to F_{4,1}$ by

$$\psi(A)X = \frac{1}{\langle A, A \rangle}(8\widehat{A} \times (A \times X) + 2\langle A, X \rangle A - \langle A, A \rangle X)$$

and also define

$$\psi(rA)=\psi(A), \qquad r\in \boldsymbol{R}-\{0\}$$

Frist of all, we must show $\psi(A)\in F_{4,1}$, that is,

Theorem 9. *For* $A\in\Pi$, $\langle A, A\rangle\neq 0$, *we have* $\psi(A)\in F_{4,1}$.

Proof First, note that ψ(A) belongs to E_6, because of $\psi(A)=\psi(A,\hat{A})\in E_6$ by Theorem 4. Next

$$\langle\psi(A)X, \psi(A)Y\rangle=(\psi(A)X, \gamma\psi(A)Y)=(\psi(A,\hat{A})X, \gamma\psi(A,\hat{A})Y)$$

Since, as is easily seen, we have $\gamma\psi(A,\hat{A})Y=\psi(\hat{A},A)\hat{Y}$ (note that $\gamma\in F_4$)

$$\begin{aligned}&=(\psi(A,\hat{A})X,\ \psi(\hat{A},A)\hat{Y})\\ &=(X,\psi(\hat{A},A)^2\hat{Y}) \qquad \text{(by Proposition 5 (2))}\\ &=(X,\hat{Y}) \qquad \text{(by Proposition 3)}\\ &=\langle X, Y\rangle\end{aligned}$$

As similar to the properties of φ in Proposition 1 the mapping ψ has the following properties.

Proposition 10. *For* $A\in\Pi$ *such that* $\langle A, A\rangle\neq 0$ *we have*

(1) $\psi(A)A=A$

(2) $\psi(A)^2=1$

(3) $\alpha\psi(A)\alpha^{-1}=\psi(\alpha A), \quad \alpha\in F_{4,1}$

(4) $\psi:\{A\in\Pi|\langle A, A\rangle\neq 0\}\to F_{4,1}$ *is injective.*

Proof (1), (2) are obvious by Proposition 5(1), Proposition 3 respectively.

(3) $\alpha\psi(A)\alpha^{-1}=\alpha\psi(A,\hat{A})\alpha^{-1}=\psi(\alpha A, {}^t\alpha^{-1}\hat{A})$ (by Proposition 5 (3))

Since ${}^t\alpha^{-1}\hat{A}={}^t\alpha^{-1}\gamma A=\gamma\alpha A=\widehat{\alpha A}$ from the condition $\alpha\in F_{4,1}$

$$=\psi(\alpha A, \widehat{\alpha A})=\psi(\alpha A)$$

(4) We must show that

$$\psi(A)X=\psi(B)X \quad \text{for all} \quad X\in\mathfrak{J} \quad \text{follows} \quad A=B$$

Now, putting $X=E_1$, then

$$\frac{1}{(1-2\alpha_1)^2}\begin{pmatrix}1 & 2a_3 & 2\bar{a}_2\\ 2\bar{a}_3 & 4\alpha_1\alpha_2 & 4\alpha_1 a_1\\ 2a_2 & 4\alpha_1\bar{a}_1 & 4\alpha_1\alpha_3\end{pmatrix}=\frac{1}{(1-2\beta_1)^2}\begin{pmatrix}1 & 2b_3 & 2\bar{b}_2\\ 2\bar{b}_3 & 4\beta_1\beta_2 & 4\beta_1 b_1\\ 2b_2 & 4\beta_1\bar{b}_1 & 4\beta_1\beta_3\end{pmatrix} \qquad \text{(i)}$$

From this we have

$$\langle A, A\rangle=(1-2\alpha_1)^2=(1-2\beta_1)^2=\langle B, B\rangle \quad \text{and} \quad a_2=b_2,\ a_3=b_3 \qquad \text{(ii)}$$

Then the condition $\psi(A)X=\psi(B)X$ is reduced to

$$4\hat{A}\times(A\times X)+(\hat{A}, X)A=4\hat{B}\times(B\times X)+(\hat{B}, X)A$$

Put $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}$ and compare (2, 3)-entries, then

$$-|a_2|^2x - |a_3|^2x + \alpha_1{}^2x + 4(a_1, x)a_1 = -|b_2|^2x - |b_3|^2x + \beta_1{}^2x + 4(b_1, x)b_1$$

Hence from (ii)

$$\alpha_1{}^2x + 4(a_1, x)a_1 = \beta_1{}^2x + 4(b_1, x)b_1$$

holds for all $x \in \mathfrak{C}$. Therefore $\alpha_1{}^2 = \beta_1{}^2$, hence $\alpha_1 = \beta_1$. If $\alpha_1 = \beta_1 \neq 0$, then $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $a_1 = b_1$ from (i), therefore we have the required result $A = B$. In the case of $\alpha_1 = \beta_1 = 0$, $A = \widehat{A}$ and $B = \widehat{B}$, hence $\varphi(A) = \psi(A, \widehat{A}) = \psi(B, \widehat{B}) = \varphi(B)$. Since φ is injective we have also $A = B$.

Using these properties of this mapping ψ we can show that the group $F_{4,1}$ is simple.

Theorem 11. *The group $F_{4,1}$ is simple and is generated by* $\{\psi(A) | A \in \Pi, \langle A, A \rangle \neq 0\}$.

Proof As is easily seen, the complexification of Lie algebra of $F_{4,1}$ is isomorphic to the complexification of the Lie algebra of F_4 (which is simple). Hence $F_{4,1}$ is simple as Lie group. The rest of the proof of this theorem is quite analogous to that of Theorem 2.

5. Space $F_{4,1}$/Spin (9)

We shall consider the subspace Int Γ of the plane Π consisting of all elements whose (1, 1)-components are greater that 1/2 :

$$\text{Int}\, \Gamma = \{A \in \Pi \mid (A, E_1) > 1/2\}$$

and call this space Int Γ the *interior of the quadratic curve* $\Gamma = \{A \in \Pi \mid \langle A, A \rangle = 0\}$.

Lemma 12. *For $\alpha \in F_{4,1}$ and $A \in$ Int Γ we have $\alpha A/\text{tr}(\alpha A) \in$ Int Γ.*

Proof First note that $\alpha A/\text{tr}(\alpha A) \in \Pi$. Now, since $F_{4,1}$ is connected (Theorem 8) there exists a path $\alpha(t)$, $0 \leqq t \leqq 1$, connecting α to 1 in $F_{4,1}$. Then, since

$$(\text{tr}(\alpha(t)A) - 2(\alpha(t)A, E_1))^2 = \langle \alpha(t)A, \alpha(t)A \rangle = \langle A, A \rangle = (1 - 2\alpha_1)^2 \neq 0$$

and $\text{tr}(\alpha(t)A) - 2(\alpha(t)A, E_1)$ is continuous with respect to t, $\text{tr}(\alpha(t)A) - 2(\alpha(t)A, E_1)$ has the constant sign. Hence $\text{tr}(\alpha A) - 2(\alpha A, E_1)$ has the same sign as $\text{tr}(A) - 2(A, E_1) = 1 - 2\alpha_1 < 0$. Therefore we have $\text{tr}(\alpha A) - 2(\alpha A, E_1) < 0$. This is the required result $(\alpha A/\text{tr}(\alpha A), E_1) > 1/2$, that is, $\alpha A/\text{tr}(\alpha A) \in$ Int Γ.

Theorem 13. ([4] Théorème 6. 8). *The group $F_{4,1}$ acts transitively on* Int Γ *and the isotropy subgroup of E_1 is Spin(9). Therefore the homogeneous space $F_{4,1}/Spin(9)$ is homeomorphic to* Int Γ :

$$F_{4,1}/Spin(9) \simeq \text{Int}\, \Gamma$$

Proof From Lemma 12 we see that the group $F_{4,1}$ acts on Int Γ by a mapping

$$\mu : F_{4,1} \times \mathrm{Int}\,\Gamma \to \mathrm{Int}\,\Gamma, \quad \mu(\alpha, A) = \alpha A/\mathrm{tr}(\alpha A)$$

We shall show that this action is transitive. For $B \in \Pi$ such that $\beta_1 = (B, E_1) > 1/2$, construct an element A of $\mathfrak{J}$ with the following components

$$\alpha_1 = \frac{\sqrt{\beta_1} + \sqrt{2\beta_1 - 1}}{2\sqrt{\beta_1}}, \qquad a_1 = \frac{b_1}{2\sqrt{\beta_1}(\sqrt{\beta_1} + \sqrt{2\beta_1 - 1})},$$

$$\alpha_2 = \frac{\beta_2}{2\sqrt{\beta_1}(\sqrt{\beta_1} + \sqrt{2\beta_1 - 1})}, \qquad a_2 = \frac{b_2}{2\beta_1},$$

$$\alpha_3 = \frac{\beta_3}{2\sqrt{\beta_1}(\sqrt{\beta_1} + \sqrt{2\beta_1 - 1})}, \qquad a_3 = \frac{b_3}{2\beta_1},$$

then $A \in \Pi$, $\langle A, A \rangle \neq 0$ and $\phi(A)E_1 = B$. This shows the transitivity of the action μ. The fact that theisotropy subgroup of E_1 is *Spin* (9) is in Theorrem 8. Thus the theorem is proved.

References

[1] H. FREUDENTHAL ; *Oktaven, Ausnahmengruppen und Oktavengeometrie*, Math. Inst. Rijks. te Utrecht, 1951 (mimeographed).

[2] ——— ; Beziehungen der E_7 und E_8 zur Oktavenebene. IV. Indag. Math. 16 (1954), 218-230.

[3] N. JACOBSON ; Some groups of transformations defined by Jordan algebras III, Jour. Reine Angew. Math. 207, 61-85 (1961).

[4] J. TITS ; Le plan projectif des octaves et les groupes de Lie exceptionnels, Bull. Acad. Roy. Belg. Sci., 39 (1953), 309-329.

[5] I. YOKOTA ; Embeddings of projective spaces into elliptic projective Lie groups, Proc. Japan Acad., 35 (1959), 291-283.

[6] ——— ; Exceptional Lie Group F_4 and its Representation Rings, Jour. Fac. Sci. Shinshu Univ., Vol. 2 (1968), 35-60.